Equimorphism types of linear orderings.

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Linear orderings - Equimorphism types

A linear ordering (a.k.a. total ordering) is a structure $\mathcal{L} = (L, \leqslant)$, where $\leqslant$ is a is transitive, reflexive, antisymmetric and $\forall x, y(x \leqslant y \vee y \leqslant x)$.

A linear ordering $\mathcal{A}$ embeds into another linear ordering $\mathcal{B}$ if $\mathcal{A}$ is isomorphic to a subset of $\mathcal{B}$. We write $\mathcal{A} \preccurlyeq \mathcal{B}$.

$\mathcal{A}$ and $\mathcal{B}$ are equimorphic if $\mathcal{A} \preccurlyeq \mathcal{B}$ and $\mathcal{B} \preccurlyeq \mathcal{A}$.
We denote this by $\mathcal{A} \sim \mathcal{B}$.

We are interested in properties of linear orderings that are preserved under equimorphisms, of course, from a logic viewpoint.

Hausdorff rank

Definition:

- Given a l.o. $\mathcal{L}$, we define another l.o. $\mathcal{L}'$ by identifying the elements of $\mathcal{L}$ which have finitely many elements in between.
- Then we define $\mathcal{L}^0 = \mathcal{L}$, $\mathcal{L}^{\alpha+1} = (\mathcal{L}^\alpha)'$, and take direct limits when α is a limit ordinal.
- $\mathrm{rk}(\mathcal{L})$, the Hausdorff rank of $\mathcal{L}$, is the least α such that $\mathcal{L}^\alpha$ is finite.

Examples: $\mathrm{rk}(\mathbb{N}) = \mathrm{rk}(\mathbb{Z}) = 1$, $\mathrm{rk}(\mathbb{Z}+\mathbb{Z}+\mathbb{Z}+\cdots) = 2$,
$\mathrm{rk}(\omega^\alpha) = \alpha$, $\mathrm{rk}(\mathbb{Q}) = \infty$.

If $\mathcal{A} \preccurlyeq \mathcal{B}$, then $\mathrm{rk}(\mathcal{A}) \leqslant \mathrm{rk}(\mathcal{B})$. So, $\mathcal{A} \sim \mathcal{B} \Rightarrow \mathrm{rk}(\mathcal{A}) = \mathrm{rk}(\mathcal{B})$

Scattered and Indecomposable linear orderings

Two other properties are preserved under equimorphism:
Definition: $\mathcal{L}$ is scattered if $\mathbb{Q} \not\preccurlyeq \mathcal{L}$.

Observation: A linear ordering $\mathcal{L}$ is scattered
$\Leftrightarrow$ for some α, $\mathcal{L}^{\alpha}$ is finite
$\Leftrightarrow$ $\text{rk}(\mathcal{L}) \neq \infty$.

Definition: $\mathcal{L}$ is indecomposable if whenever
$\mathcal{L} \preccurlyeq \mathcal{A} + \mathcal{B}$, either $\mathcal{L} \preccurlyeq \mathcal{A}$ or $\mathcal{L} \preccurlyeq B$.

Example: ω, ω^*, ω^2 are indecomposable. $\mathbb{Z}$ is not.

The structure of the scattered linear orderings

Theorem: [Laver '71] Every scattered linear ordering can be written as a finite sum of indecomposable ones.

Theorem: [Fraïssé's Conjecture '48; Laver '71]
Every indecomposable linear ordering can be written either as an ω-sum or as an ω^*-sum of indecomposable l.o. of smaller rank.

Theorem: [Fraïssé's Conjecture '48; Laver '71]
The scattered linear orderings form a WQO with respect to embeddablity.
(i.e., there are no infinite descending sequences
and no infinite antichains.)

Computable Mathematics

In Computable Mathematics we are interested in the computable aspects of mathematical theorems or objects.

Usually, countable structures can be coded as a subset of ω in a natural way.

Example

- Every countable ring $(A, +_A, \times_A)$ is isomorphic to one with $A \subseteq \omega$, $+_A \subseteq \omega^3$ and $\times \subseteq \omega^3$.
- Every countable Linear ordering $(L, \leqslant_L)$ is isomorphic to one with $L \subseteq \omega$ and $\leqslant_L \subseteq \omega^2$.

