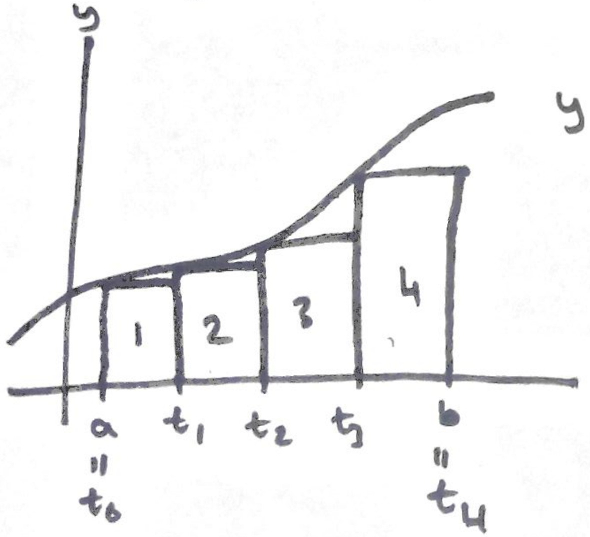
Applications of Integrals

$f(t)$ = continuous function on $[a,b]$, $f(t) \geq 0$

Recall precise definition of $\int_a^b f(t)\,dt$:

Fix n, a positive whole number. Divide $[a,b]$ into n subintervals of equal length $\Delta t = \frac{b-a}{n}$. Label the endpoints $t_0 = a, t_1, \ldots, t_{n-1}, t_n = b$



$y = f(t)$ $(n=4)$

$$\Rightarrow \int_a^b f(t)\,dt \approx \text{Area}(1) + \text{Area}(2) + \text{Area}(3) + \text{Area}(4)$$

$$= f(t_0)\Delta t + f(t_1)\Delta t + f(t_2)\Delta t + f(t_3)\Delta t$$

Approximation gets better as n increases, ie

$$\int_a^b f(t)\,dt = \lim_{n\to\infty} \left(f(t_0)\Delta t + f(t_1)\Delta t + \ldots + f(t_{n-1})\Delta t \right)$$

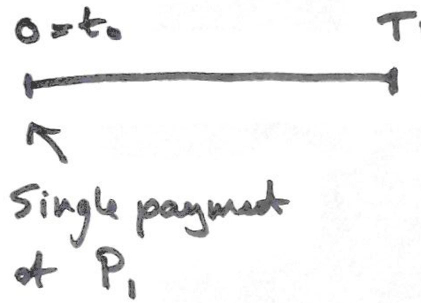
expression is called a Riemann Sum

Assume we have a bank account with annual interest rate r.

Discrete Income Streams

Assume we make regular payments into the account at equal intervals over the period $[0, T]$.

1 Payment:

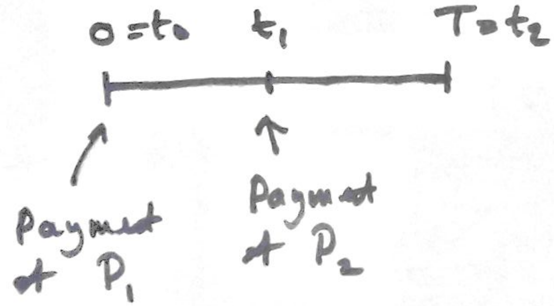


$\Rightarrow$ Total amount in account at T

$$= P_1 e^{rT} = P_1 e^{r(T-t_0)}$$

how long first payment in account

2 Payments:



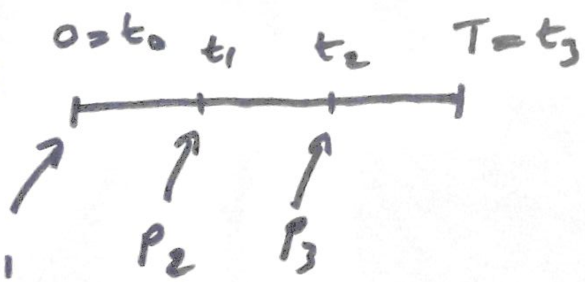
$\Rightarrow$ Total amount in account at T

$$= P_1 e^{r(T-t_0)} + P_2 e^{r(T-t_1)}$$

how long first payment is in account

how long 2nd payment is in account

3 Payments:



$\Rightarrow$ Total amount in account at T

$$= P_1 e^{r(T-t_0)} + P_2 e^{r(T-t_1)} + P_3 e^{r(T-t_2)}$$

<u>n Payments</u> : Total amount in account at time T

$$= \underbrace{P_1 e^{r(T-t_0)}}_{\text{contribution from payment 1}} + \underbrace{P_2 e^{r(T-t_1)}}_{\text{contribution from payment 2}} + \dots + \underbrace{P_n e^{r(T-t_{n-1})}}_{\text{contribution from payment } n}$$

<u>Problem</u> : Clearly hard to calculate as n grows.

