Non-hyp is a spectrum.

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Complexity vs Information

How do we measure

the complexity and information content of a set $X \subseteq \mathbb{N}$?

For complexity: we may use $deg(X)$, the Turing degree of X.

For information: we may use $deg(X)$, the Turing degree of X.

How do we measure

the complexity and information content of a structure $\mathcal{A}$?

For complexity: we may use

$$Spec(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \text{ can enumerate a copy of } \mathcal{A}\}.$$

or we may use Σ-definability, or structure-degrees,..)

For information: even less clear. one approach:

$$co\text{-}Spec(\mathcal{A}) = \{X\text{: every copy of } \mathcal{A} \text{ can enumerate } X\}$$
$$= \{X : X \leq_e \Sigma_1\text{-}tp_{\mathcal{A}}(\bar{a}), \bar{a} \in A^{<\omega}\}\text{[Knight]}.$$

Structures with Turing degree

for $X, Y, Z \subseteq \mathbb{N}$

Flower Graph: Let G_Y be the graph that contains a cycle of length n just for $n \in Y$, and all the cycles intersect at a vertex.

Obs: Z computes a copy of $G_Y \iff Y$ is c.e. in Z.
Z computes a copy of $G_{X \oplus \bar{X}} \iff X \leq_T Z$.

def: $\mathcal{A}$ has *Turing degree X* if $Spec(\mathcal{A}) = \{\mathbf{z} : X \leq_T \mathbf{z}\}$

def: $\mathcal{A}$ has *enumeration degree Y* if $Spec(\mathcal{A}) = \{\mathbf{z} : Y \text{ is c.e. in } \mathbf{z}\}$

A less direct type of information

Obs: Every stucture $\mathcal{A}$ can enumerate the family of its Σ_1-types, but not in a given order.

Def: $X \subseteq \omega$ *can enumerate a family of sets* $\mathcal{S}$ if
there is V c.e. in X with $\{V^{[i]} : i \in \omega\} = \mathcal{S}$.
$\mathcal{A}$ *codes* $\mathcal{S} \subseteq \mathcal{P}(\omega)$ if every copy of $\mathcal{A}$ can enumerate $\mathcal{S}$.
(Note that the order among the sets of $\mathcal{S}$ does not matter.)

Ex: For $\mathcal{S} \subseteq \mathcal{P}(\omega)$, let $G_{\mathcal{S}}$ be the disjoint union of G_Y for $Y \in \mathcal{S}$. Then

$$Spec(G_{\mathcal{S}}) = \{\mathbf{z} : \mathbf{z} \text{ can enumerate } \mathcal{S}\}.$$

Coding non-computability

Thm[Slaman-Wehner, 98]: There is a structure $\mathcal{A}$ with

$$Spec(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \text{ non-computable}\}.$$

Pf: Let $\mathcal{A} = G_{\mathcal{S}_0}$ where $\mathcal{S}_0$ is the family of finite sets:

$$\mathcal{S}_0 = \{\{n\} \oplus F : n \in \omega, F \subseteq_{\text{finite}} \omega, F \neq W_n\}.$$

Open Question: Can a linear ordering have this property?

Almost computable structures

Def: [Kalimullin] $\mathcal{A}$ is *almost computable* if $\lambda(Spec(\mathcal{A})) = 1$.

Obs: There are countably many almost computable structures.
Because for each such $\mathcal{A}$, there is an e with $\lambda\{X : \{e\}^X \cong \mathcal{A}\} > \frac{3}{4}$, and different structures use different e.

Cor: There are sets that compute all almost comp. structures.
Furthermore, there are measure 1 many such sets.

Q: [Kalimullin 07] How complex are these sets?

Another indirect way of coding information

Example:

Lemma: (a) If $\mathcal{C} \cong \omega$ or $\mathcal{C} \cong \omega^*$, it takes $0'$ to decide which.
(b) If $S \leq_T 0'$, then there is a computable sequence $\{\mathcal{C}_n\}_{n\in\omega}$ such that $\mathcal{C}_n \cong \begin{cases} \omega & \text{if } n \in S \\ \omega^* & \text{if } n \notin S. \end{cases}$ [Ash, Knight 90]

Def: For a graph $G = (V, E)$, and linear order $\mathcal{L}$, let $\mathcal{L}\cdot G$ be the structure obtained by attaching, to each pair $v, w \in V$, a linear ordering $\mathcal{L}_{v,w} \cong \begin{cases} \mathcal{L} & \text{if } (v,w) \in E \\ \mathcal{L}^* & \text{if } (v,w) \notin E. \end{cases}$

Cor: $Spec(\omega\cdot G) = \{\mathbf{x} : \mathbf{x}' \in Spec(G)\}$.
The information in G is coded by the jump of the information in $\omega\cdot G$.

