Theories of Hyperarithmetic Analysis.

Antonio Montalbán.
University of Chicago

Kyoto, August 2006
**Setting:** Second order arithmetic.

**Main Question:** What axioms are necessary to prove the theorems of Mathematics?

**Axiom systems:**

- **RCA**₀: Recursive Comprehension + Σ¹₀-induction + Semiring ax.
- **WKL**₀: Weak Königs lemma + RCA₀
- **ACA**₀: Arithmetic Comprehension + RCA₀
  \[ \equiv \text{“for every set } X, \text{ } X' \text{ exists”} \].
- **ATR**₀: Arithmetic Transfinite recursion + ACA₀.
  \[ \equiv \text{“ } \forall X, \forall \text{ ordinal } \alpha, X^{(\alpha)} \text{ exists”} \].
- **Π¹₁-CA**₀: Π¹₁-Comprehension + ACA₀.
A model of (the language of) second order arithmetic is a tuple

$$\langle X, M, +_X, \times_X, 0_X, 1_X, \leq_X \rangle,$$

where $M$ is a set of subsets of $X$ and $\langle X, +_X, \times_X, 0_X, 1_X, \leq_X \rangle$ is a structure in language of 1st order arithmetic.

A model of second order arithmetic is an $\omega$-model if

$$\langle X, +_X, \times_X, 0_X, 1_X, \leq_X \rangle = \langle \omega, +, \times, 0, 1, \leq \rangle.$$

$\omega$-models are determined by their second order parts, which are subsets of $\mathcal{P}(\omega)$.

We will identify subsets $M \subseteq \mathcal{P}(\omega)$ with $\omega$-models.
The class of $\omega$-models of a theory

**Observation:** $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an $\omega$-models of RCA$_0$ if $\mathcal{M}$ is closed under Turing reduction and ⊕

**Observation:** $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an $\omega$-models of ACA$_0$ if $\mathcal{M}$ is closed under Arithmetic reduction and ⊕

**Observation:** $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an $\omega$-models of ATR$_0$ if $\mathcal{M}$ is closed under Hyperarithmetic reduction and ⊕

The class of $HYP$, of hyperarithmetic sets, is not a model of ATR$_0$: There is a linear ordering $\mathcal{L}$ which isn’t an ordinal but looks like one in $HYP$ (the Harrison l.o.), so,

$$HYP \models \mathcal{L}$$

is an ordinal but $0^{(\mathcal{L})}$ does not exist.
Proposition: [Suslin-Kleene, Ash]
For a set $X \subseteq \omega$, the following are equivalent:

- $X$ is $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$.
- $X$ is computable in $0^{(\alpha)}$ for some $\alpha < \omega^1_{\text{CK}}$.

($\omega^1_{\text{CK}}$ is the least non-computable ordinal and $0^{(\alpha)}$ is the $\alpha$th Turing jump of $0$.)

- $X = \{x : \varphi(x)\}$, where $\varphi$ is a computable infinitary formula.

(Computable infinitary formulas are 1st order formulas which may contain infinite computable disjunctions or conjunctions.)

A set satisfying the conditions above is said to be hyperarithmetic.

In particular, every computable, $\Delta^0_2$, and arithmetic set is hyperarithmetic.
Definition: \( X \) is hyperarithmetic in \( Y \) (\( X \leq_H Y \)) if \( X \in \Delta^1_1(Y) \), or equivalently, if \( X \leq_T Y^{(\alpha)} \) for some \( \alpha < \omega^Y_1 \).

Let \( HYP \) be the class of hyperarithmetic sets. Let \( HYP(Y) \) be the class of set hyperarithmetic in \( Y \).

