A minimal pair of K-degrees.

Barbara F. Csima
and
Antonio Montalbán.
Cornell University.

www.math.cornell.edu/∼antonio

K-degrees

- For $\sigma \in 2^{<\omega}$, let $K(\sigma)$ be the prefix-free Kolmogorov complexity of σ.

- [Levin, Chaitin] A real $\alpha \in 2^{\omega}$ is *random* if
$$(\forall n)\ K(\alpha \upharpoonright n) \geq n - \mathcal{O}(1)$$

- A real $\alpha \in 2^{\omega}$ is *K-trivial* if
$$(\forall n)\ K(\alpha \upharpoonright n) \leq K(n) - \mathcal{O}(1)$$

- [Downey, Hirschfeldt, LaForte '01]
We use K to define a notion of relative randomness between reals:
$$\alpha \leq_K \beta \iff (\forall n)\ K(\alpha \upharpoonright n) \leq K(\beta \upharpoonright n) + \mathcal{O}(1).$$

- As usual, $\leq_K$ induces a degree structure.

Some known facts about the K-degrees

- There is a minimal K-degree.
  It consists of the K-trivial reals.

- [Yu, Ding, Downey] There are $2^{\aleph_0}$ many K-degrees, indeed $2^{\aleph_0}$ many among the K-randoms.

- [J.Miller, Yu] For every random real there is another random real strictly K-below it.

- [Downey, Hirschfeldt, LaForte] The structure of K-degrees of c.e. reals is an upper-semi lattice, and join corresponds to real addition.

- [Downey, Hirschfeldt, LaForte]
  If α and β are c.e. reals such that $\alpha <_K \beta$, there is a c.e. real γ such that $\alpha <_K \gamma <_K \beta$.

- $\leq_T \;\not\Rightarrow\; \leq_K$

- [Solovay] $\leq_K \;\not\Rightarrow\; \leq_T$

Main question.

Question: [Downey, Hirschfeldt]
Is there a minimal pair of K-degrees?

In other words:
Are there reals α and β such that

- α and β aren't K-trivial, and
- if $\gamma \leq_K \alpha$ and $\gamma \leq_K \beta$, then γ is K-trivial?

Lemma: There exists a non-decreasing, unbounded function f such that, $\forall \gamma \in 2^{\omega}$, the following are equivalent.

(1) γ is K-trivial.

(2) For a.e. n, $K(\gamma \restriction n) \leq K(n) + f(n)$.

(3) $(\forall n)\ K(\gamma \restriction n) \leq K(n) + f(n) + \mathcal{O}(1)$.

Definition: We call such a function f a *K-bounding function*.

Notation For $c \in \omega$, $\alpha \in 2^{\omega}$, we say that α is *K-trivial(c)*, if it is K-trivial with constant c. i.e.,

$$(\forall n)\ K(\alpha \restriction n) \leq K(n) + c.$$

The proof of the lemma uses only one property about K:

Theorem [Zambella]

For each c, there are finitely many K-trivials(c).

Complexity of K-bounding functions.

Looking at the proof of our Lemma, we get that there is a K-bounding function $\leq_T 0'''$.

Lemma:[Downey, J. Miller]
A K-bounding function cannot be $\leq_T 0'$.

Definition: Given $c \in \omega$,
let $G(c)$ be the number of K-trivials(c).

Question: [Downey, Nies]
What's the complexity of G?

Observation:[Downey, Nies]
There is a K-bounding function $\leq_T 0'' \oplus G$

Relativized K-bounding functions

Definition: f is *K-bounding over α* if
$(\forall n)\ K(\gamma \upharpoonright n) \leq K(\alpha \upharpoonright n) + f(n) \ \Rightarrow\ \gamma \leq_K \alpha$

Note: K-bounding = K-bouning over 0^ω.

Question: Are there K-bounding functions over other reals?

Observation: If α is such that
$$(\exists^\infty n)\ K(\alpha \upharpoonright n) \leq K(n) + \mathcal{O}(1),$$
then such a K-bounding function over α exists.

