Math 113 Homework 4

To be submitted in class on Thursday July 19

Important Information:

- This homework will be graded for completeness and correctness. Two questions will be selected at random for close scrutiny. Solutions that are unreadable or incoherent will receive no credit. Provide complete justifications for all claims that you make.
- Problems will be of varying difficulty, and do not appear in any order of difficulty. Expect to spend between 5 and 10 hours for each homework set.
- Good Luck!

Question 1

Let $N \subset G$ be a normal subgroup. Let $H \subset G$ be a normal subgroup such that $N \subset H$. Prove that the map

$$\begin{aligned} f : G/N &\to G/H \\ xN &\mapsto xH \end{aligned}$$

is a well-defined surjective homomorphism. Using, this prove that

$$(G/N)/(H/N) \cong G/H.$$

Question 2

Let $B = \{\text{upper triangular invertible matrices}\}$, a subgroup of $GL_2(\mathbb{C})$. Consider the following subgroups of B

$$T = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{C}^\times \right\}, \quad U = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{C} \right\}.$$

1. Prove that $U \subset B$ is normal in B.
2. Show that T is not normal in B.
3. Prove that $f : T \to B/U$; $t \mapsto tU$ is an isomorphism.
4. Explain why B is not isomorphic to $T \times U$.

Question 3

Let G be a finite Abelian group with $|G| = p_1 p_2 \cdots p_n$, where the p_i are distinct primes. Prove that G is cyclic.

Question 4

1. Determine, up to isomorphism, a complete list of Abelian groups of order 360.

2. Let $m = p_1^{a_1} \cdots p_n^{a_n}$, where p_i are distinct primes and $a_i \in \mathbb{N}$. Prove that, up to isomorphism, the number of Abelian groups of size m is the product $b_1 b_2 \cdots b_n$, where b_i equals the number of partitions of a_i.

Question 5

Let G be a group and assume that $H \subset G$ is a subgroup. Prove that the following defines an action of G on G/H:

$$\begin{aligned} f : G \times G/H &\rightarrow G/H \\ (x, yH) &\mapsto (xy)H \end{aligned}$$

Using this, prove that if H is a non-trivial subgroup of finite index in G, then there exists a normal subgroup of finite index $N \subset G$, such that $N \neq G$.