Indecomposable linear orderings and hyperarithmetic analysis

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Computational Prospects of Infinity
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Setting: Second order arithmetic.

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Π^1_1-CA_0: Π^1_1-Comprehension + ACA_0.

Models

A model of (the language of) second order arithmetic is a tuple

$$\langle X, \mathcal{M}, +_X, \times_X, 0_X, 1_X, \leqslant_X \rangle,$$

where $\mathcal{M}$ is a set of subsets of X and $\langle X, +_X, \times_X, 0_X, 1_X, \leqslant_X \rangle$ is a structure in language of 1st order arithmetic.

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ω-models are determined by their second order parts, which are subsets of $\mathcal{P}(\omega)$.

We will identify subsets $\mathcal{M} \subseteq \mathcal{P}(\omega)$ with ω-models.

The class of ω-models of a theory

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The class of *HYP*, of hyperarithmetic sets, is not a model of ATR_0:
There is a linear ordering $\mathcal{L}$ which isn't an ordinal but looks like one in *HYP* (the Harrison l.o.), so,

$$HYP \models \mathcal{L} \text{ is an ordinal but } 0^{(\mathcal{L})} \text{ does not exist.}$$

Hyperarithmetic sets

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For a set $X \subseteq \omega$, the following are equivalent:

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A set satisfying the conditions above is said to be hyperarithmetic.

In particular, every computable, Δ^0_2, and arithmetic set is hyperarithmetic.

Hyperarithmetic reducibility

Definition: X is hyperarithmetic in Y ($X \leqslant_H Y$) if $X \in \Delta^1_1(Y)$,
or equivalently, if $X \leqslant_T Y^{(\alpha)}$ for some $\alpha < \omega_1^Y$.

Let *HYP* be the class of hyperarithmetic sets.
Let *HYP*(Y) be the class of set hyperarithmetic in Y.

We say that an ω-model is hyperarithmetically closed is if it closed downwards under $\leqslant_H$ and is closed under $\oplus$.

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Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω-models of ATR_0 $\Rightarrow$
$\mathcal{M}$ is hyperarithmetically closed.

Question

Are there theories whose ω-models are the hyperarithmetically closed ones?

Theories of Hyperarithmetic analysis.

Definition

We say that a theory T is a theory of hyperarithmetic analysis if for every set Y, $HYP(Y)$ is the least ω-model of T containing Y, and every ω-model of T is closed under $\oplus$.

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Note that T is a theory of hyperarithmetic analysis $\Leftrightarrow$

- every ω-model of T is hyperarithmetically closed, and
- for every Y, $HYP(Y) \models$ T.

Choice and Comprehension schemes

Theorem: [Kleene 59, Kreisel 62, Friedman 67, Harrison 68, Van Wesep 77, Steel 78, Simpson 99]
The following are theories of hyperarithmetic analysis and each one is strictly weaker than the next one:

weak-Σ^1_1-AC_0 (weak Σ^1_1-choice):

$$\forall n \exists ! X(\varphi(n, X)) \Rightarrow \exists X \forall n(\varphi(n, X^{[n]})),$$

where φ is arithmetic.

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Δ^1_1-CA_0 (Δ^1_1-comprehension) :

$$\forall n(\varphi(n) \Leftrightarrow \neg\psi(n)) \Rightarrow \exists X \forall n(n \in X \Leftrightarrow \varphi(n)), \quad \text{where } \varphi \text{ and } \psi \text{ are } \Sigma^1_1.$$

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Σ^1_1-AC_0 (Σ^1_1-choice):
$$\forall n \exists X(\varphi(n, X)) \Rightarrow \exists X \forall n(\varphi(n, X^{[n]})),$$ where φ is Σ^1_1.

Σ^1_1-DC_0 (Σ^1_1-dependent choice):
$$\forall Y \exists Z(\varphi(Y, Z)) \Rightarrow \exists X \forall n(\varphi(X^{[n]}, X^{[n+1]})),$$ where φ is Σ^1_1.

The bad news

There is not theory T whose ω-models are exactly the hyperarithmetically closed ones.

Theorem: [Van Wesep 77] For every theory T whose ω-models are all hyperarithmetically closed, there is another theory T' whose ω-models are also all hyperarithmetically closed and which has more ω-models than T.

Statements of hyperarithmetic analysis

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S is a sentence of hyperarithmetic analysis if $\mathsf{RCA}_0+\mathsf{S}$ is a theory of hyperarithmetic analysis.