Lemma:[Csima, J. Miller, M.] If α is such that
$$(\forall n)\ K(\alpha \upharpoonright n) \geq K(n) + g(n) - \mathcal{O}(1),$$
where $g(n) = \min\{K(m) : m \geq n\}$, and f is a function with limit ∞,
then there exists $2^{\aleph_0}$ many γ's such that
$$(\forall n)\ K(\gamma \upharpoonright n) \leq K(\alpha \upharpoonright n) + f(n) \text{ and } \gamma \not\leq_K \alpha.$$

Main Theorem.

Theorem:
There exists a minimal pair of K-degrees.

Proof: Let f be a K-bounding function.

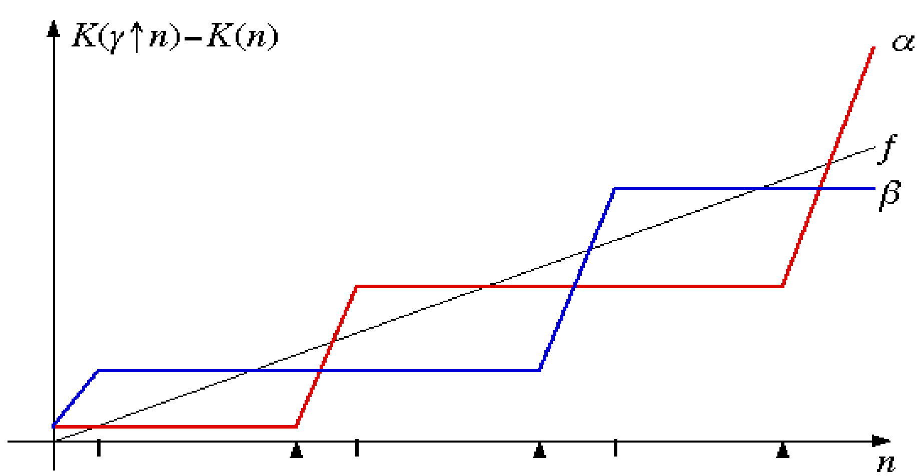
We construct two non-K-trivial reals α and β such that

$$(\forall n) \quad \min\{K(\alpha \restriction n), K(\beta \restriction n)\} \quad \leq \quad K(n) + f(n).$$

Note that $\gamma \leq_K \alpha$ and $\gamma \leq_K \beta$,

$\Rightarrow K(\gamma \restriction n) \leq K(n) + f(n) + \mathcal{O}(1)$,

$\Rightarrow \gamma$ is K-trivial.

