Equimorphism invariants for scattered linear orderings.

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Scattered linear orderings

A *linear ordering* (a.k.a. total ordering) is a structure $\mathcal{L} = (L, \leqslant)$, where $\leqslant$ is a transitive, reflexive and antisymmetric binary relation where every two elements are comparable. We say that $\mathcal{A}$ *embeds* into $\mathcal{B}$, if $\mathcal{A}$ is isomorphic to a subset of $\mathcal{B}$. We write $\mathcal{A} \preccurlyeq \mathcal{B}$.

Def: $\mathcal{L}$ is *scattered* if it doesn't contain a copy of $\mathbb{Q}$.
Theorem: [Hausdorff '08]
Let $\mathbb{S}$ be the smallest class of linear orderings such that

- $\mathbf{1} \in \mathbb{S}$;
- if $\mathcal{A}, \mathcal{B} \in \mathbb{S}$, then $\mathcal{A} + \mathcal{B} \in \mathbb{S}$; and
- if κ is a regular cardinal and $\{\mathcal{A}_\gamma : \gamma \in \kappa\} \subseteq \mathbb{S}$, then
$$\sum_{\gamma\in\kappa} \mathcal{A}_i \in \mathbb{S} \quad \text{and} \quad \sum_{\gamma\in\kappa^*} \mathcal{A}_i \in \mathbb{S}.$$

Then, $\mathbb{S}$ is the class of scattered linear orderings.

Hausdorff rank

Definition:

- Given a l.o. $\mathcal{L}$, we define another l.o. $\mathcal{L}'$ by identifying the elements of $\mathcal{L}$ which have finitely many elements in between.
- Then we define $\mathcal{L}^0 = \mathcal{L}$, $\mathcal{L}^{\alpha+1} = (\mathcal{L}^\alpha)'$, and take direct limits when α is a limit ordinal.
- rk($\mathcal{L}$), the *Hausdorff rank* of $\mathcal{L}$, is the least α such that $\mathcal{L}^\alpha$ is finite.

Examples: $\mathrm{rk}(\mathbb{N}) = \mathrm{rk}(\mathbb{Z}) = 1$, $\mathrm{rk}(\mathbb{Z}+\mathbb{Z}+\mathbb{Z}+\cdots) = 2$, $\mathrm{rk}(\omega^\alpha) = \alpha$, $\mathrm{rk}(\mathbb{Q}) = \infty$.

Observation:

1. if $\mathcal{A} \preccurlyeq \mathcal{B}$, then $\mathrm{rk}(\mathcal{A}) \leqslant \mathrm{rk}(\mathcal{B})$;
2. $\mathrm{rk}(\mathcal{A}+\mathcal{B}) = \max(\mathrm{rk}(\mathcal{A}), \mathrm{rk}(\mathcal{B}))$;
3. $\mathrm{rk}(\mathcal{A}\cdot\mathcal{B}) = \mathrm{rk}(\mathcal{A}) + \mathrm{rk}(\mathcal{B})$;
4. $\mathcal{A}$ is scattered $\Leftrightarrow$ for some α, $\mathcal{A}^\alpha$ is finite $\Leftrightarrow$ $\mathrm{rk}(\mathcal{A}) \neq \infty$.

Fraïssé's Conjecture

Theorem: [Fraïssé's Conjecture '48; Laver '71]
The scattered linear orderings form a Well-Quasi-Ordering with respect to embeddablity.
(i.e., there are no infinite descending sequences and no infinite antichains.)

Moreover, Laver proved that the class of σ-scattered linear orderings (countable union of scattered linear orderings) is Better-quasi-ordered with respect to emebeddability.

Question: What is the proof theoretic strength of Fraïssé's Conjecture?

The structure of the scattered linear orderings

Definition: A scattered $\mathcal{L}$ is *indecomposable* if whenever $\mathcal{L} \preccurlyeq \mathcal{A} + \mathcal{B}$, either $\mathcal{L} \preccurlyeq \mathcal{A}$ or $\mathcal{L} \preccurlyeq B$.

Example: ω^* and ω^3 are indecomposable, but $\mathbb{Z}$ is not.

