Math 113 **Homework 1**

To be submitted in class on Thursday June 28

Important Information:

- This homework will be graded for completeness and correctness. Two questions will be selected at random for close scrutiny. Solutions that are unreadable or incoherent will receive no credit. Provide complete justifications for all claims that you make.
- Problems will be of varying difficulty, and do not appear in any order of difficulty. Expect to spend between 5 and 10 hours for each homework set.
- Good Luck!

Question 1

Let S be a set with five elements. Find (a) the number of maps from S to itself, (b) the number of bijections from S to itself.

Question 2

Let A, B and C be three sets. If $f : A \to B$ and $g : B \to C$, we may compose them to get $g \circ f : A \to C$. Show that if both f and g are injective then so is $g \circ f$. Is the same true for surjectivity? If $g \circ f$ is injective what can be said about f and g?

Question 3

Let $f : S \to T$ be a map. Prove that f is a bijection if and only if there exists a map $g : T \to S$ such that $g \circ f = Id_S$ and $f \circ g = Id_T$.

Question 4

Outlined in Lecture notes

Using only the eight properties of $+$ and $\times$ on $\mathbb{Z}$ prove that

1. For any $a \in \mathbb{Z}$, $a \times 0 = 0$.
2. For $a \in \mathbb{Z}$, $-a = (-1) \times a$ (here $-a$ is the additive inverse of a).

Question 5

Let $a, b \in \mathbb{Z}$ be two coprime integers. Prove that there exist infinitely many $x, y \in \mathbb{Z}$ such that $ax + by = 1$. Fix two such solutions $x_0, y_0 \in \mathbb{Z}$. In terms of these two solutions write down a complete list.

Question 6

Let $c, n \in \mathbb{Z}$ and $n > 1$. Show that the equation $x^n = c$ has no rational solutions which are not in $\mathbb{Z}$.

Question 7

Let C be the set of numbers appearing on a standard clock face. For $a, b \in C$ define

$$a \sim b \;\Leftrightarrow\; b = a \;\underline{\text{or}}\; b \text{ appears next to } a \text{ on the clock face.}$$

Is an equivalence relation on C? If so, prove it; if not, explain why.

Question 8

Let S be a nonempty set and $U \subset S \times S$ an equivalence relation on S. If $a, b \in S$ and $b \in [a]$, prove that $[b] = [a]$.

Question 9

Define the equivalence relation on $\mathbb{R}$ by

$$\text{Given } a, b \in \mathbb{R},\ a \sim b \iff a - b \in \mathbb{Z}.$$

You do not need to prove this is an equivalence relation. Let us write $\mathbb{R}/\mathbb{Z}$ for the equivalence classes. Define addition (a binary operation) on $\mathbb{R}/\mathbb{Z}$ by

$$\begin{aligned} + : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} &\longrightarrow \mathbb{R}/\mathbb{Z} \\ ([x], [y]) &\longrightarrow [x + y] \end{aligned}$$

Note that the + in the square bracket indicates addition in $\mathbb{R}$.

1. Prove this is a well defined binary operation, i.e. is independent of the equivalence class representative chosen.
2. Prove that $(\mathbb{R}/\mathbb{Z}, +)$ is an abelian group.
3. Prove that we cannot define $\times$ on $\mathbb{R}/\mathbb{Z}$ in a similar way.

Question 10

Let $S = \{a, b, c\}$. How many binary operations on S give it the structure of a group?

Question 11

Let $f : S \to S'$ be a map of sets. Prove that

$$s \sim t \;\Leftrightarrow f(s) = f(t),$$

defines an equivalence relation on S. In the case of the function $f : \mathbb{R}^2 \to \mathbb{R}\ ;\ (x, y) \mapsto x - y$, describe the equivalence classes geometrically.