Obs If G_1 is the Slaman-Wehner graph relative to $0'$, then

$$Spec(\omega\cdot G_1) = \{\mathbf{x} : \mathbf{x} \text{ non-low}\}.$$

An even more indirect way of coding information

Lemma: For α a computable ordinal:
(a) If $\mathcal{C} \cong \mathbb{Z}^\alpha \cdot \omega$ or $\mathcal{C} \cong \mathbb{Z}^\alpha \cdot \omega^*$, it takes $0^{(2\alpha+1)}$ to decide which.
(b) If $S \leq_T 0^{(2\alpha+1)}$, then there is a comp. sequence $\{\mathcal{C}_n\}_{n\in\omega}$ such that $\mathcal{C}_n \cong \begin{cases} \mathbb{Z}^\alpha \cdot \omega & \text{if } n \in S \\ \mathbb{Z}^\alpha \cdot \omega^* & \text{if } n \notin S. \end{cases}$ [Ash, Knight 90]

Cor: $Spec(\mathbb{Z}^\alpha \cdot \omega \cdot G) = \{\mathbf{x} : \mathbf{x}^{(2\alpha+1)} \in Spec(G)\}$.
[Goncharov, Harizanov, Knight, McCoy, Miller and Solomon, 05]

Obs If G_α is the Slaman-Wehner graph relative to $0^{(2\alpha+1)}$, then
$$Spec(\mathbb{Z}^\alpha \cdot \omega \cdot G_\alpha) = \text{non-low}_{(2\alpha+1)}.$$

Note: if $\alpha = \beta \cdot \omega$,
$$\{\mathbf{x} : \mathbf{x} \not\leq_T 0^{(\alpha)}\} \subseteq Spec(\mathbb{Z}^\alpha \cdot \omega \cdot G_\alpha) \subseteq \{\mathbf{x} : \mathbf{x} \not\leq_T 0^{(\beta)}\}.$$

Cor:
The bound for almost comp. structures *cannot be hyperarithmetic*.

Our theorems

Theorem (Greenberg, M., Slaman – Kalimullin, Nies (Independently))

Every Π^1_1-random can compute all almost comp. structures.

In particular, Kleene's O can compute all almost comp. structures.

Kleene's O is a Π^1_1-complete set.

Theorem (Greenberg, M., Slaman)

There is a structure $\mathcal{A}$ with

$$Spec(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \textit{ non-hyperarithmetic}\} .$$

Hyperarithmetic sets.

Notation: Let ω_1^{ck} be the least non-computable ordinal.

Proposition [Suslin-Kleene] For a set $X \subseteq \omega$, T.F.A.E.:

- X is $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$.
- X is computable in $0^{(\alpha)}$ for some $\alpha < \omega_1^{ck}$.

  ($0^{(\alpha)}$ is the αth Turing jump of 0.)
- $X \in L(\omega_1^{ck})$.
- $X = \{x : \varphi(x)\}$, where φ is a computable infinitary formula.

  (*Computable infinitary formulas* are 1st order formulas which may contain infinite computable disjunctions or conjunctions.)

A set satisfying the conditions above is said to be hyperarithmetic.

The structure with non-hyp spectrum

Theorem (Greenberg, M., Slaman)

There is a structure $\mathcal{A}$ with $Spec(\mathcal{A}) = \{\mathbf{x} : \mathbf{x}$ non-hyperarithmetic$\}$.

Recall: For each $\alpha = \beta \cdot \omega < \omega_1^{ck}$ we have
$\{\mathbf{x} : \mathbf{x} \not\leq_T 0^{(\alpha)}\} \subseteq Spec(\mathbb{Z}^\alpha \cdot \omega \cdot G_\alpha) \subseteq \{\mathbf{x} : \mathbf{x} \not\leq_T 0^{(\beta)}\}$.

Let $\mathcal{A}$ be the disjoint union of

- $\mathbb{Z}^\alpha \cdot \omega \cdot G_\alpha$ for each $\alpha < \omega_1^{ck}$, and
- infinitely many copies of $\mathbb{Z}^{\omega_1^{ck}} \cdot \mathbb{Q} \cdot G$, where G is any graph.

Note: If $\mathcal{H} \cong \omega_1^{ck} + \omega_1^{ck} \cdot \mathbb{Q}$ is a Harrison linear order,
(i.e. $\mathcal{H}$ computable and every Π_1^1 subset has a least element.)
then $\mathbb{Z}^{\mathcal{H}} \cdot \omega = \mathbb{Z}^{\omega_1^{ck} + \omega_1^{ck} \cdot \mathbb{Q}} \cdot \omega = \mathbb{Z}^{\omega_1^{ck}} \cdot \mathbb{Z}^{\omega_1^{ck} \cdot \mathbb{Q}} \cdot \omega = \mathbb{Z}^{\omega_1^{ck}} \cdot \mathbb{Q}$.

A particular linear ordering

Theorem (Greenberg, M., Slaman)

There is a linear order $\mathcal{A}$ with $Spec(\mathcal{A}) = \{\mathbf{x} : \mathbf{x} \text{ non-hyp}\}$.

Key lemma [Frolov, Harizanov, Kalimullin, Kudinov, Miller 09]
There is a linear order $\mathcal{L}$ such that $Spec(\mathcal{L}) = \{\mathbf{x} : \mathbf{x} \text{ is non-low}_2\}$

Then, in the previous construction,
replace the Slaman-Wehner graph G by $\mathcal{L}$.