Theorem: [Laver '71] Every scattered linear ordering can be written as a finite sum of indecomposable ones.

Theorem: [Fraïsé's Conjecture '48; Laver '71]
Every indecomposable linear ordering can be written either as a κ-sum or as a κ^*-sum of indecomposable l.o.'s of smaller rank, for some regular cardinal κ.

Linear orderings - Equimorphism types

We say that $\mathcal{A}$ and $\mathcal{B}$ are *equimorphic* if $\mathcal{A} \preccurlyeq \mathcal{B}$ and $\mathcal{B} \preccurlyeq \mathcal{A}$.
We denote this by $\mathcal{A} \sim \mathcal{B}$.

All the properties mentioned so far are preserved under equimorphisms.
(scattered, indecomposable, rank, κ-sums, products...)

Notation: Let $\mathbb{S}$ be the class of equimorphism types of scattered linear orderings.
Let $\mathbb{H} \subset \mathbb{S}$ be the class of equimorphism types of indecomposable linear orderings.

To each $\mathcal{L} \in \mathbb{S}$ we will assign a finite object with ordinal labels, $\mathtt{Inv}(\mathcal{L})$, such that

$$\mathcal{A} \sim \mathcal{B} \quad \Leftrightarrow \quad \mathtt{Inv}(\mathcal{A}) = \mathtt{Inv}(\mathcal{B}).$$

Def: $\mathcal{A}_0 + ... + \mathcal{A}_n$ is a *minimal decomposition* of $\mathcal{L}$ if each $\mathcal{A}_i$ is indecomposable and n is minimal possible.

Theorem: [Jullien '69] Every scattered linear has a unique minimal decomposition, up to equimorphism.

To each $\mathcal{A} \in \mathbb{H}$ we will assign an invariant $\mathtt{T}(\mathcal{A})$ which is a finite tree with labels in $\mathcal{O}n \times \{+,-\}$ such that

$$\mathcal{A} \sim \mathcal{B} \quad \Leftrightarrow \quad \mathtt{T}(\mathcal{A}) = \mathtt{T}(\mathcal{B}).$$

Then, we will then define

$$\mathtt{Inv}(\mathcal{L}) = \langle \mathtt{T}(\mathcal{A}_0), ..., \mathtt{T}(\mathcal{A}_n) \rangle.$$

Definition:
$\mathcal{L}$ is *indecomposable to the right* if whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$, $\quad \mathcal{L} \preccurlyeq \mathcal{B}$.
If this is the case we let $\epsilon_{\mathcal{L}} = +$.
$\mathcal{L}$ is *indecomposable to the left* if whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$, $\quad \mathcal{L} \preccurlyeq \mathcal{A}$.
If this is the case we let $\epsilon_{\mathcal{L}} = -$.
Theorem[Jullien 69] Every scattered indecomposable linear ordering is indecomposable either to the right or to the left.

Definition: Given a countable ordinal α, let

$$\mathbb{H}_\alpha = \{\mathcal{L} \in \mathbb{H} : \mathrm{rk}(\mathcal{L}) < \alpha\}.$$

Definition: Given $\mathcal{L} \in \mathbb{H}$, let $\mathbb{I}_{\mathcal{L}} = \{\mathcal{A} \in \mathbb{H} : 1 + \mathcal{A} + 1 \prec \mathcal{L}\}$.

Note that $\mathbb{I}_{\mathcal{L}} \subseteq \mathbb{H}_{\mathrm{rk}(\mathcal{L})}$ and that $\mathbb{I}_{\mathcal{L}}$ is and *ideal*.

Theorem

For $\mathcal{A}, \mathcal{B} \in \mathbb{H}$, $\qquad \mathcal{A} \sim \mathcal{B} \Leftrightarrow \epsilon_{\mathcal{A}} = \epsilon_{\mathcal{B}}$ *and* $\mathbb{I}_{\mathcal{A}} = \mathbb{I}_{\mathcal{B}}$.

Idea of the proof:

Let $\kappa = \mathrm{cf}(\mathsf{rk}(\mathcal{A})) \quad \vee \omega$.

Lemma: $\mathbb{I}_{\mathcal{A}}$ has a cofinal subset of size κ.