So, we can talk about the computational complexity of these structures.

Sample results in Computable Mathematic

Theorem: [Friedman, Simpson, Smith 83]
There are computable rings with no computable maximal ideals.
Every computable ring has a maximal ideal computable in $0'$.

Theorem [Downey, Jockusch 94]
Every low Boolean algebra is isomorphic to a computable one.
(Recall that $X \subseteq \omega$ is low if $X' = 0'$.)

Theorem:[Spector '55]
Every hyperarithmetic well ordering is isomorphic to a computable one.

Hyperarithmetic sets.

Notation: Let ω_1^{CK} be the least non-computable ordinal.

Proposition [Suslin-Kleene, Ash] For a set $X \subseteq \omega$, T.F.A.E.:

- X is $\Delta_1^1 = \Sigma_1^1 \cap \Pi_1^1$.
- X is computable in $0^{(\alpha)}$ for some $\alpha < \omega_1^{CK}$.
  ($0^{(\alpha)}$ is the αth Turing jump of 0.)
- $X \in L(\omega_1^{CK})$.
- $X = \{x : \varphi(x)\}$, where φ is a computable infinitary formula.
  (Computable infinitary formulas are 1st order formulas which may contain infinite computable disjunctions or conjunctions.)

A set satisfying the conditions above is said to be hyperarithmetic.

Computable and arithmetic sets are hyperarithmetic.

Spector theorem for linear orderings?

Thm:[Spector] Every hyp. well ordering is isom. to a computable one.

Spector's theorem doesn't directly extend to linear orderings:
Not every hyp. linear ordering is isomorphic to a computable one.

Theorem:[Feiner '67]
There is a Δ^0_2 l.o. that is not isomorphic to any computable one.

After a sequence of results of Lerman, Jockusch, Soare, Downey, Seetapun:
Theorem: [Knight '00] For every non-computable set A, there is a linear ordering Turing equivalent to A without computable copies.

Up to equimorphism, hyperarithmetic is computable.

Obs: If α is an ordinal and $\mathcal{L} \sim \alpha$, then $\mathcal{L}$ is isomorphic to α.
Proof: $\mathcal{L} \preccurlyeq \alpha \Rightarrow \mathcal{L}$ is an ordinal and $\mathcal{L} \leqslant \alpha$.
$\alpha \preccurlyeq \mathcal{L} \Rightarrow \alpha \leqslant \mathcal{L}$ and hence $\mathcal{L} \cong \alpha$.

Theorem

Every hyperarithmetic linear ordering is equimorphic to a computable one.

Lemma

- *Every hyperarithmetic scattered l.o. has rank* $< \omega_1^{CK}$.
- *If* $\mathrm{rk}(\mathcal{L}) < \omega_1^{CK}$ *then* $\mathcal{L}$ *is equimorphic to a computable l.o.*

Equimorphism types

Definition: Let $\mathbb{L}$ be the partial ordering of equimorphism types of countable linear orderings, ordered by embeddablity.

Let $\mathbb{L}_\alpha$ be the restriction of $\mathbb{L}$ to the linear orderings of rank $< \alpha$.

Theorem

For every ordinal α*,*
$\mathbb{L}_\alpha$ *is computably presentable* $\Leftrightarrow$ $\alpha < \omega_1^{CK}$.

If $\mathrm{rk}(\mathcal{L}) < \omega_1^{CK}$ then $\mathcal{L}$ is equimorphic to a computable l.o.

1. We have $\mathbb{L}_\alpha = \{$ eq. types of rank $< \alpha\}$ is computable presentable for $\alpha < \omega_1^{CK}$.
2. We construct a computable operator
$$F\colon \mathbb{L}_{\omega_1^{CK}} \to \text{Linear Orderings}.$$
3. We use computable transfinite recursion to define $F \restriction \mathbb{L}_\alpha$.
4. **Key point:** Every indec. linear ord. of rank α is equimorphic to one of the form
$$\sum_{i\in\omega \text{ or } \omega^*} F(x_i),$$
where the sequence $\{x_i\}_{i\in\omega} \subseteq \mathbb{L}_\alpha$ is computable.