We say that an \( \omega \)-model is hyperarithmetically closed is if it closed downwards under \( \leq_H \) and is closed under \( \oplus \).
The class of \( \omega \)-models of a theory

**Observation:** \( \mathcal{M} \subseteq \mathcal{P}(\omega) \) is an \( \omega \)-models of RCA\(_0\) \iff \( \mathcal{M} \) is closed under Turing reduction and \( \oplus \)

**Observation:** \( \mathcal{M} \subseteq \mathcal{P}(\omega) \) is an \( \omega \)-models of ACA\(_0\) \iff \( \mathcal{M} \) is closed under Arithmetic reduction and \( \oplus \)

**Observation:** \( \mathcal{M} \subseteq \mathcal{P}(\omega) \) is an \( \omega \)-models of ATR\(_0\) \Rightarrow \( \mathcal{M} \) is hyperarithmetically closed.

**Question:** Are there theories whose \( \omega \)-models are the hyperarithmetically closed ones?
Theories of Hyperarithmetic analysis.

**Definition**

We say that a theory $T$ is a **theory of hyperarithmetic analysis** if for every set $Y$, $\text{HYP}(Y)$ is the least $\omega$-model of $T$ containing $Y$, and every $\omega$-model of $T$ is closed under $\oplus$.

Note that $T$ is a theory of hyperarithmetic analysis $\iff$

- every $\omega$-model of $T$ is hyperarithmetically closed, and
- for every $Y$, $\text{HYP}(Y) \models T$. 
Theorem: [Kleene 59, Kreisel 62, Friedman 67, Harrison 68, Van Wesep 77, Steel 78, Simpson 99]
The following are theories of hyperarithmetic analysis and each one is strictly weaker than the next one:

\text{weak-} \Sigma^1_1\text{-AC}_0 \ (\text{weak } \Sigma^1_1\text{-choice}):
\[\forall n \exists! X(\varphi(n, X)) \Rightarrow \exists X \forall n(\varphi(n, X^{[n]})), \quad \text{where } \varphi \text{ is arithmetic.}\]

\Delta^1_1\text{-CA}_0 \ (\Delta^1_1\text{-comprehension}):
\[\forall n(\varphi(n) \iff \neg \psi(n)) \Rightarrow \exists X \forall n(n \in X \iff \varphi(n)), \quad \text{where } \varphi \text{ and } \psi \text{ are } \Sigma^1_1.\]

\Sigma^1_1\text{-AC}_0 \ (\Sigma^1_1\text{-choice}):
\[\forall n \exists X(\varphi(n, X)) \Rightarrow \exists X \forall n(\varphi(n, X^{[n]})), \quad \text{where } \varphi \text{ is } \Sigma^1_1.\]

\Sigma^1_1\text{-DC}_0 \ (\Sigma^1_1\text{-dependent choice}):
\[\forall Y \exists Z(\varphi(Y, Z)) \Rightarrow \exists X \forall n(\varphi(X^{[n]}, X^{[n+1]})), \quad \text{where } \varphi \text{ is } \Sigma^1_1.\]
The bad news

There is not theory $T$ whose $\omega$-models are exactly the hyperarithmetically closed ones.

**Theorem:** [Van Wesep 77] For every theory $T$ whose $\omega$-models are all hyperarithmetically closed, there is another theory $T'$ whose $\omega$-models are also all hyperarithmetically closed and which has more $\omega$-models than $T$. 
Friedman [1975] introduced two statements, Arithmetic Bolzano-Weierstrass (ABW) and, Sequential Limit Systems (SL), and he mentioned they were related to hyperarithmetic analysis. Both statements use the concept of arithmetic set of reals, which is not used outside logic.

Van Wesep [1977] introduced Game-AC and proved it is equivalent to $\Sigma^1_1$-AC$_0$. It essentially says that if we have a sequence of open games such that player II has a winning strategy in each of them, then there exists a sequence of strategies for all of them.
Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{L}$ be linear orderings

- If $\mathcal{A}$ embeds into $\mathcal{B}$, we write $\mathcal{A} \preceq \mathcal{B}$.
- $\mathcal{L}$ is scattered if $\mathbb{Q} \not\preceq \mathcal{L}$.
- $\mathcal{L}$ is indecomposable if whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$, either $\mathcal{L} \preceq \mathcal{A}$ or $\mathcal{L} \preceq \mathcal{B}$.
- $\mathcal{L}$ is indecomposable to the right if for every non-trivial cut $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have $\mathcal{L} \preceq \mathcal{B}$.
- $\mathcal{L}$ is indecomposable to the left if for every non-trivial cut $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have $\mathcal{L} \preceq \mathcal{A}$.