Let $\{\mathcal{A}_\xi : \xi < \kappa\} \subseteq \mathbb{I}_{\mathcal{A}}$ be a set cofinal in $\mathbb{I}_{\mathcal{A}}$, where each memeber appears κ many times in the sequence.

Lemma: $\mathcal{A} \sim \sum_{\xi \in \kappa^{\epsilon_{\mathcal{A}}}} \mathcal{A}_\xi$.

Lemma: $\kappa = \mathrm{cf}(\mathbb{I}_{\mathcal{A}}) \vee \omega$.

So, we get that $\sum_{\xi \in \kappa^{\epsilon_{\mathcal{A}}}} \mathcal{A}_\xi$ depends only on $\epsilon_{\mathcal{A}}$ and $\mathbb{I}_{\mathcal{A}}$.

Finite Invariants

Key observation: For every ideal $\mathbb{I} \subset \mathbb{H}_\alpha$,
let $X^\alpha_{\mathbb{I}}$ be the set of minimal elements of $\mathbb{H}_\alpha \setminus \mathbb{I}$.

Since $\mathbb{H}$ is a WQO,

$X^\alpha_{\mathbb{I}}$ is finite and $\forall \mathcal{L} \in \mathbb{H}_\alpha \ (\mathcal{L} \in \mathbb{I} \iff \forall \mathcal{A} \in X^\alpha_{\mathbb{I}} \ (\mathcal{A} \not\preccurlyeq \mathcal{L}))$.

Definition

Given $\mathcal{L} \in \mathbb{H}$ of rank α, we define a finite tree $\mathtt{T}(\mathcal{L})$:
Let $X^\alpha_{\mathbb{I}_{\mathcal{L}}} = \{\mathcal{A}_0, \ldots, \mathcal{A}_k\}$ and let

$$\mathtt{T}(\mathcal{L}) = \begin{array}{ccccc} & & \langle \epsilon_{\mathcal{L}}, \alpha \rangle & & \\ \mathtt{T}(\mathcal{A}_0) & \cdots & \cdots & \cdots & \mathtt{T}(\mathcal{A}_k) \end{array}$$

Recall that $\epsilon_{\mathcal{L}} = +$ if $\mathcal{L}$ is indec. to the right and $\epsilon_{\mathcal{L}} = -$ otherwise,
and that $\mathbb{I}_{\mathcal{L}} = \{\mathcal{A} \in \mathbb{H} : 1 + \mathcal{A} + 1 \prec \mathcal{L}\}$

Observation: For $\mathcal{A}, \mathcal{B} \in \mathbb{H}$, $\qquad \mathcal{A} \sim \mathcal{B} \iff \mathtt{T}(\mathcal{A}) = \mathtt{T}(\mathcal{B})$.

Comparison of invariants for $\mathbb{H}$

The key point is that for $\mathcal{A}, \mathcal{B} \in \mathbb{H}$, $\mathcal{A} \preccurlyeq \mathcal{B}$ if and only if

- either $\tau(\mathcal{A}) \preccurlyeq \tau(\mathcal{B})$ and $\mathbb{I}_{\mathcal{A}} \subseteq \mathbb{I}_{\mathcal{B}}$,
- or $\tau(\mathcal{A}) \not\preccurlyeq \tau(\mathcal{B})$ and $\mathcal{A} \in \mathbb{I}_{\mathcal{B}}$. where $\tau(\mathcal{L}) = (\mathrm{cf}(\mathrm{rk}(\mathcal{L}) \vee \omega)^{\epsilon_{\mathcal{L}}}$

Definition

For $S = [\langle \alpha, \epsilon_S \rangle; S_0, ..., S_{l-1}]$ and $T = [\langle \beta, \epsilon_T \rangle; T_0, ..., T_{k-1}]$ we let $S \preccurlyeq T$ if,

- either $\alpha \leqslant \beta$, $\tau(S) \preccurlyeq \tau(T)$ and
  $\forall i < k \; (\mathrm{rk}(T_i) \geqslant \alpha \vee \exists j < l (S_j \preccurlyeq T_i))$,
- or $\alpha < \beta$, $\tau(S) \not\preccurlyeq \tau(T)$ and $\forall i < k \; (T_i \not\preccurlyeq S)$.

