

Differential Equations

A differential equation is an equation that involves an unknown function $y = f(x)$ and a finite number of its derivatives.

Examples $y' = y$, $y' = x$, $y'' + xy' = y$, $e^{y'} + y^2 = 0$

The order of a differential equation is the highest derivative that it involves.

Examples

$$y' = y \quad \text{first order}$$

$$y'' + y = 0 \quad \text{second order}$$

$$y''' \cdot y + x e^x = y' \quad \text{third order}$$

We will focus on first order differential equations.

Main Goal:

Given a differential equation find the most general function which satisfies it (a general solution).

$$\text{Let } u = x^2 + 3 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x} \Rightarrow$$

$$\int \frac{2x}{\sqrt{x^2+3}} dx = \int \frac{1}{\sqrt{u}} du = 2\sqrt{u} + C = 2\sqrt{x^2+3} + C$$

$\Rightarrow y = x^2 + 2\sqrt{x^2+3} + C$ is a general solution.

$$y(1) = 8 \Rightarrow 1 + 2\sqrt{4} + C = 8 \Rightarrow C = 3$$

$\Rightarrow y = x^2 + 2\sqrt{x^2+3} + 3$ is only solution with $y(1) = 8$.

The requirement that $y(x_0) = y_0$ for some fixed (x_0, y_0) is called an initial condition.

Fact: There is at most one solution to a first order differential equation which satisfies a fixed initial condition. We call it the particular solution.