Midterm Review

A group is a set G, equipped with a binary operation
$* : G \times G \to G$ such that
$(x,y) \mapsto xy$ ← notation

1. $(xy)z = x(yz)$ $\forall x,y,z \in G$ (associative)
2. $\exists e \in G$ s.t. $xe = ex = x$ $\forall x \in G$ (identity)
3. Given $x \in G$, $\exists y \in G$ s.t. $xy = yx = e$ (inverses)

written x^{-1}

Examples: $(\mathbb{Z},+)$, $(\mathbb{Q}\setminus\{0\}, \times)$, $(GL_n(\mathbb{R}), \times)$, $(\mathbb{Z}/n\mathbb{Z}, +)$

Facts
1. The identity and inverses are unique
2. If $xy = yx$ $\forall x,y \in G$ we say G Abelian.

$H \subset G$ a subgroup $\iff$
1. $e \in H$
2. $x,y \in H \Rightarrow xy \in H$
3. $x \in H \iff x^{-1} \in H$

$\Big\}$ Makes H a group under induced composition.

left coset of H containing x.

Given $x \in G$, $xH = \{xh \mid h \in H\} \subset G$

Facts
1. $\{xH \mid x \in G\}$ form a partition of G
2. $yH = xH \iff x^{-1}y \in H$
3. $|H| < \infty \Rightarrow |H| = |xH|$ $\forall x \in H$

1, 3, $\Rightarrow$ Lagrange's Theorem : $|G| < \infty \Rightarrow |H| \mid |G|$

Let $X \subset G$ (just a subset). Define $X^{-1} = \{x^{-1} \mid x \in X\}$

$gp(X) :=$ All finite compositions of elements of $X \cup X^{-1} \cup \{e\}$

subgroup generated by X

Facts : $gp(X) \subset G$ is a subgroup.

We say G is finitely generated (f.g.) if $\exists X \subset G$ s.t. 1/ $|X| < \infty$ and 2/ $gp(X) = G$.

G cyclic $\iff \exists x \in G$ s.t. $G = gp(\{x\})$ $= \{x^a \mid a \in \mathbb{Z}\}$

All Abelian

G cyclic $\Rightarrow G \cong (\mathbb{Z}, +)$ or $G \cong (\mathbb{Z}/n\mathbb{Z}, +)$

Given $x \in G$, $ord(x) = |gp(\{x\})| = \min m \in \mathbb{N}$ s.t. $x^m = e$

$|G| < \infty$ and cyclic $\iff \exists x \in G$ s.t. $ord(x) = |G|$

Facts 1/ G cyclic and $m \mid |G| \Rightarrow \exists! H \subset G$ a subgroup s.t. $|H| = m$

2/ Every subgroup of cyclic group is cyclic

Let G and H be two groups. A homomorphism from G to H is a map $\phi : G \to H$ s.t. $\phi(xy) = \phi(x)\phi(y)$ $\forall x, y \in G$

composition in G composition in H

$\phi : G \longrightarrow H$ a homomorphism $\Rightarrow \phi(e_G) = e_H$ and $\phi(x^{-1}) = \phi(x)^{-1}$ $\forall x \in G$

Isomorphism = Bijective homomorphism

$G \cong H \iff \exists \phi : G \to H$ an isomorphism (say G isomorphic to H)

$Im\, \phi = \{\phi(g) \mid g \in G\} \subset H$ is a subgroup

ϕ injective $\Rightarrow G \cong Im\, \phi$.

Let S be a set.

binary operation = composition of functions

Permutation group of $S := \Sigma(S) = \{f : S \to S \mid f \text{ bijection}\}$.

$\mathrm{Sym}_n = \Sigma(\{1, 2, \dots, n\})$ for $n \in \mathbb{N}$

n^{th} finite symmetric group

An action of G on S is a map $\begin{array}{l} G \times S \to S \\ (g, s) \mapsto g(s) \end{array}$

s.t. 1) $e(s) = s$ and 2) $(gh)(s) = g(h(s))$ $\forall h, g \in G, s \in S$.

Equivalently, an action of G on S is a homomorphism:

$$\varphi : G \to \Sigma(S)$$

Say action is <u>faithful</u> if φ injective.

<u>Example</u>: Left regular representation: $\begin{array}{l} G \times G \to G \\ (g, x) \longmapsto gx \end{array}$

composition in G

$\Rightarrow$ <u>Cayley's Theorem</u>: G is isomorphic to a subgroup of $\Sigma(G)$. $|G| = n \Rightarrow G$ isomorphic to a subgroup of Sym_n.

