Complete sets of Π_n^c relations.

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Intrinsically Π_n relations.

Let $\mathcal{A}$ be an $\mathcal{L}$-structure.

Def: A relation R on $\mathcal{A}$ is *relatively intrinsically computable* if in every copy $(\mathcal{B}, Q) \cong (\mathcal{A}, R)$, Q is computable in $\mathcal{B}$.

Def: A relation R on $\mathcal{A}$ is *relatively intrinsically c.e.* Σ^0_n Π^0_n if in every copy $(\mathcal{B}, Q) \cong (\mathcal{A}, R)$, Q is c.e. Σ^0_n Π^0_n in $\mathcal{B}$.

Ex: on a lin.ord. let *Succ* $= \{(a, b) \in \mathcal{A}^2 : a < b \,\&\, \nexists c\ (a < c < b)\}$. Then *Succ* is relatively intrinsically co-c.e.

Q: Given $\mathcal{A}$, which relations on $\mathcal{A}$ are the relatively intrinsically Π_n?

Computably infinitary formulas.

A *computably infinitary $\mathcal{L}$-formula* is a
1st order $\mathcal{L}$-formula where infinite conjunctions or disjunctions are allowed, so long as they are c.e.

A Σ^c_1 Π^c_1 *$\mathcal{L}$-formula* is one of the form $\bigvee_{j\in\omega} \exists \bar{y}$ $\bigwedge_{j\in\omega} \forall \bar{y}$ $\psi_j(\bar{z}, \bar{y})$
where $\{\psi_j : j \in \omega\}$ is a comp. list of finitary quantifier-free $\mathcal{L}$-formulas.

A Σ^c_{n+1} Π^c_{n+1} *$\mathcal{L}$-formula* is one of the form $\bigvee_{j\in\omega} \exists \bar{y}$ $\bigwedge_{j\in\omega} \forall \bar{y}$ $\psi_j(\bar{z}, \bar{y})$
where $\{\psi_j : j \in \omega\}$ is a comp. list of Π^c_n Σ^c_n $\mathcal{L}$-formulas.

Thm: [Ash,Knight, Mennasse,Slaman] [Chisholm]
Let R be a relation on $\mathcal{A}$. TFAE

- R is relatively intrinsically Π^0_n.
- R is definable in $\mathcal{A}$ by a Π^c_n $\mathcal{L}$-formula.

Complete sets of Π^c_n relations

Let P_0, P_n,... be uniformly Π^c_n relations on $\mathcal{A}$.

Definition ([M])

$\{P_0, P_1, ...\}$ is a *complete set of Π^c_n relations on* $\mathcal{A}$ if every Π^c_n $\mathcal{L}$-form. is unif- equivalent to a $\Sigma^{c,0'}_1$ $(\mathcal{L} \cup \{P_0, ...\})$form.

Obs: There is always a complete set of Π^c_n relations.
Namely the set of all Π^c_n relations.

Lemma: (1) $\implies$ $\iff$ (2)
(1) $\{P_0, P_1, ...\}$ is a complete set of Π^c_n relations on $\mathcal{A}$
(2) whenever R is rel. intrinsically Π^c_n on $\mathcal{A}$
in any copy $\mathcal{B}$ of $(\mathcal{A}, P_0, P_1, ...)$, $R^{\mathcal{B}} \leq_T \mathcal{B}$. (2)
whenever $R_0, R_1, ..$ are uniformly rel. intrinsically Π^c_n on $\mathcal{A}$
in any copy $\mathcal{B}$ of $(\mathcal{A}, P_0, P_1, ...)$, $\bigoplus_i R_i^{\mathcal{B}} \leq_T \mathcal{B}$.

Example: Π^c_1 relations on Linear orderings

Let $\mathcal{A}$ be a linear ordering.
Let $Succ = \{(a, b) \in \mathcal{A}^2 : a < b \,\&\, \nexists c\ (a < c < b)\}$.

Example:
On a linear order, $\{Succ\}$ is a complete set of Π^c_1 relations.

Every Σ^c_2 relation on $\mathcal{A}$ is a $0'$-disjunction of
Σ_1 finitary formulas in the language $\{\leq, Succ\}$.

Example: Π^c_2 relations on Linear orderings

Let $D_1 = \{(a,b) \in \mathcal{A}^2 : a < b \ \& \nexists c_0, c_1 \text{in between}, Succ(c_0,c_1)\}$
Let $D_n = \{(a,b) \in \mathcal{A}^2 : a < b \ \&$
$\nexists c_0, ..., c_n \text{in between}, \bigwedge_{i<n} Succ(c_i, c_{i+1})\}$
Let $D_n^{+\infty} = \{a \in \mathcal{A}^2 : a < b \ \&$
$\nexists c_0, ..., c_n > a, \bigwedge_{i<n} Succ(c_i, c_{i+1})\}$

Example:
The relations $\{Succ, D_1, D_2, D_3, ..., D_1^{+\infty}, ...D_1^{-\infty}...\}$ are a complete set of Π^c_2 relations.