An extension

Lemma [Liang Yu, 06]
Every Σ^1_1 scattered linear ordering has rank $< \omega_1^{CK}$.

Putting this together with

Lemma

If $\mathrm{rk}(\mathcal{L}) < \omega_1^{CK}$ *then* $\mathcal{L}$ *is equimorphic to a computable l.o.*

we get:

Theorem:
Every Σ^1_1 linear ordering is equimorphic to a computable one.

1. Equimorphism types of Linear Orderings
2. Computable Mathematics
3. Reverse Mathematics
4. Equimorphism invariants

Reverse Mathematics

Main Question: What axioms are necessary to prove the theorems of Mathematics?

Setting: Second order arithmetic.

Main question revisted

1. Fix a base theory.
   (We use RCA_0 that essentially says that the computable sets exists)
2. Pick a theorem T.
3. What axioms do we need to add to RCA_0 to prove T?
4. Suppose we found axioms $A_0, ..., A_k$ such that
   RCA_0 proves A_0 & ... & $A_k \Rightarrow T$.
   How do we know these are necessary?
5. It's often the case that RCA_0 also proves $T \Rightarrow A_0$ & ... & A_k
6. Then, we know that $\mathrm{RCA}_0 + A_0, ..., A_k$ is the least system (extending RCA_0) where T can be proved.

The "big five" systems

Axiom systems:

RCA_0: Recursive Comprehension + Σ^0_1-induction + Semiring ax.
WKL_0: Weak König's lemma + RCA_0
ACA_0: Arithmetic Comprehension + RCA_0
$\Leftrightarrow$ "for every set X, X' exists".
ATR_0: Arithmetic Transfinite recursion + ACA_0.
$\Leftrightarrow$ " $\forall X$, $\forall$ ordinal α, $X^{(\alpha)}$ exists".
Π^1_1-CA_0: Π^1_1-Comprehension + ACA_0.
$\Leftrightarrow$ "$\forall X$, the hyper-jump of X exists".

Fraïssé's Conjecture

Theorem [Fraïssé's Conjecture '48; Laver '71]
FRA: The countable linear orderings form a WQO with respect to embeddablity.
(i.e., there are no infinite descending sequences and no infinite antichains.)

Theorem[Shore '93]
FRA implies ATR_0 over RCA_0.

Conjecture:[Clote '90][Simpson '99][Marcone]
FRA is equivalent to ATR_0 over RCA_0.

$$\begin{tikzcd}
\Pi^1_2\text{-}\mathsf{CA}_0 \arrow[d] \arrow[ddr] & \\
\Pi^1_1\text{-}\mathsf{CA}_0 \arrow[dd] & \\
 & \mathsf{FRA} \arrow[dl] \\
\mathsf{ATR}_0 \arrow[d] & \\
\mathsf{ACA}_0 \arrow[d] & \\
\mathsf{WKL}_0 \arrow[d] & \\
\mathsf{RCA}_0 &
\end{tikzcd}$$

Fraïssé's conjecture again.

Claim

RCA_0+FRA is the least system where it is possible to develop a reasonable theory of equimorphism types of linear orderings.

Theorem

The following are equivalent over RCA_0

- *FRA;*
- *Every scattered linear ordering can be written as a finite sum of indecomposable ones;*
- *Every indecomposable linear ordering can be written either as an ω-sum or as an ω^* sum of indecomposable l.o. of smaller rank.*

Extendability

Lemma: Every well-founded poset has a well-ordered linearization.

Definition: A linear ordering $\mathcal{L}$ is extendible if every poset $\mathcal{P} = (P, \leqslant_P)$ such that $\mathcal{L} \not\preccurlyeq \mathcal{P}$, has a linearization $\mathcal{Q} = (P, \leqslant_Q)$ such that $\mathcal{L} \not\preccurlyeq \mathcal{Q}$.

Example: ω^*, ω, $\mathbb{Z}$, $\mathbb{Q}$, and ω^α are extendible.
$\mathbf{1} + \mathbf{1}$, and $\omega + \omega^*$ are **not** extendible.