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Friedman [1975] introduced two statements, Arithmetic Bolzano-Weierstrass (ABW) and, Sequential Limit Systems (SL), and he mentioned they were related to hyperarithmetic analysis.
Both statements use the concept of arithmetic set of reals, which is not used outside logic.

Van Wesep [1977] introduced Game-AC and proved it is equivalent to Σ^1_1-AC_0.
It essentially says that if we have a sequence of open games such that player *II* has a winning strategy in each of them, then there exists a sequence of strategies for all of them.

The indecomposability statement

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{L}$ be linear orderings

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Theorem[Jullien '69] INDEC: Every scattered indecomposable linear ordering is indecomposable either to the right or to the left.

Δ^1_1-$\mathsf{CA}_0 \vdash$INDEC

Proof:(Δ^1_1-CA$_0$) Let $\mathcal{A}$ be scattered and indecomposable.

We want to show that $\mathcal{A}$ is indecomposable either to the left or to the right

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Proof:(Δ^1_1-CA_0) Let $\mathcal{A}$ be scattered and indecomposable.

1. For every $x \in \mathcal{A}$, either $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$ or $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leqslant a)}$.

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2. For no x we could have both $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$ and $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leqslant a)}$.
   Otherwise $\mathcal{A} \succcurlyeq \mathcal{A} + \mathcal{A} \succcurlyeq \mathcal{A} + \mathcal{A} + \mathcal{A} \succcurlyeq \mathcal{A} + 1 + \mathcal{A}$.
   So, $\mathcal{A} \succcurlyeq \mathcal{A} + 1 + \mathcal{A}$

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   Following this procedure we could build an embedding $\mathbb{Q} \preccurlyeq \mathcal{A}$.

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3. Using Δ^1_1-CA$_0$ define
   $L = \{x \in \mathcal{A} : \mathcal{A} \preccurlyeq \mathcal{A}_{(>x)}\}$ and $R = \{x \in \mathcal{A} : \mathcal{A} \preccurlyeq \mathcal{A}_{(\leqslant x)}\}$.

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4. If $L = \emptyset$, then A is indecomposable to the right.
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4. If $L = \emptyset$, then A is indecomposable to the right.
   If $R = \emptyset$, then A is indecomposable to the left.
5. Suppose this is not the case and assume $\mathcal{A} \preccurlyeq L$. Then
   $\mathcal{A} + 1 \preccurlyeq L + 1 \preccurlyeq \mathcal{A} \preccurlyeq L$. So, for some $x \in L$, $\mathcal{A} \preccurlyeq A_{(<x)}$.
   Therefore $\mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A}$, again contradicting $\mathbb{Q} \not\preccurlyeq \mathcal{A}$.

An equivalent formulation

$\mathcal{A}$ is weakly indecomposable if for every $a \in \mathcal{A}$, either $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$ or $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leqslant a)}$.

Looking at the proof of Δ^1_1-$\mathsf{CA}_0 \vdash$INDEC carefully, we can observe the following:

Theorem

The following are equivalent over RCA_0:

1. *INDEC*
2. *If $\mathcal{A}$ is a scattered, weakly indecomposable linear ordering, then there exists a cut $\langle L, R\rangle$ of A such that $L = \{a \in \mathcal{A} : \mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}\}$ and $R = \{a \in \mathcal{A} : \mathcal{A} \preccurlyeq \mathcal{A}_{(\leqslant a)}\}$*

strength of INDEC

Theorem

INDEC implies ACA_0 over RCA_0.

Proof:

1. Construct a computable linear ordering $\mathcal{A}$ such that in RCA_0,
    - $\mathcal{A}$ is infinite,
    - $\forall x \in \mathcal{A}$, either $\mathcal{A}_{(<x)}$ or $\mathcal{A}_{(>x)}$ is finite,
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strength of INDEC

Theorem

INDEC implies ACA_0 over RCA_0.

Proof:

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   For instance, given $s > t \in \mathbb{N}$, let $s \leqslant_k t$ if t looks like a true for the enumeration of $0'$ at time s.
   Let $\mathcal{A} = \langle \mathbb{N}, \leqslant_k \rangle$.

   Note that $\mathcal{A}$ is isomorphic $\omega + \omega^*$, and that $\mathcal{A}$ is weakly indecomposable. But RCA_0 cannot prove this.