Every Σ^c_3 relation on $\mathcal{A}$ is a $0'$-disjunction of Σ_1 finitary formulas in the language $\{\leq, Succ, D_1, D_2, ...\}$.

Q: Is there a finite complete set of Π^c_2 relations on linear ords.?
Q: Are there natural complete set of Π^c_3 relations on linear ords.?

Jump Inversion

Lemma

Let P_0, P_1,... be a complete set of Π^c_1 relations on $\mathcal{A}$.
If $Y \geq_T 0'$ computes a copy of $(\mathcal{A}, P_0, P_1, ...)$, then
$\exists$ X that computes a copy of $\mathcal{A}$ and $X' \equiv_T Y$.

So, $$DegSp(\mathcal{A}, P_0, P_1, ...) \cap \mathcal{D}_{(\geq 0')} = \{\mathbf{x}' : \mathbf{x} \in DegSp(\mathcal{A})\}$$

Other similar Jump Inversion Theorems have been considered by I. Soskov, A. Soskova, V. Baleva, A. Stukachev.

Jump of a structure

Lemma

Let $P_0, P_1, \ldots$, and $R_0, R_1 \ldots$ be complete sets of Π^c_n relations on $\mathcal{A}$. Then, there is a uniform computable procedure that in any presentation $\mathcal{B}$ of $(\mathcal{A}, P_0, P_1, \ldots)$ computes $R_0^{\mathcal{B}}, R_1^{\mathcal{B}}, \ldots$, and vise-versa.

Definition ([M])

Let $\mathcal{A}$ be an $\mathcal{L}$-structure.
The *jump of $\mathcal{A}$* is an $\mathcal{L}_1$-structure $\mathcal{A}'$ where:
$\mathcal{L}_1$ is $\mathcal{L} \cup \{P_0, P_1, \ldots\}$,
$P_0, P_1, \ldots$ is a complete set of Π^c_1 relations on $\mathcal{A}$
and $\mathcal{A}' = (\mathcal{A}, P_0, P_1, \ldots)$.

Boolean algebras

Theorem ([Harris, M] rels. used by Downey-Jockusch, Thurber, Knight-Stob)

Let $\mathcal{B}$ be a Boolean algebra.

$\mathcal{B}' = (\mathcal{B}, At^{\mathcal{B}})$

$\mathcal{B}'' = (\mathcal{B}, At^{\mathcal{B}}, Inf^{\mathcal{B}}, Atless^{\mathcal{B}})$.

$\mathcal{B}''' = (\mathcal{B}, At^{\mathcal{B}}, Inf^{\mathcal{B}}, Atless^{\mathcal{B}}, atomic^{\mathcal{B}}, 1\text{-}atom^{\mathcal{B}}, atominf^{\mathcal{B}})$.

$$\mathcal{B}^{(4)} = (\mathcal{B}, At^{\mathcal{B}}, Inf^{\mathcal{B}}, Atless^{\mathcal{B}}, atomic^{\mathcal{B}}, 1\text{-}atom^{\mathcal{B}}, atominf^{\mathcal{B}}, \sim\text{-}inf^{\mathcal{B}}, Int(\omega+\eta)^{\mathcal{B}}, infatomicless^{\mathcal{B}}, 1\text{-}atomless^{\mathcal{B}}, nomaxatomless^{\mathcal{B}}).$$

Furthermore, $\forall n$ there is a finite complete set of Π^c_n relations

For $\mathcal{B}^{(5)}$ we use 27 relations. For $\mathcal{B}^{(6)}$ we use 1578 relations.

Q: Does every low$_n$-BA have a computable copy? [DJ 94]

Thm: [Knight Stob 00] Every low$_4$ BA has a computable copy.

[Harris, M] The case $n = 5$ is essentially different than the previous cases.

Example: Boolean algebras

Thm: [Downey, Jockush] [Thurber] [Knight, Stob] For $n = 1, 2, 3, 4$, Every low$_n$ BA is $0^{(n+2)}$-isomorphic to a computable one.

Theorem ([Harris, M.])

There is a low$_5$ BA not $0^{(7)}$-isomorphic to any computable one.

Lemma ([Harris, M.] Diagonalization Lemma)

Given a computable list of 5-BAs, there is a computable 5-BA that is not $0^{(2)}$-isomorphic to any BA in the list.

A 5-BA, is a Boolean algebra together with its set of complete Π^c_5-relations.

