Extensions of Embeddings in the Δ^0_2 Turing Degrees.

Antonio Montalbán.
U. of Chicago

Nanjing,
May 2008

Joint work with Rod Downey, Noam Greenberg and Andy Lewis.

Basic definitions

We let $\mathbf{D}_{(\leq 0')}$ be the set of degrees below $0'$, and

$$\mathcal{D}_{(\leq 0')} = (\mathbf{D}_{(\leq 0')}, \leq_T, \vee).$$

Question: How does the upper-semi-lattice $\mathcal{D}_{(\leq 0')}$ look like?

Extensions of Embeddings problem

Let $\mathcal{L}$ be a finite language and $\mathcal{A}$ be a $\mathcal{L}$-structure. Ex: $\mathcal{A} = (\mathbf{D}_{(\leq 0')}, \leq_T, \vee)$.

Def: The *extensions of embedding problem* for $\mathcal{A}$ is:

Given a pair of finite $\mathcal{L}$-structures $\mathcal{P} \subseteq \mathcal{Q}$,
does every embedding $\mathcal{P} \hookrightarrow \mathcal{A}$ have an extension $\mathcal{Q} \hookrightarrow \mathcal{A}$?

$$\begin{tikzcd} \mathcal{P} \arrow[r, hook] \arrow[d, hook] & \mathcal{A} \\ \mathcal{Q} \arrow[ur, hook, dotted] & \end{tikzcd}$$

Def: Let $\mathbb{E}^{\mathcal{A}} = \{(\mathcal{P}, \mathcal{Q}) :$ the answer is YES $\}$.

Question: Is $\mathbb{E}^{\mathcal{A}}$ computable?

Extensions of embeddings vs two-quantifier theroy

Suppose $\mathcal{A}$ is an upper-semi-lattice (*usl*).

Lemma: The $\exists - Th(\mathcal{A})$ is decidable $\iff$
The *substructure problem* is decidable
i.e. the set of finite usl $\mathcal{P}$ which embed into $\mathcal{A}$ is computable.

Lemma: The $\forall\exists - Th(\mathcal{A})$ is decidable $\iff$
the *multi-extensions of embeddings problem* is decidable
i.e. given usls $(\mathcal{P}, \mathcal{Q}_1, ..., Q_m)$, it is decidable whether
every embedding $\mathcal{P} \hookrightarrow \mathcal{A}$ has an extension $\mathcal{Q}_i \hookrightarrow \mathcal{A}$ for some i

$$\begin{tikzcd}
 & & & \mathcal{P} \arrow[r, hook] \arrow[dlll, hook] \arrow[dll, hook] \arrow[d, hook] \arrow[drr, hook] & & & \mathcal{A} \\
\mathcal{Q}_1 & \mathcal{Q}_1 & \cdots & \mathcal{Q}_i \arrow[urrr, hook, dotted] & \cdots & \mathcal{Q}_m &
\end{tikzcd}$$

$$\begin{array}{ccc} \text{Substructures Problem} & \iff & \exists - Th(\mathcal{D}_{(\leq_T 0')}) \\ \bigcap & & \\ \text{Extension of embeddings prob.} & & \\ \bigcap & & \\ \text{Multi-extension of embeddings} & \iff & \forall\exists - Th(\mathcal{D}_{(\leq_T 0')}) \end{array}$$

Question: How does the upper-semi-lattice $\mathcal{D}_{(\leq 0')}$ look like?

- $\mathcal{D}_{(\leq 0')}$ is complicated
  $\mathrm{Th}(\mathbf{D}_{(\leq 0')}, \leq_T)$ is undecidable. [Epstein 79][Lerman 83]
- Not that complicated
  $\exists - \mathrm{Th}(\mathbf{D}_{(\leq 0')}, \leq_T)$ is decidable. [Kleene, Post '54]

Question: Which fragments of $\mathrm{Th}(\mathbf{D}_{(\leq 0')}, \leq_T, \vee)$ are decidable?

This question has been widely studied for $\mathcal{D}$, $\mathcal{R}$ and $\mathcal{D}_{(\leq 0')}$ among other structures.

History of Decidability Results in **D**.