Let G act on S.

orbit of s

Given $s \in S$, $\mathrm{orb}(s) = \{g(s) \mid g \in G\} \subset S$

<u>Fact:</u> The orbits form a partition of S

ie $t \in \mathrm{orb}(s) \iff \mathrm{orb}(t) = \mathrm{orb}(s)$

We say the action is <u>transitive</u> $\iff \mathrm{orb}(s) = S \;\forall s \in S$

$\iff$ given $s, t \in S$, $\exists g \in G$ s.t. $g(s) = t$.

$\mathrm{Stab}(s) = \{g \in G \mid g(s) = s\} \subset G$.

stabilizer subgroup of s.

Orbit-Stabilizer: $|G| < \infty \Rightarrow |G| = |\mathrm{Stab}(s)| \cdot |\mathrm{orb}(s)|$.

$\Rightarrow |\mathrm{orb}(s)| \mid |G|$ <u>and</u> $|\mathrm{Stab}(s)| \mid |G|$

O-S. Lagrange

$|Sym_n| = n!$

cycle of length r

Cycle notation : $\sigma = (a_1 a_2 \cdots a_r)(a_{r+1} \cdots a_s) \cdots (a_m \cdots a_n)$

<u>Fact</u> : Every $\sigma \in Sym_n$ can be written as a product of disjoint cycles. The lengths of the cycles give a partition of n, called the cycle structure of σ.

E.g. $\sigma = (123)(45) \in Sym_5$ has cycle structure 3, 2

$\sigma = (32)(1)(5)(4) \in Sym_5$ has cycle structure 2, 1, 1, 1

ord (σ) = LCM of cycle lengths in disjoint decomposition.

Lowest common multiple

$(a\,b)$ = transposition.

<u>Fact</u> : $\sigma \in Sym_n$ can be written as a product of transpositions. The number transpositions must always either be <u>odd</u> or <u>even</u>. We call σ <u>odd</u> or <u>even</u> respectively.

$Alt_n = \{\sigma \in Sym_n \mid \sigma \text{ even}\} \subset Sym_n$

alternating subgroup

generated by cycles of length 3 $(a\,b\,c)$

$|Alt_n| = \dfrac{|Sym_n|}{2} = \dfrac{n!}{2}$

$Isom(\mathbb{R}^n) = \{\phi : \mathbb{R}^n \to \mathbb{R}^n \mid d(\underline{x}, \underline{y}) = d(\phi(\underline{x}), \phi(\underline{y})) \; \forall \underline{x}, \underline{y} \in \mathbb{R}^n\}$

group of isometries of $\mathbb{R}^n$

$X \subset \mathbb{R}^n$ (just a subset)

$\mathrm{Sym}(X) := \{ \sigma \in \mathrm{Isom}(\mathbb{R}^n) \mid \sigma \text{ permutes } X \}$

Symmetry group of X.

$D_n = \mathrm{Sym}(\text{Regular } n\text{-gon}) = n^{th}$ Dihedral Group

E.g. $D_3 = \mathrm{Sym}(\triangle)$

$$D_n = \{ \underbrace{e, \sigma, \sigma^2, \dots, \sigma^{n-1}}_{Rot(X) = \text{ Subgroup of rotations}}, \underbrace{\tau, \tau\sigma, \tau\sigma^2, \dots, \tau\sigma^{n-1}}_{\tau Rot(X) = \text{ Left coset of reflections}} \}$$

σ: rotation by $\frac{2\pi}{n}$ clockwise

τ: Reflection

$\mathrm{ord}(\sigma) = n$, $\mathrm{ord}(\tau) = 2$, $\sigma^d \tau = \tau \sigma^{-d} \quad \forall d \in \mathbb{Z}$

$\Rightarrow D_n$ non-Abelian and $|D_n| = 2n$

Let $N \subset G$ be a subgroup. We say N <u>normal</u> (written $N \triangleleft G$) in G if $n \in N, g \in G \Rightarrow gng^{-1} \in N$.

<u>Facts</u> $N \triangleleft G \Leftrightarrow gN = Ng$ (right coset) $\forall g \in G$

We say G is simple if $N \triangleleft G \Rightarrow N = \{e\}$ or G

<u>Facts</u> $N \triangleleft G \Rightarrow$
$$G/N \times G/N \longrightarrow G/N$$
$$(xN, yN) \longmapsto (xy)N$$
is well-defined and gives G/N the structure a group. We call it the quotient group.

The map $\phi: G \longrightarrow G/N$, $x \longrightarrow xN$ is called the quotient homomorphism.