<u>Continuous Income Streams</u> :

Imagine now that money is being <u>continuously</u> added to account.

Q/ : What is the total amount in account at time T ?

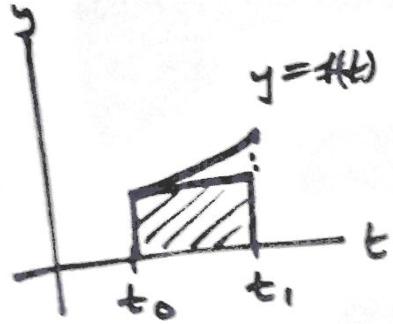
$F(t)$ = total amount of money paid into account over $[0,t]$

$f(t) = F'(t)$ = annual rate of income (t in years)

E.g. $f(t) = 30{,}000 \Rightarrow$ Steady continuous income of \$30,000 per year

<u>Strategy</u> : Approximate continuous income with discrete income.

P_1 = Total income over $[t_0, t_1] = F(t_1) - F(t_0) = \int_{t_0}^{t_1} f(t)\,dt$



$$\Rightarrow P_1 \approx \text{Area}(\ ) = f(t_0)(t_1 - t_0) = f(t_0)\Delta t$$

↑ crude approximation

$$P_2 = \text{Total income over } [t_1, t_2] = F(t_2) - F(t_1) = \int_{t_1}^{t_2} f(t)\,dt \approx f(t_1)\Delta t$$

$$P_3 = \text{Total income over } [t_2, t_3] \approx f(t_2)\Delta t$$

$\vdots$

$$P_n = \text{Total income over } [t_{n-1}, t_n] \approx f(t_{n-1})\Delta t$$

$\Rightarrow$ Total amount of money in account at time T $\approx P_1 e^{r(T-t_0)} + \dots + P_n e^{r(T-t_{n-1})}$

$$= f(t_0)\Delta t\, e^{r(T-t_0)} + f(t_1)\Delta t\, e^{r(T-t_1)} + \dots + f(t_{n-1})\Delta t\, e^{r(T-t_{n-1})}$$

↑ Riemann Sum! for $f(t)e^{r(T-t)}$

Now let n tend to infinity $\Rightarrow$

Total amount of money in account at time T $= \displaystyle\int_0^T f(t)\, e^{r(T-t)}\, dt$

Conclusion:

annual interest rate

Accumulated value of continuous income stream over $[0,T]$ $= \displaystyle\int_0^T f(t)\, e^{r(T-t)}\, dt = e^{rT} \int_0^T f(t)\, e^{-rt}\, dt$

annual income rate

Q/ How much would we need to add to account in a single payment at $t=0$ to end up with the same amount at time T?

Need P_1 such that $P_1 e^{rT} = \displaystyle\int_0^T f(t)\, e^{r(T-t)} = e^{rT} \int_0^T f(t)\, e^{-rt}\, dt$

$\Rightarrow \quad P_1 = \displaystyle\int_0^T f(t)\, e^{-rt}\, dt$

Conclusion:

"Present" value of continuous income stream over $[0,T]$ $= \displaystyle\int_0^T f(t)\, e^{-rt}\, dt$

This is a way to talk about the value at $t=0$ of an asset which will provide a continuous income with rate $f(t)$ over $[0,T]$.

<u>Example</u> A company expects to have a continuous income over the next five years with rate $20000t$. They intend to invest it in an account with interest rate 1%. How much money will be in the account after 5 years? What is the present value?

$$\text{Accumulated value} = \int_0^5 20000t\, e^{0.01(5-t)}\, dt = 20000 e^{0.05} \int_0^5 t e^{(-0.01)t}\, dt$$