..,where $\mathrm{rk}(T) = \beta$ and $\tau(T) = \mathrm{cf}(\beta)^{\epsilon_T}$.

Proposition

For $\mathcal{A}, \mathcal{B} \in \mathbb{H}$, $\mathcal{A} \preccurlyeq \mathcal{B}$ if and only if $\mathtt{T}(\mathcal{A}) \preccurlyeq \mathtt{T}(\mathcal{B})$.

Comparison of invariants for $\mathbb{S}$

Key point: If $\mathcal{A} = \mathcal{A}_0 + ... + \mathcal{A}_l$ and $\mathcal{B} = \mathcal{B}_0 + ... + \mathcal{B}_k$ then

$$\mathcal{A} \preccurlyeq \mathcal{B} \Leftrightarrow \mathcal{A}_0 + ... + \mathcal{A}_{i_1 - 1} \preccurlyeq \mathcal{B}_0 \quad \& \quad \cdots \quad \& \quad \mathcal{A}_{i_k} + ... + \mathcal{A}_l \preccurlyeq \mathcal{B}_k,$$

for some $0 = i_0 \leqslant ... \leqslant i_k \leqslant i_{k+1} = l + 1$..

Definition

Now, given $S = \langle S_0, ..., S_l \rangle$ and $T = \langle T_0, ..., T_k \rangle$ we let $S \preccurlyeq T$ if

$$\bigvee_{0 = i_0 \leqslant ... \leqslant i_k \leqslant i_{k+1} = l+1} \left(\bigwedge_{n \leqslant k} \langle S_{i_n}, S_{i_n + 1}, ..., S_{i_{n+1} - 1} \rangle \preccurlyeq T_n \right).$$

Proposition

Let $\mathcal{A}, \mathcal{B} \in \mathbb{S}$. Then, $\mathtt{Inv}(\mathcal{A}) \preccurlyeq \mathtt{Inv}(\mathcal{B})$ if and only if $\mathcal{A} \preccurlyeq \mathcal{B}$.

Recognizing invariants

Def: Let $\mathcal{Tr} = \{\mathtt{T}(\mathcal{L}) : \mathcal{L} \in \mathbb{H}\}$ and $\mathcal{In} = \{\mathtt{Inv}(\mathcal{L}) : \mathcal{L} \in \mathbb{S}\}$.

We are interested in characterizing $\mathcal{Tr}$ and $\mathcal{In}$.

Obs: $\mathcal{A}_0 + ... + \mathcal{A}_n$ is a minimal decomposition of $\mathcal{L} \in \mathbb{S}$, iff for no $i < n$ we have

- $\mathcal{A}_i + \mathcal{A}_{i+1} \sim \mathcal{A}_i$ or
- $\mathcal{A}_i + \mathcal{A}_{i+1} \sim \mathcal{A}_{i+1}$.

Obs: For $\mathcal{L} \in \mathbb{H}$ of rank α, we have

- $\mathbb{I}_{\mathcal{L}} \subseteq \mathbb{H}_\alpha$ has elements of arbitrary large rank $< \alpha$.
- $\mathbb{I}_{\mathcal{L}}$ has the same cofinality as α, if infinite.

Obs: $\mathcal{A}_i + \mathcal{A}_{i+1} \sim \mathcal{A}_i$ iff $\mathcal{A}_i$ is indec. to the left and $\mathcal{A}_{i+1} \in \mathbb{I}_{\mathcal{A}_i}$.

Proposition

Let $T = \langle T_0, \ldots, T_k \rangle \in \mathcal{T}r^{<\omega}$. Then, $T \in \mathcal{I}n$ if and only if for no $i < k$ we have that

1. *either $\epsilon_i = -$ and $T_{i+1} \in \mathcal{I}_{T_i}$,*
2. *or $\epsilon_{i+1} = +$ and $T_i \in \mathcal{I}_{T_{i+1}}$,*

where $\mathcal{I}_T = \mathcal{I}^{\alpha}_{T_0,\ldots,T_{k-1}}$.

Recognizing $\mathcal{T}r$, the invariants for $\mathbb{H}$.

