Overview

Aim Find general solution to Diff. Eq.

All Diff. Eqs. ← Far to many

↓

1^{st} Order

- Separable: $\frac{dy}{dx} = p(x)q(y)$
  - Autonomous: $\frac{dy}{dx} = q(y)$ ↓ Graphical methods (if separation fails)
  - constant: $q(y) = 0$
  - non-constant: $\int \frac{1}{q(y)}\,dy = \int p(x)\,dx$
- Linear: $\frac{dy}{dx} + a(x)y = b(x)$

$$y = \frac{1}{e^{A(t)}}\left(\int e^{A(t)}\, b(t)\, dt\right)$$

$$A'(t) = a(t)$$

- Everything Else → No method.

Applications of Differential Equations

Natural Population Growth / Radioactive Decay / Continuous compound Interest :

$$y(t) = \begin{cases} \text{pop. size at time } t \\ \text{mass of material at time } t \\ \text{account balance at time } t \end{cases} \Rightarrow \frac{dy}{dt} = ky \quad \text{where}$$

constant

$$k = \begin{cases} \text{population growth constant } (k>0) \\ \text{Decay constant } (k<0) \\ \text{interest rate } (k>0 \end{cases}$$

Calculated using separation of variable method.

Explicit solution : $y(t) = y_0 e^{kt}$ where

$$y_0 = \begin{cases} \text{initial population} \\ \text{initial mass} \\ \text{initial balance.} \end{cases}$$

Linear

Population growth with immigration / emigration :

natural growth

emigration

$f(t) =$ immigration rate

$g(t) =$ emigration rate $\Rightarrow \dfrac{dy}{dt} = ky + f(t) - g(t)$

$k =$ growth constant

immigration

Example A population with growth constant 1 has immigration rate $2t$ and emigration rate 4. If the initial population is 1000 what is the population a year later?

$$\frac{dy}{dt} = y + 2t - 1 \Rightarrow \frac{dy}{dt} + (-1)y = 2t - 1$$

$$\Rightarrow a(t) = -1, \quad b(t) = 2t - 1 \quad . \quad \text{Let } A(t) = -t$$

$$\Rightarrow y(t) = \frac{1}{e^{-t}} \int e^{-t}(2t-1)\,dt = \frac{1}{e^{-t}}\left(2\int t e^{-t}dt - \int e^{-t}dt\right)$$

$$\int \underset{f(t)}{t}\,\underset{g(t)}{e^{-t}}\,dt \underset{\text{I.B.P.}}{=} -te^{-t} + \int e^{-t}\,dt = -te^{-t} - e^{-t} + \text{Constant}$$

$$\Rightarrow y(t) = \frac{1}{e^{-t}}\left(-2te^{-t} - 2e^{-t} + e^{-t} + C\right)$$

$$= -2t - 1 + Ce^{t}$$

$$y(0) = 1000 \Rightarrow -1 + C = 1000 \Rightarrow C = 1001$$

$$\Rightarrow \quad y(t) = -2t - 1 + 1001e^t$$

$$\Rightarrow \quad y(1) = 1001 \cdot e - 3.$$

Continuous income / withdrawls from savings account :

$f(t) =$ income rate

$g(t) =$ withdrawl rate

$r =$ interest rate

$$\Rightarrow \quad \frac{dy}{dt} = \overset{\text{Interest}}{ry} + \underset{\text{income}}{f(t)} - \overset{\text{withdrawls}}{g(t)}$$

Linear.

<u>Remark</u> If $g(t) = 0$ and $y(0) = 0$ we are in same situation as accumulated value.

<u>Awesome Exercise</u> : Try solving $\frac{dy}{dt} = ry + f(t)$, $y(0) = 0$

You should find $y(T) =$ accumulate value formula.

Example : <u>Paying of a Loan</u>.

You take out a loan of \$4000. Interest on the loan is 10%. You arrange to make payments at a rate

$g(t) = 500 \leftarrow$ 500 dollars per year continuously.

How long will it take to pay of the loan.