$f(t) = t \quad g(t) = e^{-0.01t}$

$f'(t) = 1 \quad G(t) = \frac{1}{-0.01} e^{-0.01t} = -100 e^{-0.01t}$

$$\Rightarrow \int t e^{-0.01t}\, dt = -100 t e^{-0.01t} + 100 \int e^{-0.01t}\, dt$$

$$= -100t\,e^{-0.01t} - 10000\,e^{-0.01t} + C$$

$$= -(10000 + 100t)\,e^{-0.01t} + C$$

$$\Rightarrow \int_0^5 t\,e^{-0.01t}\,dt = -(10000 + 100t)\,e^{-0.01t}\Big|_0^5$$

$$= 10000 - 10500\,e^{-0.05}$$

$$\Rightarrow \text{Accumulated value} = 20000 \cdot e^{0.05}\left(10000 - 10500\,e^{-0.05}\right)$$

$$= 20000\left(10000\,e^{0.05} - 10500\right)$$

$$\approx \$254\,219.27$$

$$\text{Present Value} = \int_0^5 20000t\,e^{-0.01t}\,dt = 20000\left(10000 - 10500\,e^{-0.05}\right)$$

$$\approx \$241\,820.35$$

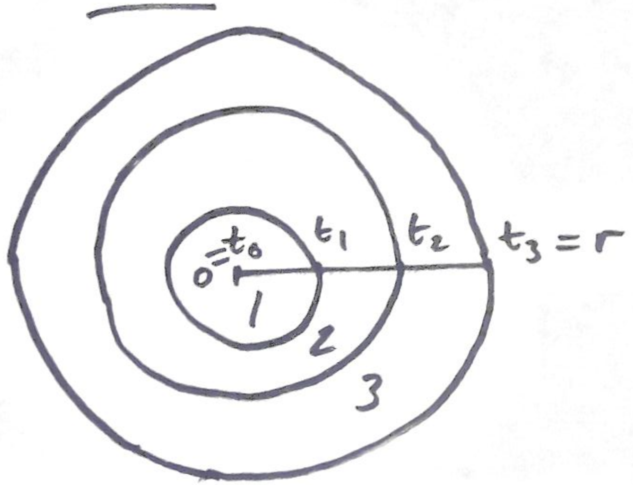
Population Density

$D(t)$ = population density (people per square mile) at distance t from city center.

Q: What is the total population living within distance r of the center?

Strategy: Approximate using concentric rings.

E.g. $n = 3$



$$\text{Total Population within radius } r = Pop(1) + Pop(2) + Pop(3)$$

$$\approx D(t_0) \times Area(1) + D(t_1)\,Area(2) + D(t_2)\,Area(3)$$

Lets further approximate these areas:

$$Area(2) = \pi(t_2)^2 - \pi(t_1)^2 = \pi(t_1 + \Delta t)^2 - \pi t_1^2$$

$$= 2\pi t_1 \Delta t + \pi(\Delta t)^2 \underset{(\Delta t \text{ small})}{\approx} 2\pi t_1 \Delta t$$

Similarly $\quad \text{Area}(1) \approx 2\pi t_0 \Delta t$

$\text{Area}(3) \approx 2\pi t_2 \Delta t$

$\Rightarrow$ Total Population within radius r $\approx D(t_0) \cdot 2\pi t_0 \Delta t + D(t_1) 2\pi t_1 \Delta t + D(t_2) 2\pi t_2 \Delta t$

More generally,

$$\text{Total Pop} \approx \underbrace{D(t_0) \cdot 2\pi t_0 \Delta t + \dots + D(t_{n-1}) 2\pi t_{n-1} \Delta t}_{\text{Riemann Sum for } D(t)2\pi t}$$

$$\Rightarrow \boxed{\text{Total population within radius } r = \int_0^r D(t) \cdot 2\pi t \, dt}$$

Example In 1940 the population density t miles from the center of New York was $120e^{-0.2t}$ (thousand people per square mile). What was the number of people who lived within 2 miles of the center?

$$\text{Population within 2 miles} = \int_0^2 120e^{-0.2t} \cdot 2\pi t \, dt = 240\pi \int_0^2 t e^{-0.2t} \, dt$$

$f(t) = t \quad g(t) = e^{-0.2t}$

$f'(t) = 1 \quad G(t) = -5e^{-0.2t}$

$$\Rightarrow \int t e^{-0.2t} dt = -5te^{-0.2t} + 5\int e^{-0.2t}$$
$$= -5te^{-0.2t} - 25e^{-0.2t} + C$$
$$= -(25+5t)e^{-0.2t} + C$$

$$\Rightarrow \text{Population within 2 miles} = 240\pi\left(25 - 35e^{-0.4}\right) \approx 1160$$

$\Rightarrow$ In 1940 rough 1160000 people lived within 2 miles of the center of New York.