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   - any infinite descending sequence in $\mathcal{A}$ computes $0'$.
2. For each $x \in \mathcal{A}$, let $\mathcal{B}_x$ be such that
   $$\mathcal{B}_x \cong \begin{cases} \omega^x & \text{if } \mathcal{A}_{(<x)} \text{ is finite} \\ (\omega^x)^* & \text{if } \mathcal{A}_{(>x)} \text{ is finite.} \end{cases}$$
   Let $\mathcal{C} = \sum_{x\in\mathcal{A}} \mathcal{B}_x$.

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   Let $\mathcal{C} = \sum_{x\in\mathcal{A}} \mathcal{B}_x$.
3. $\mathcal{C}$ is scattered and weakly indecomposable. Then, by INDEC, the middle cut of $\mathcal{C}$ exists, and from it we can compute a descending sequence in $\mathcal{A}$. Therefore $0'$ exists.

ω-models of INDEC

Theorem

INDEC is a statement of hyperarithmetic analysis.

Let $\mathcal{M} \models$ INDEC.
We want to show that $\mathcal{M}$ is hyperarithmetically closed.

We do it by proving that for every $X \in \mathcal{M}$,
if $\alpha \in \mathcal{M}$ is an ordinal and $\forall \beta < \alpha (X^{(\beta)} \in \mathcal{M})$ then $X^{(\alpha)} \in \mathcal{M}$.
By transfinite induction, this implies that if $Y \leqslant_H X$, then $Y \in \mathcal{M}$.

The successor steps follow from ACA_0. For the limit steps we construct a linear ordering using the recursion theorem and results that Ash and Knight proved using the Ash's method of α-systems.

The Jump Iteration statement

Let JI be the statement that says:
$\forall X \forall \alpha(\alpha$ an ordinal & $\forall \beta(0^{(\beta)}$ exists) $\Rightarrow 0^{(\alpha)}$ exists)

Conjecture: (RCA_0) INDEC implies JI.

Theorem

JI is a statement of hyperarithmetic analysis.

Incomparable statements

Observation: INDEC is Π^1_2-conservative over ACA_0
(because Σ^1_1-AC_0 is Π^1_2-conservative over ACA_0[Barwise, Schlipf 75]).

Therefore, for instance, INDEC is incomparable with Ramsey's theorem.

Also, INDEC is incomparable with $\mathsf{ACA}_0{}^+$.

($\mathsf{ACA}_0{}^+$ essentially says that for every X, $X^{(\omega)}$ exists.)

Hence, INDEC is incomparable with the statement that says that elementary equivalence invariants for boolean algebra exists, which is equivalent to $\mathsf{ACA}_0{}^+$ [Shore 04].

Finitely terminating games

GAME STATEMENTS.

Finitely terminating games

To each well founded tree $T \subseteq \omega^{<\omega}$, we associate a game $G(T)$ which is played as follows. Player I starts by playing a number $a_0 \in \mathcal{N}$ such that $\langle a_0 \rangle \in T$. Then player II plays $a_1 \in \mathcal{N}$ such that $\langle a_0, a_1 \rangle \in T$, and then player I plays $a_2 \in \mathcal{N}$ such that $\langle a_0, a_1, a_2 \rangle \in T$. They continue like this until they get stuck. The first one who cannot play *looses*.

We will refer to games of the form $G(T)$, for T well-founded, as finitely terminating games

Observation Finitely terminating games are in 1-1 correspondence with clopen games.

Finitely terminating games

- Let $T_I = \{\sigma \in T : |\sigma| \text{ is even}\}$, $T_{II} = \{\sigma \in T : |\sigma| \text{ is odd}\}$.
  A strategy for I in $G(T)$ is a function $s\colon T_I \to \mathcal{N}$.
  A strategy s for I is a winning strategy if whenever I plays following the s, he wins.
  A game $G(T)$ is determined if there is a winning strategy for one of the two players.

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  A game $G(T)$ is determined if there is a winning strategy for one of the two players.
- We say that a game is completely determined if there is a map $d\colon T \to \{\mathtt{W}, \mathtt{L}\}$ such that for every $\sigma \in T$,
  - $d(s) = \mathtt{W} \Leftrightarrow I$ has a winning strategy in $G(T_\sigma)$, and
  - $d(s) = \mathtt{L} \Leftrightarrow II$ has a winning strategy in $G(T_\sigma)$.