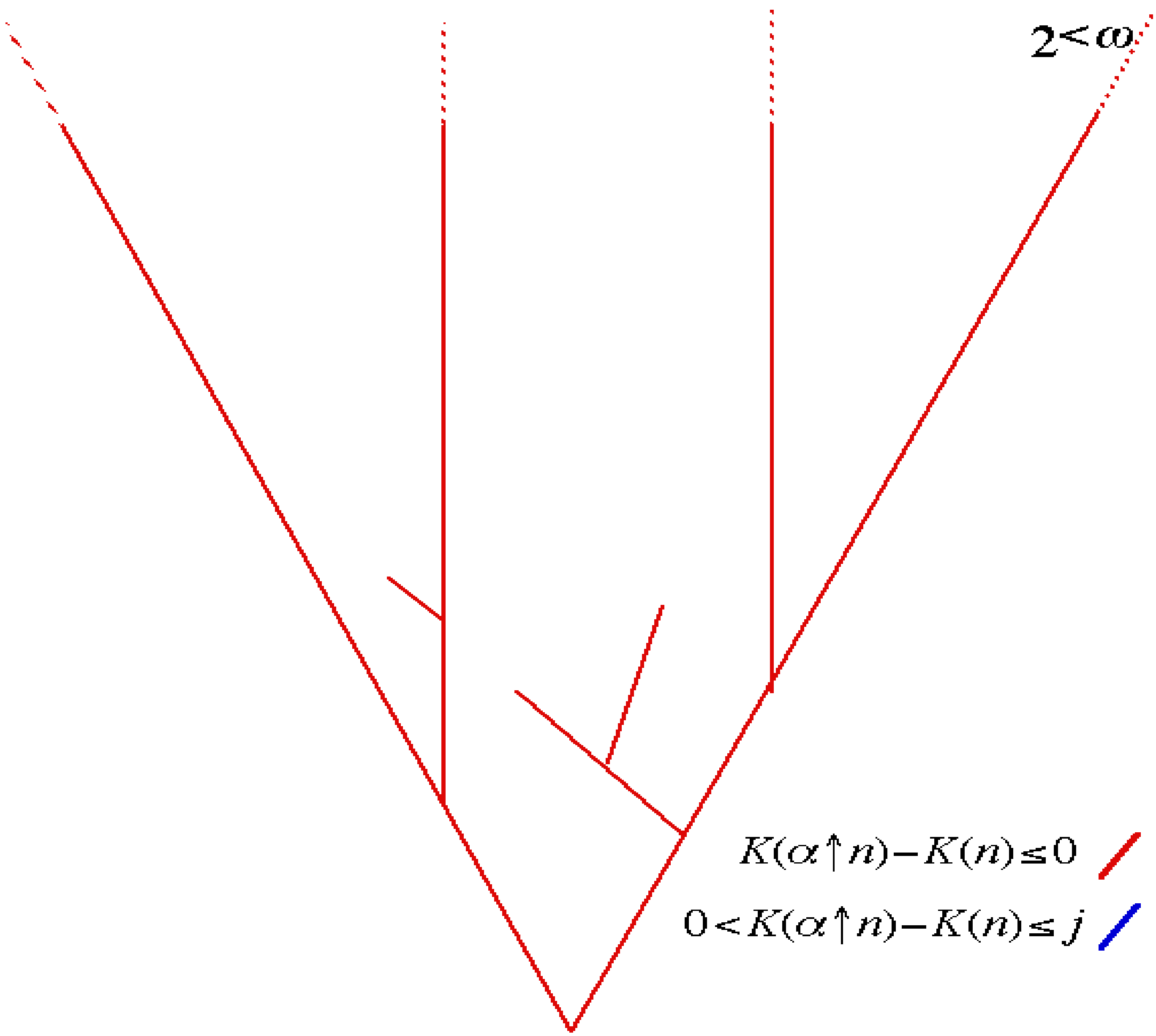


We want to construct a K-bounding function f.
i.e., a non-decreasing, unbounded function f such that

