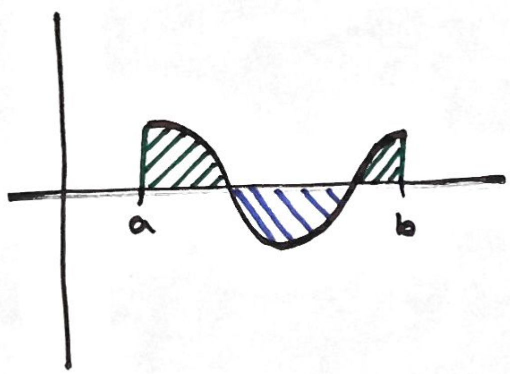
Double Integrals

$f(x)$ = single-variable function, $[a,b]$ closed interval

$$\int_a^b f(x)\,dx = \begin{array}{l}\text{Area }\underline{\text{above}}\\ x\text{-axis bounded}\\ \text{by } y=f(x) \text{ between}\\ a \text{ and } b\end{array} - \begin{array}{l}\text{Area }\underline{\text{below}}\\ x\text{-axis bounded}\\ \text{by } y=f(x) \text{ between}\\ a \text{ and } b\end{array}$$

E.g.



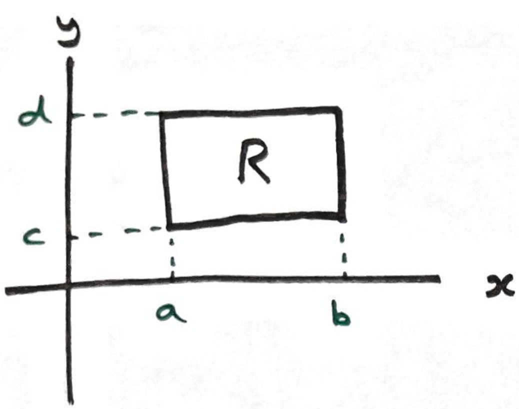
$$\Rightarrow \int_a^b f(x)\,dx = \text{Area }(\ ) - \text{Area }(\ )$$

Fundamental Theorem of Calculus: $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

$(F'(x) = f(x))$

<u>Aim</u>: Generalize to 2-variable functions.

1/ Replace closed interval $[a,b]$ with <u>closed rectangle</u> in xy-plane. Call it R.



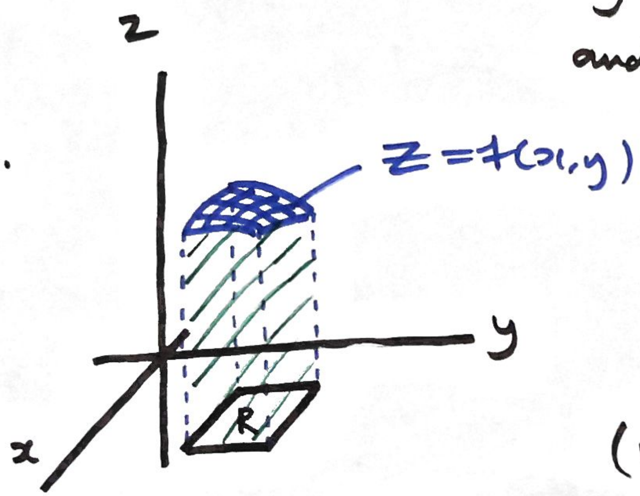
2/ Replace $f(x)$, a continuous function on $[a,b]$, with $f(x,y)$, a continuous function on R.

Definition (Intuitive)

"double integral"

$$\iint_R f(x,y)\,dx\,dy = \underline{\text{Volume above}}\ \text{xy-plane bounded by } z=f(x,y) \text{ and } R. - \underline{\text{Volume below}}\ \text{xy-plane bounded by } z=f(x,y) \text{ and } R.$$

E.g.



$$\Rightarrow \iint_R f(x,y)\,dx\,dy = \mathrm{Vol}(\text{▨})$$

(we're assuming $f(x,y) \geq 0$ on R)

Q/ : How can we calculate $\iint_R f(x,y)\,dx\,dy$?

A/ Treating y like a <u>constant</u> calculate $\int_a^b f(x,y)\,dx$ using fundamental theorem. This will give an expression only involving y.

B/ Now calculate the usual integral $\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$, again using the fundamental theorem.