$y(t) =$ amount still owed at time t

$\Rightarrow \dfrac{dy}{dt} = \underset{\uparrow \text{ interest}}{0.1y} - \underset{\uparrow \text{ payments}}{500} \quad \Rightarrow \quad \dfrac{dy}{dt} + (-0.1)y = -500$

$y(0) =$ [redacted] $4000 \quad \Rightarrow \quad a(t) = -0.1$

$b(t) = -500 \qquad A(t) = -0.1t$

$\Rightarrow y(t) = \dfrac{1}{e^{-0.1t}} \cdot \displaystyle\int (e^{-0.1t} \cdot -500)\, dt$

$= e^{0.1t} \displaystyle\int -500 e^{-0.1t}\, dt = e^{0.1t}\left(5000 e^{-0.1t} + C\right)$

$= 5000 + Ce^{0.1t}$

$y(0) =$ [redacted] $4000 \Rightarrow 5000 + C =$ [redacted] $4000 \Rightarrow C =$ [redacted] -1000

$\Rightarrow \quad y(t) = 5000 - 1000e^{0.1t}$

$y(t) = 0 \Rightarrow 5000 - 1000e^{0.1t} = 0 \Rightarrow e^{0.1t} = 5 \Rightarrow t = \frac{\ln(5)}{0.1}$.

This is how long it will take to pay off the loan.

What would happen if the initial loan was \$10000 instead?

$y(0) = 10000 \Rightarrow 5000 + C = 10000 \Rightarrow C = 5000$

$\Rightarrow \quad y(t) = 5000 + 5000e^{0.1t}$

$y(t) = 0 \Rightarrow e^{0.1t} = -1$ ← no solutions

$\Rightarrow$ You <u>never</u> pay off the loan.

Logistic Model of Population Growth :

Assumptions :

1/ Population experiences natural growth if small $\Rightarrow$ $\frac{dy}{dt} \approx ky$ for small y.

2/ There is number M, called the carrying capacity, such that the population must decrease if greater than M $\Rightarrow$
If $y > M$ then $\frac{dy}{dt} < 0$

Aim : Find a single differential equation that does both 1/ and 2/

Simplest Example : $\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$ — "Logistic differential Eq."

Remarks

1/ Logistic equation is autonomous, so separable. The problem is we cannot calculate $\int \frac{1}{ky\left(1-\frac{y}{M}\right)}\,dy$. We'll have to use graphic methods.

2/ The logistic equation turns up in all sorts of unexpected applications. For example it can be used to describe how new technologies replace old ones:

$z(t)$ = market share of new product at time t

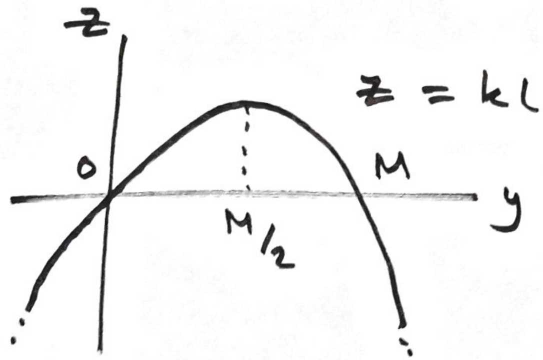
($z=0 \Rightarrow$ no one has product, $z=1 \Rightarrow$ Everyone has product)

$1-z(t)$ = market share of old product at time t.

$$\Rightarrow \quad \frac{dz}{dt} = kz(1-z)$$

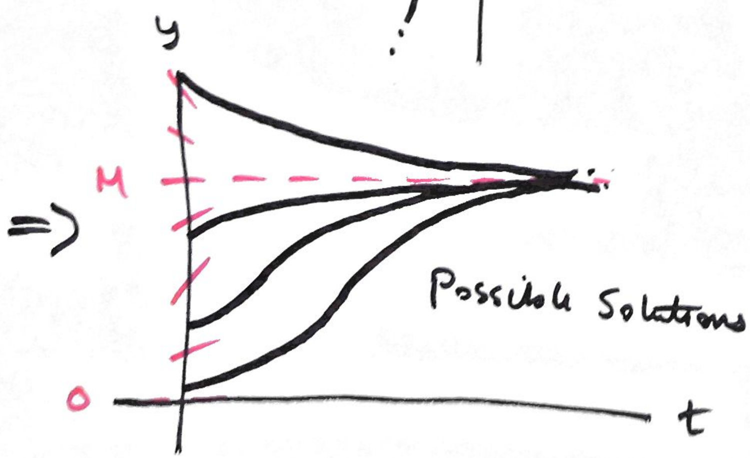
Graphic Solution to $\dfrac{dy}{dt} = ky\left(1-\dfrac{y}{M}\right) = q(y)$

$z = q(y)$:



$z = k(y)\left(1-\frac{y}{M}\right) \leftarrow$ quadratic with 0 and M as zeroes.

$\Rightarrow$



$\Rightarrow$ For any non-zero starting population the population approaches M over time

<u>Fact</u>: An explicit formula for a general non-zero solution is

$$y(t) = \frac{M}{1 + A e^{-kt}}$$ where A is any constant.