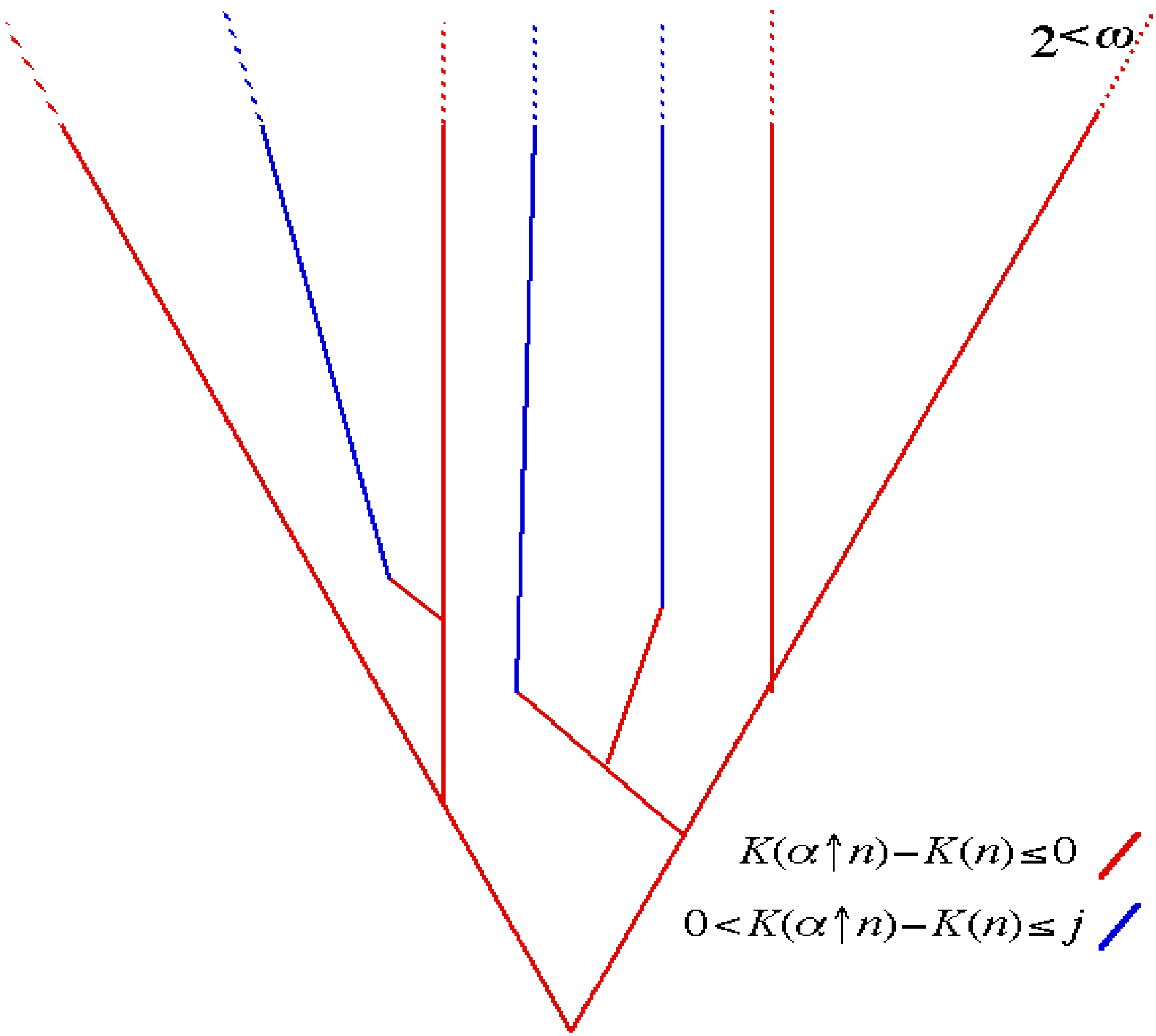
$$\text{if } K(\alpha \upharpoonright n) \leq K(n) + f(n) + \mathcal{O}(1) \;\Rightarrow\; \alpha \text{ is } K\text{-trivial.}$$

We start by constructing f_0 such that

$$\text{if } K(\alpha \upharpoonright n) \leq K(n) + f_0(n), \text{ then } \alpha \text{ is } K\text{-trivial}(0).$$