Question: Which fragments of $\mathrm{Th}(\mathbf{D}, \leq_T, \vee, ', 0)$ are decidable?

	$\exists$	$\forall\exists$	$\exists\forall\exists$
$(\mathbf{D}, \leq_T)$	✓	✓	× [Schmerl]
$(\mathbf{D}, \leq_T, \vee)$	✓[Kleene Post 54]	✓[Jockusch Slaman 93]	×
$(\mathbf{D}, \leq_T, ')$	✓[Hinman Slaman 91]	**?**	×
$(\mathbf{D}, \leq_T, \vee, ')$	✓[M. 03]	× [Shore Slaman 06]	×
$(\mathbf{D}, \leq_T, ', 0)$	✓[Lerman 08?]	**?**	×
$(\mathbf{D}, \leq_T, \vee, ', 0)$	**?**	×	×

Extensions of Embeddings in the Upper-Semi-Lattice $(\mathbf{D}, \leq_T, \vee)$

Thm: [Lerman 71] Every finite usl embedds as an initial segment of $\mathcal{D}$.

Def: Given usls $\mathcal{P} \subseteq \mathcal{Q}$, we say that $\mathcal{Q}$ is an *end extension* of $\mathcal{P}$ if

$$\forall \mathbf{x}, \mathbf{y} \in \mathcal{Q}\ (\mathbf{x} \leq \mathbf{y}\ \&\ \mathbf{y} \in \mathcal{P} \implies \mathbf{x} \in \mathcal{P}).$$

Thm:[Jockusch Slaman 93] $\mathcal{Q}$ end extension of $\mathcal{P} \Longrightarrow (\mathcal{P}, \mathcal{Q}) \in \mathbb{E}^{(\mathbf{D}, \leq, \vee)}$.

i.e Every embedding $\mathcal{P} \hookrightarrow \mathcal{D}$ extends to $\mathcal{Q} \hookrightarrow \mathcal{D}$.

Corollary: $\exists\forall - Th(\mathbf{D}, \leq_T, \vee)$ is decidable.

Proof: Given $\mathcal{P}, \mathcal{Q}_1, ..., \mathcal{Q}_k$ such that $\mathcal{P} \subseteq \mathcal{Q}_j$, we have that every embedding $\mathcal{P} \hookrightarrow \mathcal{D}$ extends to $\mathcal{Q}_i \hookrightarrow \mathcal{D}$ for some i $\iff$ for some i, $\mathcal{Q}_i$ is an end extension of $\mathcal{P}$.

Decidability Results in $\mathcal{R}$

Question: Which fragments of $\mathrm{Th}(\mathcal{R}, \leq_T, \vee, \wedge)$ are decidable?

	$\exists$	$\forall\exists$	$\exists\forall\exists$
$(\mathcal{R}, \leq_T)$	✓	**?**	× [Lempp, Nies, Slaman 98]
$(\mathcal{R}, \leq_T, \vee)$	✓ [Sacks 63]	**?**	×
$(\mathcal{R}, \leq_T, \vee, \wedge)$	**?**	× [Miller, Nies, Shore 04]	×

$\wedge$ is the partial function that give the Greatest Lower Bound.

Thm:[Slaman Soare 01]
The extension of embeddings problem for $(\mathcal{R}, \leq_T)$ is decidable.

Decidability results in $\mathcal{D}_{(\leq 0')}$

Question: Which fragments of $\mathrm{Th}(\mathbf{D}_{(\leq 0')}, \leq_T, \vee, \wedge)$ are decidable?

	$\exists$	$\forall\exists$	$\exists\forall\exists$
$(\mathbf{D}_{(\leq 0')}, \leq_T)$	$\checkmark$	$\checkmark$ [Lerman Shore 88]	$\times$ [Lerman 83][Schmerl]
$(\mathbf{D}_{(\leq 0')}, \leq_T, \vee)$	$\checkmark$ [Kleene Post 54]	**?**	$\times$
$(\mathbf{D}_{(\leq 0')}, \leq_T, \vee, \wedge)$	$\checkmark$ [Lachlan Lebeuf 76]	$\times$ [Miller, Nies, Shore 04]	$\times$

Extensions of Embeddings in the Partial Ordering $(\mathbf{D}_{(\leq 0')}, \leq_T, 0')$

Thm [Lerman 83]: Every finite poset is an initial segment of $\mathcal{D}_{(\leq 0')}$.