**Theorem**[Jullien ’69] \textbf{INDEC}: Every scattered indecomposable linear ordering is indecomposable either to the right or to the left.
\( \Delta^1_1 \)-CA\(_0 \vdash \text{INDEC} \)

**Proof:** (\( \Delta^1_1 \)-CA\(_0 \)) Let \( A \) be scattered and indecomposable.

We want to show that \( A \) is indecomposable either to the left or to the right

1. For every \( x \in A \), either \( A \preceq A(>a) \) or \( A \preceq A(\leq a) \).
2. For no \( x \) we could have both \( A \preceq A(>a) \) and \( A \preceq A(\leq a) \). Otherwise \( A \preceq A + A \preceq A + A + A \preceq A + 1 + A \).
   So, \( A \preceq A + 1 + A \preceq (A + 1 + A) + 1 + (A + 1 + A) \) ...
   Following this procedure we could build an embedding \( \mathbb{Q} \preceq A \).
3. Using \( \Delta^1_1 \)-CA\(_0 \) define
   \[ L = \{ x \in A : A \preceq A(>x) \} \quad \text{and} \quad R = \{ x \in A : A \preceq A(\leq x) \}. \]
4. If \( L = \emptyset \), then \( A \) is indecomposable to the right.
   If \( R = \emptyset \), then \( A \) is indecomposable to the left.
5. Suppose this is not the case and assume \( A \preceq L \). Then \( A + 1 \preceq L + 1 \preceq L \). So, for some \( x \in L \), \( A \preceq A(>x) \). 

Antonio Montalbán. University of Chicago

Theories of Hyperarithmetic Analysis.
An equivalent formulation

\( \mathcal{A} \) is weakly indecomposable if for every \( a \in \mathcal{A} \), either \( \mathcal{A} \preceq \mathcal{A}(>a) \) or \( \mathcal{A} \preceq \mathcal{A}(\leq a) \).

Looking at the proof of \( \Delta^1_1 \)-CA\(_0\) \( \vdash \) INDEC carefully, we can observe the following:

**Theorem**

The following are equivalent over RCA\(_0\):

1. INDEC
2. If \( \mathcal{A} \) is a scattered, weakly indecomposable linear ordering, then there exists a cut \( \langle L, R \rangle \) of \( \mathcal{A} \) such that 
   
   \[ L = \{ a \in \mathcal{A} : \mathcal{A} \preceq \mathcal{A}(>a) \} \quad \text{and} \quad R = \{ a \in \mathcal{A} : \mathcal{A} \preceq \mathcal{A}(\leq a) \} \]
Theorem

\textit{INDEC} implies ACA$_0$ over RCA$_0$.

Proof:

1. Construct a computable linear ordering $A$ such that in RCA$_0$,
   - $A$ is infinite,
   - $\forall x \in A$, either $A_{(<x)}$ or $A_{(>x)}$ is finite,
   - any infinite descending sequence in $A$ computes $0'$.

   For instance, given $s > t \in \mathbb{N}$, let $s \leq_k t$ if $t$ looks like a true for the enumeration of $0'$ at time $s$.

   Let $A = \langle \mathbb{N}, \leq_k \rangle$.

   Note that $A$ is isomorphic $\omega + \omega^*$, and that $A$ is weakly indecomposable. But RCA$_0$ cannot prove this.

2. For each $x \in A$, let $B_x$ be such that

   $B_x \cong \begin{cases} \omega^x & \text{if } A_{(<x)} \text{ is finite} \\ (\omega^x)^* & \text{if } A_{(>x)} \text{ is finite} \end{cases}$
Theorem

INDEC is a statement of hyperarithmetic analysis.

Let $\mathcal{M} \models \text{INDEC}$. We want to show that $\mathcal{M}$ is hyperarithmetically closed.