- $0^{(5)}$ can list all computable BAs and calculate their Π^c_5-relations.
- Given this $0^{(5)}$-list $\{\mathcal{B}^{(5)}_0, \mathcal{B}^{(5)}_1, \ldots\}$, there is
  a $0^{(5)}$-comp 5-BA $\mathcal{B}^{(5)}$ not $0^{(7)}$-isomorphic to any $\mathcal{B}_i$.
- By the jump inversion, there is a low$_5$ degree that computes a copy of $\mathcal{B}$.

Strongly-complete set of Π^c_n relations

There is always a complete set of Π^c_n-relations, namely the set of all the Π^c_n-relations.

Q: Which structures have natural complete set of Π^c_n-relations?

Let $\mathcal{K}$ be a class of structures and $\mathcal{P}_0, P_1, \ldots$ be a list of Π^c_n-formulas.

Definition ([M])

$\{P_0, P_1, \ldots\}$ is a *strongly-complete set of Π^c_n formulas for $\mathcal{K}$* if every $\Pi^{c,Z}_n$ $\mathcal{L}$-form. is unif- equivalent to a $\Sigma^{c,Z'}_1$ $(\mathcal{L} \cup \{P_0, \ldots\})$form.

Examples:

- $\{Succ\}$ is a strongly-complete set of Π^c_1 formula for lin. orders.
- $\{Succ, D_1, D_2, D_3, \ldots., D_1^{+\infty}, \ldots D_1^{-\infty} \ldots\}$ is a strongly-complete set of Π^c_2 formulas for the class of lin. orders.
- $\{Atom\}$ is a strongly-complete set of Π^c_1 formulas for BAs.
- ...

Back and forth relations

Theorem[Ash, Knight] Let $\mathcal{A}$ and $\mathcal{B}$ be structures. TFAE

1. Given $\mathcal{C}$ that's isomorphic to either $\mathcal{A}$ or $\mathcal{B}$,
   deciding whether $\mathcal{C} \cong \mathcal{A}$ is $\mathbf{\Sigma}^0_n$-hard.
2. All the infinitary Π_n sentences true in $\mathcal{A}$ are true in $\mathcal{B}$.

If so we say that $\mathcal{A}$ is *back-and-forth below* $\mathcal{B}$
$\mathcal{A} \leq_n \mathcal{B}$.

If $\mathcal{A} \leq_n \mathcal{B}$ and $\mathcal{B} \leq_n \mathcal{A}$, then $\mathcal{A} \equiv_n \mathcal{B}$.

Negative examples

Theorem ([M] Work in progress.)

If there are uncountably many $\equiv_n$-equivalent classes in $\mathcal{K}$, then there is no countable strongly-complete set of Π^c_n form for $\mathcal{K}$.

Obs: There are $2^{\aleph_0}$ many $\equiv_3$-equivalent classes of linear orders, for example for each increasing $f:\mathbb{N}\to\mathbb{N}$ consider

$$\mathbb{Q}+f(1)+\mathbb{Q}+f(2)+\mathbb{Q}+f(3)+\mathbb{Q}+\dots.$$

Corollary: There is NO countable strongly-complete set of Π^c_3 formulas for linear orderings.

Questions

Q: What are natural classes that have natural complete set of Π^c_n-formulas?

Q: Is there a finite complete set of Π^c_2 formulas for linear orderings?

Section 2.

Strongly-complete set of Π_n^c relations for Boolean algebras.

Theorem for BAs - restated

Theorem ([Harris, M])

$\forall n$ there is a finite strongly-complete set of Π^c_n relations for Boolean algebras

Back-and-forth relations are key to prove this Lemma.

Back-and-Forth relations

Notation: $a_1, \ldots, a_k$ is a partition of a BA $\mathcal{B}$ if

$$a_0 \vee \ldots \vee a_k = 1 \text{ and } \forall i \neq j \ (a_i \wedge a_j = 0).$$

We write $\mathcal{B} \upharpoonright a$ for the BA whose domain is $\{x \in \mathcal{B} : x \leq a\}$.

Theorem[Ash, Knight] TFAE

1. $\mathcal{A} \leq_n \mathcal{B}$.
2. Given $\mathcal{C}$ that's isomorphic to either $\mathcal{A}$ or $\mathcal{B}$, deciding whether $\mathcal{C} \cong \mathcal{A}$ is $\mathbf{\Sigma}^0_n$-hard.
3. All the infinitary Σ_n sentences true in $\mathcal{B}$ are true in $\mathcal{A}$.
4. for every partition $(b_i)_{i \leq k}$ of $\mathcal{B}$, there is a partition $(a_i)_{i \leq k}$ of $\mathcal{A}$ such that $\forall i \leq k$
$$\mathcal{B} \upharpoonright b_i \leq_{n-1} \mathcal{A} \upharpoonright a_i \ .$$

The bf-types

Obs: $\equiv_n$ is an equivalence relation on the class of BAs.