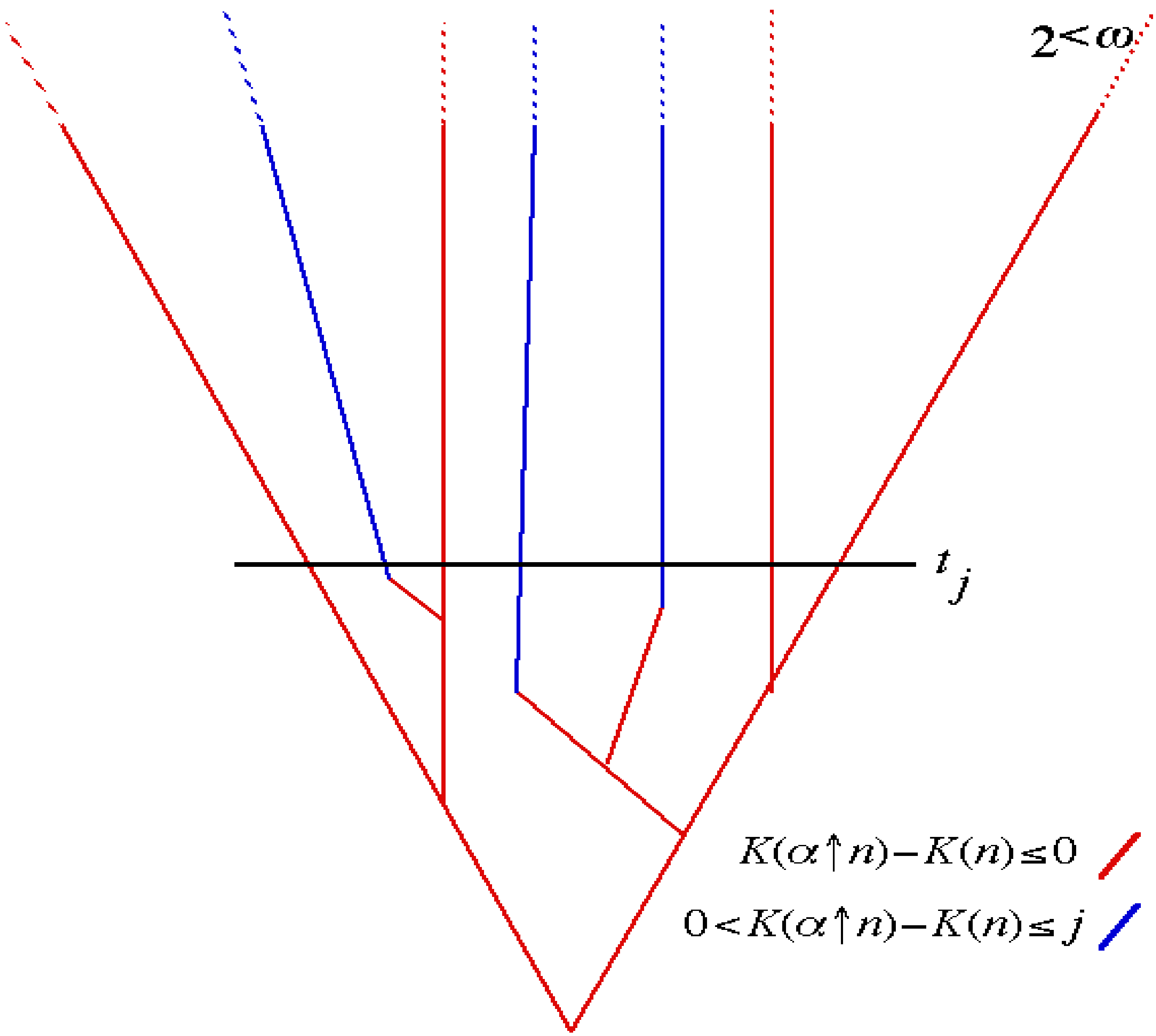


For each $j \in \omega$, there are only finitely many reals which are K-tivial(j).