Let $\mathcal{T}r_\alpha = \{T \in \mathcal{T}r : \mathrm{rk}(T) < \alpha\}$.
Suppose we already know how to recognize the elements of $\mathcal{T}r_\alpha$.

Proposition

A tree $T = [\langle \alpha, \epsilon \rangle; T_0, ..., T_{k-1}]$ with labels in $\mathcal{O}n \times \{+, -\}$ belongs to $\mathcal{T}r$ if and only if

1. *for each i, $T_i \in \mathcal{T}r_\alpha$;*
2. *$T_0, .., T_{k-1}$ are mutually $\preccurlyeq$-incomparable;*
3. *for no i, $\tau(T_i) \prec \tau(T)$.*
4. $\mathrm{rk}(\mathcal{I}^\alpha_{T_0,...,T_{k-1}}) = \alpha$;
5. $\mathrm{cf}(\mathcal{I}^\alpha_{T_0,...,T_{k-1}}) \vee \omega = \mathrm{cf}(\alpha) \vee \omega$;

where $\mathcal{I}^\alpha_{T_0,...,T_{k-1}} = \{S \in \mathcal{T}r_\alpha : \mathrm{rk}(S) < \alpha \ \&\ \forall i < k (T_i \not\preccurlyeq S)\}$.
Given and ideal $\mathcal{I} \subset \mathcal{T}r$, let $\mathrm{rk}(\mathcal{I}) = \sup\{\mathrm{rk}(T) + 1 : T \in \mathcal{I}\}$.

Minimal Ideals

So, to be able to recognize the elements of $\mathcal{I}n$ we need to recognize the ideals $\mathbb{I} \subseteq \mathbb{H}_\alpha$ of rank α.

Laver proved that $\mathbb{H}$ is a better-quasi-ordered (BQO), a stronger notion than wqo.
Remark: The set of ideals of a BQO is also a BQO.

So, the ideals of $\mathbb{H}_\alpha$ form, in particular, a WQO.
Hence, there exists a finite set of minimal ideals of $\mathbb{H}_\alpha$ of rank α.

If we found them we could tell whether an ideal has rank α by comparing it with these finitely many ideals.

Minimal equimorphism types

Theorem: [Hausdorff] Let κ be a regular cardinal and $\mathcal{L}$ a scattered linear ordering. Then

$$\kappa \leqslant |\mathcal{L}| \Leftrightarrow \text{ either } \kappa \preccurlyeq \mathcal{L} \text{ or } \kappa^* \preccurlyeq \mathcal{L}.$$

Equivalently: κ and κ^* are the minimal linear orderings of rank $\geqslant \kappa$.

For each α we want to find the minimal linear ord. of rank α.

(From these we can get the minimal ideals of $\mathbb{H}_{\alpha+1}$ of rank α.)

Minimal linear orderings of rank $\beta = \omega^\delta$

Consider an ordinal β of the form ω^δ,
and let $\{\beta_\xi : \xi < \lambda\}$ be an increasing sequence cofinal in β.
Note that $\omega^\beta = \sum_{\xi\in\lambda} \omega^{\beta_\xi}$.

Def: Given $\epsilon_0, \epsilon_1 \in \{+.-\}$, let $\omega^{\langle\beta,\epsilon_0,\epsilon_1\rangle} = \sum_{\xi\in\lambda^{\epsilon_1}} (\omega^{\beta_\xi})^{\epsilon_0}$.

Examples: $\omega^{\langle\omega,+,+\rangle} \sim \mathbf{1} + \omega + \omega^2 + ... = \omega^\omega$
and $\omega^{\langle\omega,+,-\rangle} \sim ... + \omega^2 + \omega + \mathbf{1}$.

(Up to equimorphism, $\omega^{\langle\beta,\epsilon_0,\epsilon_1\rangle}$ doesn't depend on the cofinal sequence.)