We call the equivalence classes *n-bf-types*.

We study the following family of *ordered monoids*

$$(BAs/\equiv_n\ ,\ \leq_n\ ,\ \oplus)$$

where $\mathcal{A} \oplus \mathcal{B}$ is the product BA with coordinatewise operations,
together with the projections $(\cdot)_{n-1}\ :\ BAs/\equiv_n \longrightarrow BAs/\equiv_{n-1}$.

Theorem

$BAs/\equiv_n$ is countable.

The invariants

For each n we define a set $\mathbf{INV}_n$ of finite objects, and an invariant map $T_n\colon BAs \to \mathbf{INV}_n$ such that

$$\mathcal{A} \equiv_n \mathcal{B} \quad \Longleftrightarrow \quad T_n(\mathcal{A}) = T_n(\mathcal{B})$$

Moreover, on $\mathbf{INV}_n$ we define $\leq_n$ and $+$ so that

$$(BAs/\equiv_n, \leq_n, \oplus) \cong (\mathbf{INV}_n, \leq_n, +),$$

We also define a computable operator $\mathcal{B}_{::}\colon \mathbf{INV}_n \to BAs$

$$(\forall \sigma \in \mathbf{INV}_n)\ T_n(\mathcal{B}_\sigma) = \sigma.$$

In general such operator doesn't need to be computable

Def: For each $\sigma \in \mathbf{INV}_n$ we define a unary relation R_σ on a BA $\mathcal{A}$:

$$\mathcal{A} \models R_\sigma(x) \iff \mathcal{A}\restriction x \geq_n \mathcal{B}_\sigma.$$

Theorem

In the case of BAs, the relations R_σ for $\sigma \in \mathbf{INV}_n$ are uniformly Π^c_n.

In general these relations are infinitary Π_n, but not computable.

Theorem

$\{R_\sigma : \sigma \in \mathbf{INV}_n\}$ is a strongly-complete set of Π^c_n relations.

Idea: For every Π^c_n sentence ψ,
notice that if $\sigma \in \mathbf{INV}_n$, $\mathcal{B}_\sigma \models \psi$, and $\mathcal{A} \models R_\sigma$, then $\mathcal{A} \models \psi$.
So,

$$\psi \iff \bigvee_{\sigma : \mathcal{B}_\sigma \models \psi} R_\sigma,$$

which is $\Sigma_1^{c,0^{(n)}}$.

Indecomposable Boolean Algebras

Definition

A BA $\mathcal{A}$ is *n-indecomposable* if for every partition $a_1, ..., a_k$ of $\mathcal{A}$, there is an $i \leq k$ such that $\mathcal{A} \equiv_n \mathcal{A} \restriction a_i$.

Theorem

1. *Every BA is a finite sum of n-indecomposable BAs.*
2. *There are finitely many $\equiv_n$-equivalence classes among the n-indecomposable BAs.*

Let $\mathbf{BF}_n = \{\sigma : \mathcal{B}_\sigma \text{ is } n\text{-indecomposable}\} \subset \mathbf{INV}_n$.

$\mathbf{BF}_n$ is a finite generator of $(\mathbf{INV}_n, \leq_n, +)$.

n	1	2	3	4	5	6	...
$\lvert\mathbf{BF}_n\rvert$	2	3	5	9	27	1578	...

Boolean Algebra Predicates

Def: For each $\sigma \in \mathbf{INV}_n$ we define a unary relation R_σ on a BA $\mathcal{A}$:

$$\mathcal{A} \models R_\sigma(x) \iff \mathcal{A}\restriction x \geq_n \mathcal{B}_\sigma.$$

Theorem

$\{R_\alpha : \alpha \in \mathbf{BF}_n\}$ *is a strongly-complete set of* Π^c_n *formulas for BAs.*

Picture - Levels 1, 2 and 3

bf-relations for 1- and 2-indecomposable bf-types

$$\begin{array}{cccc}
\underline{\text{projection}} & \underline{a_0} & \underline{b_0} & \underline{b_1} \\
 & & & \\
 & b_0 & c_0 & c_2 \\
 & | & & | \\
 & b_1 & & c_1
\end{array}$$

bf-relations for 3-indecomposable bf-types

$$\begin{array}{cccc}
\underline{\text{projection}} & \underline{c_0} & \underline{c_1} & \underline{c_2} \\
 & & & \\
 & d_0 & d_1 & d_4 \\
 & & | & \\
 & & d_2 & \\
 & & | & \\
 & & d_3 &
\end{array}$$

Picture - Level 4

bf-relations for 4-indecomposable bf-types

projection	d_0	d_1	d_2	d_3	d_4
	e_0	e_1	e_3	e_7	e_8
			e_2	e_4	
				e_5	
				e_6	

bf-relations for 5-indecomposable bf-types