Pierre Jullien gave a characterization of the countable extendible linear orderings in 1969.

Question:[Downey, Remmel '00]
What is the proof-theoretic strength of Jullien's Thm?

Extendability of $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$.

Theorem:[Downey, Hirschfeldt, Lempp, Solomon '03][Becker][J. Miller]

$$\begin{tikzcd}
 & \Pi^1_1\text{-CA}_0 \arrow[dl] \arrow[d, Rightarrow] & \\
ext(\mathbb{Q}) \arrow[ddr, Rightarrow] & \text{ATR}_0 \arrow[r, leftrightarrow] \arrow[d, Rightarrow] & ext(\mathbb{Z}) \\
 & \text{ACA}_0 \arrow[r] \arrow[d, Rightarrow] & ext(\mathbb{N}) \arrow[dl, Rightarrow] \\
 & \text{WKL}_0 \arrow[d, Rightarrow] & \\
 & \text{RCA}_0 &
\end{tikzcd}$$

Theorem ([J. Miller][M.])

The extendibility of $\mathbb{Q}$ is equivalent to ATR_0 over Σ^1_1-$Choice_0$ + Σ^1_1-IND.

Jullien's theorem

Definition

- $\mathcal{L}$ is indecomposable to the right if whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have that $\mathcal{L} \preccurlyeq \mathcal{B}$.
- $\mathcal{L}$ is bad if $\mathcal{L} = \mathbf{1} + \mathbf{1}$ or $\mathcal{L} = \mathcal{C} + \mathcal{D}$ where $\mathcal{C}$ is indec. to the right and $\mathcal{D}$ is indec. to the left.
- If $\mathcal{L} = \mathcal{A} + \mathcal{B} + \mathcal{C}$, $\mathcal{B}$ is an essential segment of $\mathcal{L}$ if whenever $\mathcal{L} \preccurlyeq \mathcal{A} + \mathcal{B}' + \mathcal{C}$, we have that $\mathcal{B} \preccurlyeq \mathcal{B}'$.

Theorem:[Jullien '69]
$\mathcal{L}$ is extendible $\Leftrightarrow$ it has no bad essential segments.

Theorem

Jullien's Thm. is equivalent to FRA over $RCA_0 + \Sigma^1_1$*-induction.*

A Partition theorem

Theorem:[Folklore] If we color $\mathbb{Q}$ with finitely many colors, there exists an embedding $\mathbb{Q} \to \mathbb{Q}$ whose image has only one color.

Theorem:[Laver '72]
For every ctble $\mathcal{L}$, there exists $n \in \mathbb{N}$, such that:
if $\mathcal{L}$ is colored with finitely many colors, there is an embedding $\mathcal{L} \to \mathcal{L}$ whose image has at most n many colors.

Theorem

FRA is implied by Laver's Theorem above over RCA_0.

Conjecture

FRA is equivalent to Laver's Theorem above over RCA_0.

Hyperarithmetic analysis.

Consider $HYP = \{$ hyperarithmetic sets $\}$, as an ω-model of second order arithmetic.

Theories that have HYP as their least ω-models have been studied since the seventies.
Examples: Δ^1_1-CA_0, Σ^1_1-AC_0, Σ^1_1-DC_0 and weak-Σ^1_1-AC_0.

Definition: We say that a sentence S is a sentence of hyperarithmetic analysis if for every set Y, the least ω-model of RCA_0+S containing Y is $HYP(Y)$.

The indecomposability statement

- $\mathcal{L}$ is scattered if $\mathbb{Q} \not\preccurlyeq \mathcal{L}$.
- $\mathcal{L}$ is indecomposable if whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$, either $\mathcal{L} \preccurlyeq \mathcal{A}$ or $\mathcal{L} \preccurlyeq B$.
- $\mathcal{L}$ is indecomposable to the right if for every non-trivial cut $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have $\mathcal{L} \preccurlyeq \mathcal{B}$.
- $\mathcal{L}$ is indecomposable to the left if for every non-trivial cut $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have $\mathcal{L} \preccurlyeq \mathcal{A}$.

Theorem[Jullien '69] INDEC: Every scattered indecomposable linear ordering is indecomposable either to the right or to the left.

