Math 113 Homework 3

To be submitted in class on Thursday July 12

Important Information:

- This homework will be graded for completeness and correctness. Two questions will be selected at random for close scrutiny. Solutions that are unreadable or incoherent will receive no credit. Provide complete justifications for all claims that you make.
- Problems will be of varying difficulty, and do not appear in any order of difficulty. Expect to spend between 5 and 10 hours for each homework set.
- Good Luck!

Question 1

Prove that $Alt_n \subset Sym_n$ is a normal subgroup. How many distinct conjugacy classes of Sym_6 are contained in Alt_6?

Question 2

Prove that the intersection of two normal subgroups is normal.

Question 3

Let G be a group and $H \subset G$ be a subgroup. Given $g \in G$, let

$$gHg^{-1} = \{ghg^{-1} | h \in H\}.$$

Prove that gHg^{-1} is a subgroup of G. Let $Sub(G)$ denote the set of all subgoups of G. Prove that this construction defines and action of G on $Sub(G)$.

Question 4

Determine all possible subgroups of Sym_3. Let Sym_3 act on $Sub(Sym_3)$ as in question 3. Describe all the orbits of this action. Calculate the stabiliser subgroups of $gp(\{(12)\})$ and $gp(\{(123)\})$.

Question 5

Determine all possible orders of elements on Sym_5. How many elements have order 2? How many have order 3?

Question 6

Prove that if every subgroup of a group G is normal then elements of coprime order commute. (Hint: $x, y \in G$ commute if and only if $x^{-1}y^{-1}xy = e$).

Question 7

For $x \in \mathbb{R}^2$ we denote by $Rot(x) \subset Isom(\mathbb{R}^2)$ the set of all rotations about x. Let $X \subset \mathbb{R}^2$ with the property that $Rot(x) \subset Sym(X)$. Prove that $Sym(X)$ contains a reflection. Is it true that $Sym(\{x\}) = Sym(X)$?

Question 8

Recall that in the first homework we defined the group $(\mathbb{R}/\mathbb{Z}, +)$. Let $\mathbb{Q}/\mathbb{Z} := \{[x] \in \mathbb{R}/\mathbb{Z} | x \in \mathbb{Q}\}$. Find a subset $X \subset \mathbb{R}^2$ such that $Sym(X) \cong \mathbb{Q}/\mathbb{Z}$. Hint: $\mathbb{R}/\mathbb{Z}$ is isomorphic to the group of rotational symmetric of a circle.

Question 9

Let G be a group and $H \subset G$ a subgroup. Recall that the right cosets of H in G are subsets of the form

$$Hg = \{h * g | h \in H\}.$$

Prove that H is normal if and only if $Hg = gH$ for all $g \in G$. Using this, or otherwise, prove that any subgroup of index two must be normal.

Question 10

Let $n \in \mathbb{N}$, $n > 2$. Determine all of the conjugacy classes of D_n. How many are there?

Question 11

Determine all subgroups of D_4. Which ones are normal?