Note that completely determined games are determined.

Known results

Theorem [Steel 1976] The following are equivalent over RCA_0.

- ATR_0;
- Every finitely terminating game is determined;
- Every finitely terminating game is completely determined.

New statements

- CDG-CA: Given a sequence $\{T_n : n \in \mathcal{N}\}$ of completely determined trees, there exists a set X such that
$$\forall n \; (n \in X \text{ iff } I \text{ has a winning strategy in } G(T_n)).$$

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- CDG-AC: Given a sequence $\{T_n : n \in \mathcal{N}\}$ of completely determined trees, there exists a sequence $\{d_n : n \in \mathcal{N}\}$ where for each n, $d_n\colon T \to \{\mathtt{W}, \mathtt{L}\}$ is a winning function for $G(T_n)$.

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- DG-CA: Given a sequence $\{T_n : n \in \mathcal{N}\}$ of determined trees, there exists a set X such that
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- DG-AC: Given a sequence $\{T_n : n \in \mathcal{N}\}$ of determined trees, there exists a sequence $\{s_n : n \in \mathcal{N}\}$ of winning strategies for the T_n's.

Implications between statements

Theorem

$$
\begin{tikzcd}
\Sigma^1_1\text{-}DC_0 \arrow[drr, Rightarrow] & & & & \\
& & \Sigma^1_1\text{-}AC_0 \arrow[ull, bend right, "\times"] \arrow[d, Rightarrow] & & \\
& & DG\text{-}AC \arrow[dl, Rightarrow] & & \\
& DG\text{-}CA \arrow[r, Leftrightarrow] \arrow[dl, Rightarrow] & \Delta^1_1\text{-}CA_0 \arrow[uu, bend right, "\times"] \arrow[drr, Rightarrow] & & \\
weak\ \Sigma^1_1\text{-}AC_0 \arrow[ur, bend right, "\times"] \arrow[dr, Rightarrow] & & & & INDEC \\
& CDG\text{-}AC \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & CDG\text{-}CA & & \\
& JI. \arrow[ur, "\times"] & & &
\end{tikzcd}
$$

over RCA_0.

JI doesn't imply CDG-CA

To prove this non-implication we construct an ω-model of JI using Steel's method of forcing with tagged trees [Steel 76].

Steel used his method to prove that $\Delta^1_1\text{-CA}_0 \not\Rightarrow \Sigma^1_1\text{-AC}_0$.

Maybe, similar arguments can be used to prove other non-implications between statements of hyperarithmetic analysis.

DG-CA implies Δ^1_1-CA_0

Let φ and ψ be Σ^1_1 formulas such that $\forall n(\varphi(n) \Leftrightarrow \neg\psi(n))$.

There exists sequences of trees $\{S_n : n \in \mathbb{N}\}$ and $\{T_n : n \in \mathbb{N}\}$ such that for every n,

$$\varphi(n) \Leftrightarrow S_n \text{ is not well founded,}$$
$$\psi(n) \Leftrightarrow T_n \text{ is not well founded.}$$

For each n consider the game G_n where I plays nodes in S_n and II plays nodes in T_n. The first one who cannot move looses.

Since for every n, either S_n or T_n is well founded, this is a finitely terminating game. Moreover, each G_n is determined and I wins the game iff T_n is well founded. Therefore, I wins G_n iff $\varphi(n)$.

Then, by DG-CA, the set $\{n : \varphi(n)\}$ exists.

CDG-AC implies JI.

It is not hard to show that CDG-AC implies ACA_0.

Let α be a limit ordinal and suppose that $\forall \beta < \alpha$, $0^{(\beta)}$ exists. By recursive transfinite induction, we construct a family of finitely terminating games $\{G_{\beta,n} : \beta < \alpha, n \in \mathcal{N}\}$, such that

$$n \in 0^{(\beta)} \Leftrightarrow I \text{ has a winning strategy in } G_{\beta,n}.$$

Moreover, we claim that, using our assumption that for every $\beta < \alpha$, $0^{(\beta)}$ exists, we can prove that each game $G_{\beta,n}$ is completely determined:

By CDG-CA, there exits a set X such that

$$\langle \beta, n \rangle \in X \Leftrightarrow I \text{ wins } G_{\beta,n}.$$

This X is $0^{(\alpha)}$.