3^{rd} Isomorphism Theorem: $N \triangleleft G$.

1. There is a bijection

$$\left\{ \begin{matrix} \text{Subgroups of } G \\ \text{containing } N \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{subgroups} \\ \text{of } G/N \end{matrix} \right\}$$

$$H \longmapsto H/N$$

2. $H/N \triangleleft G/N \iff H \triangleleft G$ and $(G/N)/(H/N) \cong G/H$

Let $\phi: G \to H$ be a homomorphism

$\ker\phi = \{ g \in G \mid \phi(g) = e_H \} \subset G$

kernel of ϕ

Fact: $\ker\phi \triangleleft G$

1^{st} Isomorphism Theorem: $G/\ker\phi \to \operatorname{Im}\phi$

$x\ker\phi \longmapsto \phi(x)$

is a well-defined isomorphism.

$\Rightarrow$ If $|G| < \infty$ then $|G| = |\ker\phi| \cdot |\operatorname{Im}\phi|$.

Direct Products

Let $H_1, \dots, H_r$ be groups.

$H_1 \times H_2 \times \dots \times H_r$ has a group structure given by

Composition in H_r

$(h_1, h_2, \dots, h_r) * (g_1, g_2, \dots, g_r) := (h_1 g_1, h_2 g_2, \dots, h_r g_r)$

$(H_1 \times H_2 \times \dots \times H_r, *) =$ Direct Product of $H_1, \dots, H_r$

$H_i \cong \underbrace{\{(e_{H_1}, e_{H_2}, \dots, e_{H_{i-1}}, h_i, e_{H_{i+1}}, \dots, e_{H_r}) \mid h_i \in H\}}_{\text{Subgroups of } H_1 \times H_2 \times \dots \times H_r} \subset H_1 \times \dots \times H_r$

Direct Sums

subgroups

$H_1, \ldots, H_r \subset G$

$G = H_1 \oplus H_2 \oplus \cdots \oplus H_r \iff$

1/ Given $g \in G$ $\exists!\, h_i \in H_i$ such that $g = h_1 h_2 h_3 \cdots h_r$

2/ $h_i h_j = h_j h_i \quad \forall\, h_i \in H_i,\ h_j \in H_j,\ i \neq j$

Fact : $H_1 \oplus \cdots \oplus H_r \cong H_1 \times H_2 \times \cdots \times H_r$

Finitely Generated Abelian Groups

G - f.g. Abelian group

$tG = \{ x \in G \mid \mathrm{ord}(x) < \infty \} \subset G$

torsion subgroup

G/tG f.g. and torsion-free $\Rightarrow$ G/tG is f.g. free-Abelian

$\Rightarrow$ G/tG has a $\mathbb{Z}$-basis

$\mathrm{rank}(G) := \mathrm{rank}(G/tG)$ $\Leftarrow$ size of a $\mathbb{Z}$-basis

Fact : $\exists\, F \subset G$ a f.g. free-Abelian group s.t. $\mathrm{rank}(F) = \mathrm{rank}(G) = n$ and

$$G = F \oplus tG \cong \mathbb{Z}^n \times tG$$

$\Rightarrow$ $tG \cong G/F$ $\Rightarrow$ tG f.g. and torsion $\Rightarrow |tG| < \infty$

Let $|tG| = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, p_i distinct primes, $\alpha_i \in \mathbb{N}$

$\Rightarrow$ $tG = (tG)_{p_1} \oplus \cdots \oplus (tG)_{p_k}$ where

$$(tG)_{p_i} = \{ x \in tG \mid p_i^{\alpha_i} x = 0 \} \subset tG$$

$\Rightarrow (tG)_{p_i} = C_1 \oplus \dots \oplus C_d$ where C_i are cyclic subgroups. The C_i are unique up to isomorphism. $|C_i| =$ power of p_i.

$\Rightarrow (tG)_{p_i} \cong \mathbb{Z}/p_i^{a_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p_i^{a_d}\mathbb{Z}$

Putting all this together gives:

<u>Structure Theorem for Finitely Generated Abelian Groups</u>

Let G be a finitely generated Abelian group. Then $\exists\ C_1, C_2, \dots, C_n$ cyclic groups such that

1) $G \cong C_1 \times C_2 \times \dots \times C_n$

2) C_i is either $(\mathbb{Z}, +)$ or $(\mathbb{Z}/p^k\mathbb{Z}, +)$ for p prime and $k \in \mathbb{N}$

3) If $G \cong D_1 \times \dots \times D_m$ is another such expression then $n = m$ and after reordering $C_i = D_i$.