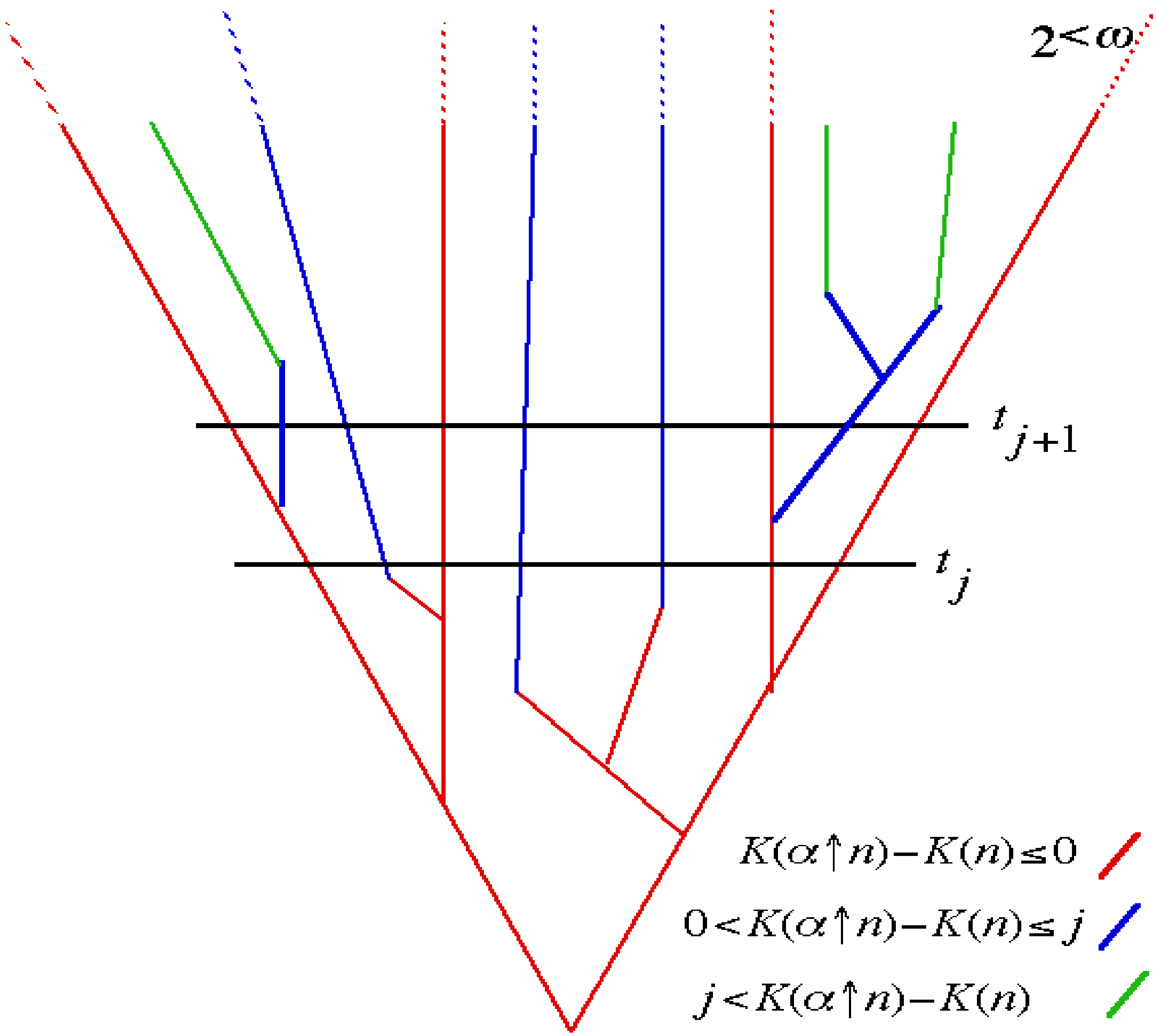


For each $j \in \omega$, there are only finitely many reals which are K-tivial(j).

Let t_j be such that every α that is K-trivial(j), but not K-trival(0), is already not K-trivial(0) by t_j.