We do it by proving that for every $X \in \mathcal{M}$,
if $\alpha \in \mathcal{M}$ is an ordinal and $\forall \beta < \alpha (X^{(\beta)} \in \mathcal{M})$ then $X^{(\alpha)} \in \mathcal{M}$. By transfinite induction, this implies that if $Y \leq_H X$, then $Y \in \mathcal{M}$.

The successor steps follow from ACA$_0$. For the limit steps we construct a linear ordering using the recursion theorem and results that Ash and Knight proved using the Ash’s method of $\alpha$-systems.
Let JI be the statement that says:
\[ \forall X \forall \alpha (\alpha \text{ an ordinal } \& \ \forall \beta (0^{(\beta)} \text{ exists}) \Rightarrow 0^{(\alpha)} \text{ exists}) \]

**Conjecture:** \((\text{RCA}_0) \text{ INDEC implies JI.}\)
**Observation:** INDEC is $\Pi^1_2$-conservative over ACA$_0$ (because $\Sigma^1_1$-AC$_0$ is $\Pi^1_2$-conservative over ACA$_0$ [Barwise, Schlipf 75]).

Therefore, for instance, INDEC is incomparable with Ramsey’s theorem.

Also, INDEC is incomparable with ACA$_0^+$. (ACA$_0^+$ essentially says that for every $X$, $X^{(\omega)}$ exists.) Hence, INDEC is incomparable with the statement that says that elementary equivalence invariants for boolean algebra exists, which is equivalent to ACA$_0^+$ [Shore 04].
Finitely terminating games

GAME STATEMENTS.
Finitely terminating games

To each well founded tree $T \subseteq \omega^\omega$, we associate a game $G(T)$ which is played as follows. Player I starts by playing a number $a_0 \in \mathbb{N}$ such that $\langle a_0 \rangle \in T$. Then player II plays $a_1 \in \mathbb{N}$ such that $\langle a_0, a_1 \rangle \in T$, and then player I plays $a_2 \in \mathbb{N}$ such that $\langle a_0, a_1, a_2 \rangle \in T$. They continue like this until they get stuck. The first one who cannot play loses.

We will refer to games of the form $G(T)$, for $T$ well-founded, as finitely terminating games.

**Observation** Finitely terminating games are in 1-1 correspondence with clopen games.
Finitely terminating games

- Let $T_I = \{ \sigma \in T : |\sigma| \text{ is even} \}$, $T_{II} = \{ \sigma \in T : |\sigma| \text{ is odd} \}$.  
- A strategy for $I$ in $G(T)$ is a function $s : T_I \to \mathbb{N}$. 
- A strategy $s$ for $I$ is a winning strategy if whenever $I$ plays following the $s$, he wins. 
- A game $G(T)$ is determined if there is a winning strategy for one of the two players. 

- We say that a game is completely determined if there is a map $d : T \to \{W, L\}$ such that for every $\sigma \in T$, 
  - $d(s) = W \iff I$ has a winning strategy in $G(T_\sigma)$, and 
  - $d(s) = L \iff II$ has a winning strategy in $G(T_\sigma)$. 

Note that completely determined games are determined.
Theorem [Steel 1976] The following are equivalent over RCA$_0$.

- ATR$_0$;
- Every finitely terminating game is determined;
- Every finitely terminating game is completely determined.
New statements

- **CDG-CA**: Given a sequence \( \{ T_n : n \in \mathcal{N} \} \) of completely determined trees, there exists a set \( X \) such that
  \[ \forall n (n \in X \iff I \text{ has a winning strategy in } G(T_n)). \]

- **CDG-AC**: Given a sequence \( \{ T_n : n \in \mathcal{N} \} \) of completely determined trees, there exists a sequence \( \{ d_n : n \in \mathcal{N} \} \) where for each \( n \), \( d_n : T \to \{ W, L \} \) is a winning function for \( G(T_n) \).