Theorem

Let $\mathcal{L} \in \mathbb{S}$ and $\beta = \omega^\delta$. Then

$$\mathrm{rk}(\mathcal{L}) \geqslant \beta \quad \Leftrightarrow \quad (\exists \epsilon_0, \epsilon_1 \in \{+,-\})\ \omega^{\langle\beta,\epsilon_0,\epsilon_1\rangle} \preccurlyeq \mathcal{L}$$

So $\mathbb{F}_\beta = \{\omega^{\langle\beta,+,+\rangle}, \omega^{\langle\beta,+,-\rangle}, \omega^{\langle\beta,-,+\rangle}, \omega^{\langle\beta,-,-\rangle}\}$ is the set of minimal equimorphism types of rank β.

Theorem

Let $\mathcal{L} \in \mathbb{S}$ and α have Cantor Normal Form $\omega^{\alpha_0} + ... + \omega^{\alpha_n}$. Then $\mathrm{rk}(\mathcal{L}) \geqslant \alpha$ *IFF* *there exist $\epsilon_0, ..., \epsilon_{2n+1} \in \{+, -\}$ such that*

$$\omega^{\langle \omega^{\alpha_0}, \epsilon_0, \epsilon_1 \rangle} \cdot \omega^{\langle \omega^{\alpha_1}, \epsilon_2, \epsilon_3 \rangle} \cdot ... \cdot \omega^{\langle \omega^{\alpha_n}, \epsilon_{2n}, \epsilon_{2n+1} \rangle} \preccurlyeq \mathcal{L}.$$

So $\mathbb{F}_\alpha = \{\omega^{\langle \omega^{\alpha_0}, \epsilon_0, \epsilon_1 \rangle} \cdot ... \cdot \omega^{\langle \omega^{\alpha_n}, \epsilon_{2n}, \epsilon_{2n+1} \rangle} : \epsilon_0, ..., \epsilon_{2n+1} \in \{+, -\}\}$ is the set of minimal equimorphism types of rank α.

Computing representatives for members of $\mathcal{I}$

We say that $\mathcal{L}$ is *finitely alternating* if it is of the form

$$\omega^{\langle \omega^{\alpha_0}, \epsilon_0, \epsilon_1 \rangle} \cdot \ldots \cdot \omega^{\langle \omega^{\alpha_n}, \epsilon_{2n}, \epsilon_{2n+1} \rangle}.$$

Theorem

We find:

1. *The set of minimal elements of $\mathbb{H}_\alpha \setminus \mathbb{I}_{\mathcal{L}}$, for each finitely alternating $\mathcal{L}$, which is a set of finitely alternating linear orderings.*
2. *The invariant $\mathtt{T}(\mathcal{L})$ for each finitely alternating $\mathcal{L}$.*
3. *The set of minimal ideals of $\mathbb{H}_\alpha$ of rank α, for each α. which is a set of finitely alternating linear orderings.*

Corollary: We can decide whether an ideal in $\mathcal{T}r_\alpha$ has rank α via a finite algorithm that compares ordinals and their cofinalities.

(To decide whether a tree $T \in \mathcal{T}r$ we still need to be able to compute cofinalities of ideals of $\mathcal{T}r_\alpha$.)

An application

Corollary

For every computable ordinal α, $(\mathcal{I}n_\alpha, \preccurlyeq)$ is computable.

Corollary

*For every $\alpha < \omega_1^{CK}$,
there exists a computable transformation lin that assigns a linear ordering $\mathrm{lin}(a)$ to each $a \in \mathcal{I}_\alpha$, such that $\mathrm{inv}(\mathrm{lin}(a)) = a$.*

Theorem

Every linear ordering of Hausdorff rank $< \omega_1^{CK}$ is equimorphic to a computable one.

Corollary

Every hyperarithmetic linear ordering is equimorphic to a computable one.

Open Questions

Question: Given a tree with labels in $\mathcal{O}n \times \{+, -\}$, is it possible to decide if it belongs to $\mathcal{T}r$ via a finite manipulation of the symbols in the tree, using some basic operations on ordinals?

Question: What about computing the invariant of the product of two linear orderings?

Definition: We say that $\mathcal{L}$ is *σ-scatteered* if it is a countable union of scattered linear orderings.

Versions of all of Laver's results were proved for this class, including Fraïssé's conjecture.

Question: Can we define invariants of this sort for the class of σ-scattered linear ordering?

Question: Is it consistent that the class of σ-scattered linear ordering is the well-founded part of the whole class of linear orderings?