Def: Given partial orderings with top element $(\mathcal{P}, \leq, \mathbf{1}) \subseteq (\mathcal{Q}, \leq, \mathbf{1})$ we say that $\mathcal{Q}$ is an*end extension* of $\mathcal{P}$ if

$$\forall \mathbf{x}, \mathbf{y} \in \mathcal{Q}(\mathbf{x} \leq \mathbf{y} \ \&\ \mathbf{y} \in \mathcal{P} \setminus \mathbf{1} \implies \mathbf{x} \in \mathcal{P}).$$

Thm[Lerman Shore 88]: $\mathcal{Q}$ end extension of $\mathcal{P} \implies (\mathcal{P}, \mathcal{Q}) \in \mathbb{E}^{(\mathbf{D}_{(\leq 0')}, \leq)}$.

Corollary: The $\exists\forall - Th(\mathbf{D}_{(\leq 0')}, \leq_T, 0')$ is decidable.

Extensions of embeddings below c.e. degrees

Def: Let $\mathbb{E}^{jump} = \{(\mathcal{P}, \mathcal{Q})$ usls: every embedding $h : \mathcal{P} \hookrightarrow \mathcal{D}$ with $h(\mathbf{1}) \equiv_T h(\mathbf{0})'$, has an extension to $\mathcal{Q} \hookrightarrow \mathcal{D}$ $\}$.

($\mathcal{P}$ and $\mathcal{Q}$ have top element $\mathbf{1}$ and bottom element $\mathbf{0}$).

Def: Let $\mathbb{E}^{c.e.} = \{(\mathcal{P}, \mathcal{Q})$ usls: every embedding $h : \mathcal{P} \hookrightarrow \mathcal{D}$ where $h(\mathbf{1})$ is c.e. in $h(\mathbf{0})$, has an extension to $\mathcal{Q} \hookrightarrow \mathcal{D}$ $\}$.

Given $\mathcal{P}$, let $\mathcal{P}^*$ be $\mathcal{P} \cup \{\mathbf{0}_{P^*}\}$ where $\mathbf{0}_{P^*} < \mathbf{0}_{\mathcal{P}}$.

It looks likely that, if decidable and proofs are relativizable,

$$(\mathcal{P}, \mathcal{Q}) \in \mathbb{E}^{c.e.} \iff (\mathcal{P}^*, \mathcal{Q}^*) \in \mathbb{E}^{jump} \iff (\mathcal{P}^*, \mathcal{Q}^*) \in \mathbb{E}^{(\mathcal{D}_{(\leq 0')})}.$$

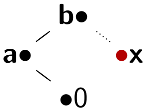
End extensions

Thm: [Lerman 83]
Every finite usl is an initial segment below any c.e. degree.

Corollary: $(\mathcal{P}, \mathcal{Q}) \in \mathbb{E}^{c.e.} \implies \mathcal{Q}$ end extension of $\mathcal{P}$.

A degree unlike $0'$

Thm:[Slaman Steel 89] There exists c.e. degrees $0 <_T \mathbf{a} <_T \mathbf{b}$ such that $\not\exists \mathbf{x} <_T \mathbf{b}\ (\mathbf{x} \vee \mathbf{a} \equiv_T \mathbf{b})$.



Contiguous degrees

Thm: [Downey 87] For every c.e. $\mathbf{b}$, there exists c.e. $\mathbf{a}$ such that

$$\forall \mathbf{x}\ (\mathbf{x} \vee \mathbf{a} \geq_{wtt} \mathbf{b} \implies \mathbf{x} \geq_{wtt} \mathbf{b}).$$

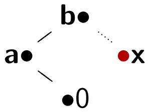
Thm: [Downey 87] There exists a c.e. $\mathbf{b}$ such that

$$\forall \mathbf{x}\ (\mathbf{x} \equiv_T \mathbf{b} \implies \mathbf{x} \equiv_{wtt} \mathbf{b}).$$

Such degrees $\mathbf{b}$ are called *strongly contiguous degrees*.