<u>Fact</u> :

$$\iint_R f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

<u>Example</u> If $a=1$, $b=2$, $c=1$, $d=2$ and $f(x,y) = xy^2$

then $\int_a^b f(x,y)\,dx = \int_1^2 x\underset{\substack{\uparrow \\ \text{constant}}}{y^2}\,dx = y^2 \int_1^2 x\,dx = y^2 \cdot \frac{1}{2}x^2 \Big|_1^2$

$$= y^2\left(2 - \frac{1}{2}\right) = \frac{3y^2}{2}$$

$$\Rightarrow \iint_R x y^2 \, dx\, dy = \int_{1 \leftarrow c}^{2 \leftarrow d} \frac{3y^2}{2}\, dy = \left.\frac{y^3}{2}\right|_1^2 = 4 - \frac{1}{2} = \frac{7}{2}.$$

Fact: $$\iint_R f(x,y)\, dx\, dy = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx$$

Can do x or y first.

Definition Average value of $f(x,y)$ on R $= \dfrac{1}{\text{Area}(R)} \cdot \iint_R f(x,y)\, dx\, dy$

just $(b-a)\cdot(d-c)$

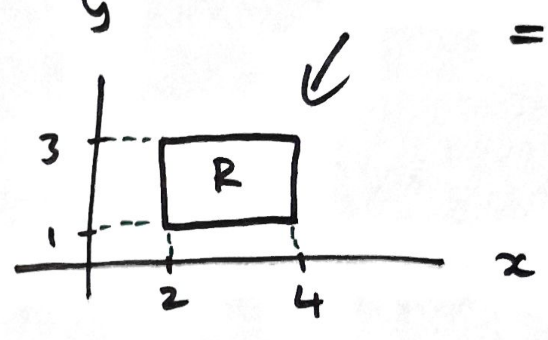
Example A company's total cost for operating two warehouses is $C(x,y) = x^2 + y^2 + 3$, where x = number of units stored at first warehouse and y = number of units stored at second warehouse.

What is the average cost if the first warehouse can have between 2 and 4 units, while the second can have between 1 and 3?

$\Rightarrow \quad 2 \leq x \leq 4$ and $1 \leq y \leq 3$

$\Rightarrow \quad (x, y)$ in rectangle R :
(actually a square)

Area $= 2 \times 2$
$= 4$



$$\Rightarrow \quad \text{Average cost} = \frac{1}{4} \iint_R (x^2 + y^2 + 3)\,dx\,dy$$

$$\int_2^4 x^2 + y^2 + 3 \, dx = \frac{1}{3}x^3 + y^2 x + 3x \Big|_2^4 = \left(\frac{64}{3} + 4y^2 + 12\right) - \left(\frac{8}{3} + 2y^2 + 6\right)$$

$$= \frac{74}{3} + 2y^2$$

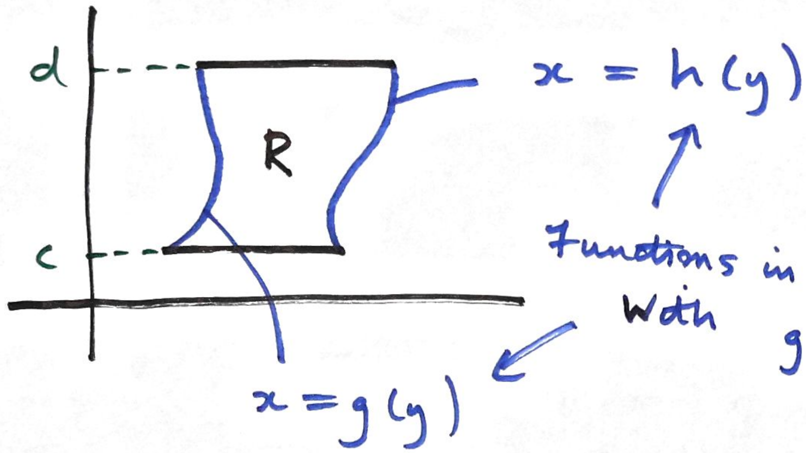
$$\Rightarrow \iint_R x^2 + y^2 + 3 \, dx\,dy = \int_1^3 \frac{74}{3} + 2y^2 \, dy = \frac{74}{3}y + \frac{2}{3}y^3 \Big|_1^3$$

$$= (74 + 18) - \left(\frac{74}{3} + \frac{2}{3}\right) = 66\ 2/3$$

$$\Rightarrow \text{ Average cost } = \frac{66\ 2/3}{4} = 16\ 2/3$$

We can define $\iint_R f(x,y)\,dx\,dy$ for more complicated regions R:

1/



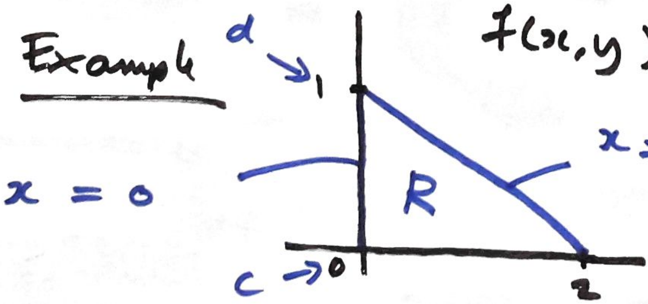
Functions in y with $g(y) \leq h(y)$

If $h(y) = b$ and $g(y) = a$ we get the same rectangle as before.

$$\Rightarrow \iint_R f(x,y)\,dx\,dy = \int_c^d \left(\int_{g(y)}^{h(y)} f(x,y)\,dx \right) dy$$

same as before

<u>Example</u>



$f(x,y) = xy$

$x = 2 - 2y$

(ie $y = 1 - \frac{1}{2}x$)

$c = 0$, $d = 1$

$\rightsquigarrow$ $g(y) = 0$, $h(y) = 2 - 2y$

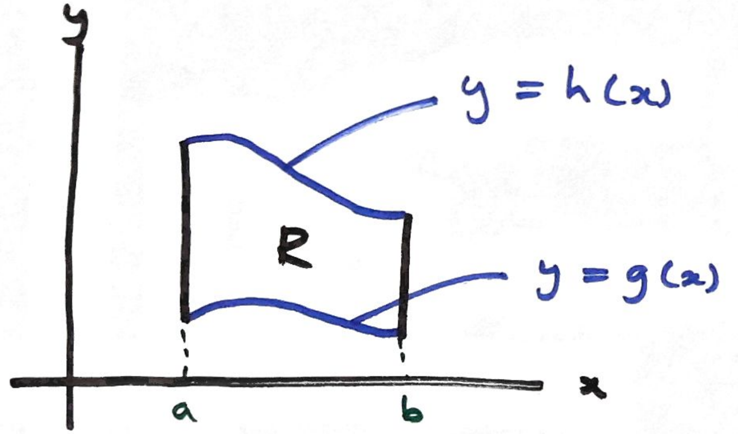
$$\Rightarrow \iint_R xy \, dx\, dy = \int_0^1 \left(\int_0^{2-2y} xy \, dx \right) dy$$

$$\int_0^{2-2y} xy \, dx = \frac{1}{2} x^2 y \Big|_0^{2-2y} = \frac{1}{2}(2-2y)^2 y = 2y - 4y^2 + 2y^3$$

$$\Rightarrow \iint_R xy \, dx\, dy = \int_0^1 2y - 4y^2 + 2y^3 \, dy = y^2 - \frac{4}{3}y^3 + \frac{2}{4}y^4 \Big|_0^1$$

$$= 1 - \frac{4}{3} + \frac{2}{4} = 1/6 .$$

2/



Functions in x with $g(x) \le h(x)$

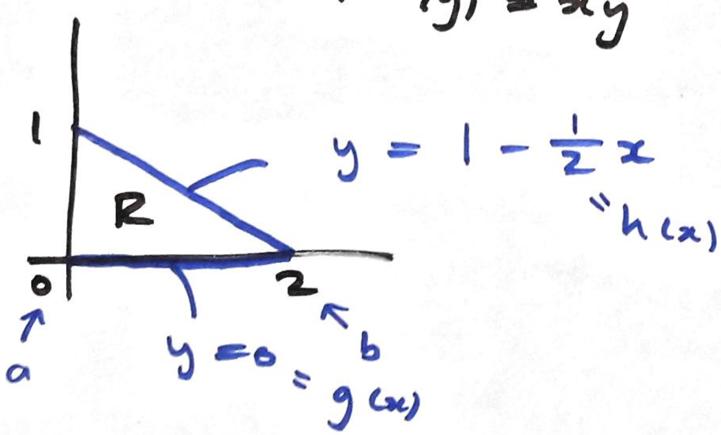
Warning: $$\iint_R f(x,y)\,dx\,dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx$$

Must do y first in general.

Sometimes we can view R as either type 1/ or 2/

Example

$f(x,y) = xy$



$$\Rightarrow \iint_R xy\,dx\,dy = \int_0^2 \left(\int_0^{1-\frac{1}{2}x} xy\,dy \right) dx$$

$$= \int_0^2 \left(\frac{xy^2}{2} \Big|_0^{1-\frac{1}{2}x} \right) dx$$

$$= \int_0^2 x/_2 - \frac{x^2}{2} + \frac{1}{8}x^3\,dx$$

$$= \frac{x^2}{4} - \frac{x^3}{6} + \frac{x^4}{32} \Big|_0^2 = 1/_6 \text{ (as expected)}$$

It's crucial to do dy first in case 2/.