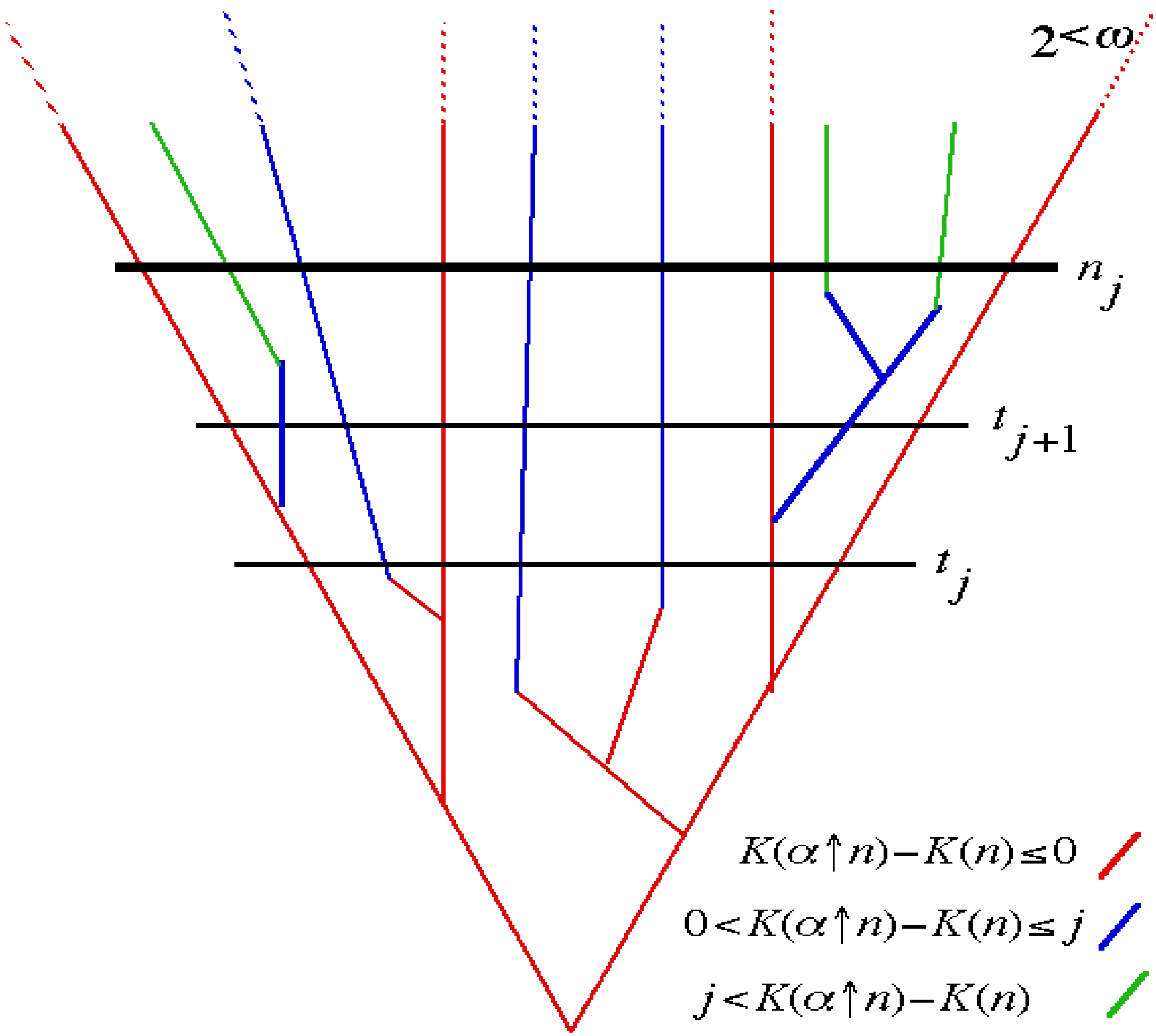
Now, every real α which is not K-trival(0), but that it stops being K-trivial(0) after t_j, is not K-trivial(j). So, at some n, $K(\alpha \restriction n) - K(n) > j$.



Now, every real α which is not K-trival(0), but that it stops being K-trivial(0) after t_j, is not K-trivial(j). So, at some n, $K(\alpha \restriction n) - K(n) > j$.

Let n_j be such that every such α has stopped being K-trivial(j) by n_j.

We define f_0 such that for every $m \leq n_j$, $f_0(m) \leq j$.



Construction of f.

Let $f_0(n) = j$ for every m, $n_{j-1} < m \leq n_j$.

For every α which is not K-trivial(0), there exists j such that α stops being K-trvial(0) between t_j and t_{j+1}. So, at some $m \leq n_j$, α stops being K-trivial(j). Then,

$$K(\alpha \restriction m) \quad \geq \quad K(m) + j + 1 \quad > \quad K(m) + f_0(m).$$

We have constructed f_0 such that

if $K(\alpha \restriction n) \leq K(n) + f_0(n)$, then α is K-trivial(0).

We want to define f such that

if $K(\alpha \restriction n) \leq K(n) + f(n) + \mathcal{O}(1) \ \Rightarrow \alpha$ is K-trivial.

For each $e \in \omega$, let f_e be a non-decreasing unbounded function such that $f_e(0) = e$ and

if $K(\alpha \restriction n) \leq K(n) + f_e(n) \ \Rightarrow \alpha$ is K-trivial(e).

Let $f(n) = \min\{f_{2e}(n) - e : e \in \omega\}$.

If α is a real such that

$K(\alpha \restriction n) \leq K(n) + f(n) + e$ for some constant e,

Then $K(\alpha \restriction n) \leq K(n) + f_{2e}(n)$, and hence α is K-trivial($2e$).