- **DG-CA**: Given a sequence \( \{ T_n : n \in \mathcal{N} \} \) of determined trees, there exists a set \( X \) such that
  \[ \forall n (n \in X \iff I \text{ has a winning strategy in } G(T_n)). \]

- **DG-AC**: Given a sequence \( \{ T_n : n \in \mathcal{N} \} \) of determined trees, there exists a sequence \( \{ s_n : n \in \mathcal{N} \} \) of winning strategies for the \( T_n \)'s.
Implications between statements

**Theorem**

\[
\Sigma_1^1-DC_0 \leftrightarrow \Sigma_1^1-AC_0 \\
\Sigma_1^1-AC_0 \downarrow \leftrightarrow DG-AC \times \\
DG-AC \downarrow \leftrightarrow DG-CA \leftrightarrow \Delta_1^1-CA_0 \\
weak \Sigma_1^1-AC_0 \leftrightarrow CDG-AC \leftrightarrow CDG-CA \\
\Downarrow \leftrightarrow JI. \\
\rightarrow \leftrightarrow \overline{INDEC} \\
\overline{over \ RCA_0}.
\]
JI doesn’t imply CDG-CA

To prove this non-implication we construct an $\omega$-model of JI using Steel’s method of forcing with tagged trees [Steel 76].

Steel used his method to prove that $\Delta^1_1$-$CA_0 \not\iff \Sigma^1_1$-$AC_0$.

Maybe, similar arguments can be used to prove other non-implications between statements of hyperarithmetic analysis.
DG-CA implies $\Delta^1_1$-CA$_0$

Let $\varphi$ and $\psi$ be $\Sigma^1_1$ formulas such that $\forall n(\varphi(n) \iff \neg \psi(n))$.

There exists sequences of trees $\{S_n : n \in \mathbb{N}\}$ and $\{T_n : n \in \mathbb{N}\}$ such that for every $n$,

- $\varphi(n) \iff S_n$ is not well founded,
- $\psi(n) \iff T_n$ is not well founded.

For each $n$ consider the game $G_n$ where I plays nodes in $S_n$ and II plays nodes in $T_n$. The first one who cannot move looses.

Since for every $n$, either $S_n$ or $T_n$ is well founded, this is a finitely terminating game. Moreover, each $G_n$ is determined and I wins the game iff $T_n$ is well founded. Therefore, I wins $G_n$ iff $\varphi(n)$.

Then, by DG-CA, the set $\{n : \varphi(n)\}$ exists.
CDG-AC implies JI.

It is not hard to show that CDG-AC implies ACA\(_0\).

Let \(\alpha\) be a limit ordinal and suppose that \(\forall \beta < \alpha, 0(\beta)\) exists. By recursive transfinite induction, we construct a family of finitely terminating games \(\{G_{\beta,n} : \beta < \alpha, n \in \mathbb{N}\}\), such that

\[
n \in 0(\beta) \iff I \text{ has a winning strategy in } G_{\beta,n}.
\]

Moreover, we claim that, using our assumption that for every \(\beta < \alpha, 0(\beta)\) exists, we can prove that each game \(G_{\beta,n}\) is completely determined:

By CDG-CA, there exits a set \(X\) such that

\[
\langle \beta, n \rangle \in X \iff I \text{ wins } G_{\beta,n}.
\]

This \(X\) is \(0(\alpha)\).
Work in Progress

$\Pi^1_1$-sep: Given $\Pi^1_1$ formulas $\varphi(n)$ and $\psi(n)$, $\forall n (\neg \varphi(n) \lor \neg \psi(n)) \Rightarrow \exists X \forall n (\varphi(n) \Rightarrow n \in X \land \psi(n) \Rightarrow n \notin X)$

Observation [Tanaka]
$\Sigma^1_1$-AC$_0 \Rightarrow \Pi^1_1$-sep $\Rightarrow \Delta^1_1$-CA$_0$

Theorem (Work in progress)
$\Pi^1_1$-sep is strictly in between $\Sigma^1_1$-AC$_0$ and $\Rightarrow \Delta^1_1$-CA$_0$