Theorem

INDEC is a statement of hyperarithmetic analysis.

1. Equimorphism types of Linear Orderings

2. Computable Mathematics

3. Reverse Mathematics

4. Equimorphism invariants

Signed Trees.

Definition

- A *signed tree* is a well founded tree $T \subset \omega^{<\omega}$ together with a map $s_T \colon T \to \{+,-\}$.
- Given a signed tree T, let $T_i = \{\sigma : i^\frown\sigma \in T\}$
  If $s_T(\emptyset) = +$, let
  $$\mathrm{lin}(T) = \mathrm{lin}(T_0) + (\mathrm{lin}(T_0) + \mathrm{lin}(T_1)) + \ldots$$
  If $s_T(\emptyset) = -$, let
  $$\mathrm{lin}(T) = \ldots + (\mathrm{lin}(T_1) + \mathrm{lin}(T_0)) + \mathrm{lin}(T_0).$$
- Let $\mathrm{lin}(\emptyset) = \mathbf{1}$.
- Linear orderings of the form $\mathrm{lin}(T)$ are called *h-indecomposable*.

Hereditarily Indecomposables.

Example:
$$\mathrm{lin}\left(\begin{array}{ccccc} & & + & & \\ & / & & \backslash & \\ - & & & & + \end{array}\right) \sim \omega+\omega^*+\omega+\omega^*\dots$$

Theorem: (follows from [Laver '71]) For scattered $\mathcal{L}$,
$\mathcal{L}$ is indecomposable iff $\mathcal{L}$ is h-indecomposable.

Theorem

The theorem above and FRA are equivalent over RCA_0.

An apparently simpler statement.

Definition

Let T and T' be signed trees.

- $h\colon T \longrightarrow T'$ is a *homomorphism* if $\forall \sigma, \tau \in T$
  $$\sigma \subsetneq \tau \Rightarrow h(\sigma) \subsetneq h(\tau) \text{ and } s_{T'}(h(\sigma)) = s_T(\sigma).$$
- Let $T \preccurlyeq T'$ if such an h exists.
- Let *WQO(ST)* be the statement:
  The signed trees form a WQO under $\preccurlyeq$.

Observation:

- $T \preccurlyeq T' \Leftrightarrow \mathrm{lin}(T) \preccurlyeq \mathrm{lin}(T')$.
- WQO(ST) follows from FRA.

Theorem

FRA and WQO(ST) are equivalent over RCA_0.

The structure of the indecomposables.

Definition: Let $\mathbb{H}$ be the set of signed trees, up to equimorphism. Given a countable ordinal α, let $\mathbb{H}_\alpha = \{T \in \mathbb{H} : \mathrm{rk}(T) < \alpha\}$.

Observations

- $\mathbb{H}_\alpha$ is isomorphic to the set of indecomposables up to equimorphism, ordered by embeddability.
- $\mathrm{rk}(T) \leqslant \alpha$ iff $\forall i \ (T_i \in \mathbb{H}_\alpha)$

Definition: Given T, let $\mathbb{I}_T$ be the downwards closure of the branches of T. i.e. $\mathbb{I}_T = \{S \in \mathbb{H} : \exists i \ (S \preccurlyeq T_i)\}$.

Observation: $T \sim S \Leftrightarrow s_T(\emptyset) = s_S(\emptyset) \ \& \ \mathbb{I}_S = \mathbb{I}_T$.

Finite Invariants

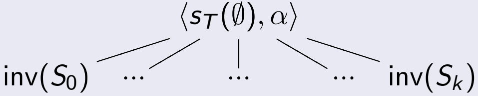
Key observation: For every ideal $\mathbb{I} \subset \mathbb{H}_\alpha$, let $X^\alpha_\mathbb{I}$ be the set of minimal elements of $\mathbb{H}_\alpha \smallsetminus \mathbb{I}$ Since $\mathbb{H}$ is a WQO, $X^\alpha_\mathbb{I}$ is finite and

$$\forall T \in \mathbb{H}_\alpha \left(T \in \mathbb{I} \iff \forall S \in X_\mathbb{I} \left(S \not\leq T\right)\right).$$

Definition

Given $T \in \mathbb{H}$ of rank α, we define a finite tree $\text{inv}(T)$:
Let $X^\alpha_{\mathbb{I}_T} = \{S_0, \dots, S_k\}$ and

$$\text{inv}(T) =$$



$$\langle s_T(\emptyset), \alpha \rangle$$

$$\text{inv}(S_0) \quad \cdots \quad \cdots \quad \cdots \quad \text{inv}(S_k)$$

Observation: For $T, S \in \mathbb{H}$, $\quad S \sim T \iff \text{inv}(T) = \text{inv}(S).$

Comparison of invariants

Def: Let $\mathcal{I}n = \{\mathrm{inv}(T) : T \in \mathbb{H}\}$ and $\mathcal{I}n_\alpha = \{\mathrm{inv}(T) : T \in \mathbb{H}_\alpha\}$.

Let $\mathrm{inv}(S) = [\langle \alpha, \epsilon_S \rangle; a_0, ..., a_l]$ and $\mathrm{inv}(T) = [\langle \beta, \epsilon_T \rangle; b_0, ..., b_k]$
then $S \preccurlyeq T \;\; \Leftrightarrow \;\; \alpha \leqslant \beta$ and

- either $*_S = *_T$ and $\forall i \leqslant k \; (\mathrm{rk}(b_i) \geqslant \alpha \vee \exists j \leqslant l(a_j \preccurlyeq b_i))$,
- or $*_S \neq *_T$, $\alpha < \beta$ and $\forall i \leqslant k \; (b_i \not\preccurlyeq \mathrm{inv}(S))$.

Problem: It is not easy to identify the members of $\mathcal{I}n$:
Consider $a = [\langle \alpha, \epsilon \rangle; a_0, ..., a_{l-1}]$ and suppose $a_i = \mathrm{inv}(S_i) \in \mathcal{I}n_\alpha$.
Let $\mathbb{I}_a = \{T \in \mathbb{H}_\alpha : \forall i = 0, ..., l-1 \; (S_i \not\preccurlyeq T)\}$.
Then $a \in \mathcal{I}n \Leftrightarrow \mathrm{rk}(\mathbb{I}_a) = \alpha$.   where $\mathrm{rk}(\mathbb{I}) = \sup\{\mathrm{rk}(T) + 1 : T \in \mathbb{I}\}$

Identifying the members of $\mathcal{I}n$

So, to be able to recognize the elements of $\mathcal{I}n$ we need to recognize the ideals $\mathbb{I} \subseteq \mathbb{H}_\alpha$ of rank α.

Laver proved that $\mathbb{H}$ is a better-quasi-ordered (BQO), a stronger notion than wqo.
Remark: The set of ideals of a BQO is also a BQO.

So, the ideals of $\mathbb{H}_\alpha$ form, in particular, a WQO.
Hence, there exists a finite set of minimal ideals of $\mathbb{H}_\alpha$ of rank α.

If we found them we could tell whether an ideal has rank α by comparing it with these finitely many ideals.

Identifying the members of $\mathcal{I}n$ computably

Lemma

Suppose that $\mathcal{I}n_\alpha$ is computable.
Then, computably uniformly in α we can list the finitely many minimal ideals of $\mathcal{I}n_\alpha$ of rank α. We list them giving the finite set of minimal elements of their complement.

Corollary

Suppose that $\mathcal{I}n_\alpha$ is computable.
Computably uniformly in α, we can identify the ideals of $\mathcal{I}n_\alpha$ of rank α, and hence identify the members of $\mathcal{I}n_{\alpha+1}$.

Corollary

For every computable ordinal α, $(\mathcal{I}n_\alpha, \preccurlyeq)$ is computable.

Computing representatives for members of $\mathcal{I}n$

Corollary

For every $\alpha < \omega_1^{CK}$,
there exists a computable transformation lin *that assigns a linear ordering* $\mathrm{lin}(a)$ *to each* $a \in \mathcal{I}n_\alpha$, *such that* $\mathrm{inv}(\mathrm{lin}(a)) = a$.

Theorem

Every linear ordering of Hausdorff rank $< \omega_1^{CK}$ *is equimorphic to a computable one.*