Cor: There exists c.e. degrees $0 <_T \mathbf{a} <_T \mathbf{b}$ such that

$$\nexists \mathbf{x} <_T \mathbf{b}\ (\mathbf{x} \vee \mathbf{a} \equiv_T \mathbf{b})$$



These results extend previous results of [Ladner Sasso 75] for c.e. degrees

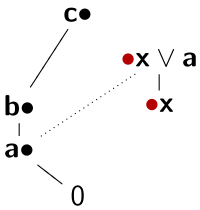
Contiguous pairs

Theorem

There exists a c.e. $\mathbf{b} <_T \mathbf{c}$ *such that*
$\forall \mathbf{y}\ (\mathbf{b} \leq_T \mathbf{y} \leq_T \mathbf{c} \implies \mathbf{b} \leq_{wtt} \mathbf{y})$.

Cor: There exists c.e. degrees $0 <_T \mathbf{a} <_T \mathbf{b} <_T \mathbf{c}$ such that

$$\nexists \mathbf{x} \leq_T \mathbf{c}\ (\mathbf{x} \vee \mathbf{a} \geq_T \mathbf{b} \quad \& \quad \mathbf{x} \not\geq_T \mathbf{b}).$$

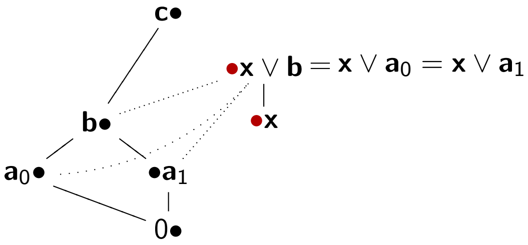


Contiguous pair

Theorem

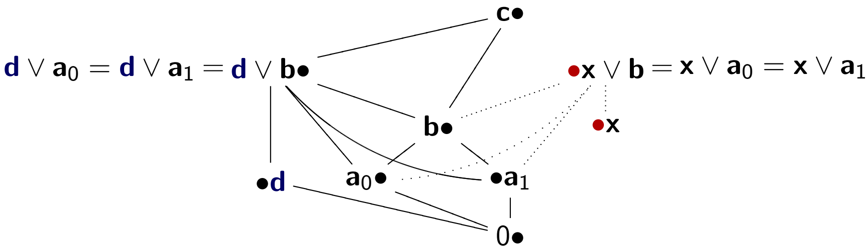
For every c.e. $\mathbf{b}$, *there exists c.e.* $\mathbf{a}_0, \mathbf{a}_1$ *such that*
$\forall \mathbf{x}\ (\mathbf{x} \vee \mathbf{a}_0 \geq_{wtt} \mathbf{b}\ \&\ \mathbf{x} \vee \mathbf{a}_1 \geq_{wtt} \mathbf{b} \implies \mathbf{x} \geq_{wtt} \mathbf{b})$.

Cor: There exists c.e. degrees $\mathbf{0} <_T \mathbf{a} <_T \mathbf{b} <_T \mathbf{c}$ such that
$\nexists \mathbf{x} \leq_T \mathbf{c}\ (\mathbf{x} \vee \mathbf{a}_0 \geq_T \mathbf{b}\ \&\ \mathbf{x} \vee \mathbf{a}_1 \geq_T \mathbf{b} \quad \& \quad \mathbf{x} \not\geq_T \mathbf{b})$.



The anti-cupping condition

Def: $(\mathcal{P}, \mathcal{P}[\mathbf{x}])$ satisfies the *anti-cupping condition* if
for every $\mathbf{b} \in \mathcal{P}$, $\mathbf{x} \not\geq \mathbf{b}$,
there exists $\mathbf{d} \in \mathcal{P}$, $\mathbf{x} \vee \mathbf{d} \not\geq \mathbf{b}$ such that
$\forall \mathbf{a} \in \mathcal{P}, \mathbf{a} \leq \mathbf{b}\ (\mathbf{x} \vee \mathbf{a} \geq \mathbf{b} \implies \mathbf{d} \vee \mathbf{a} \geq \mathbf{b})$.



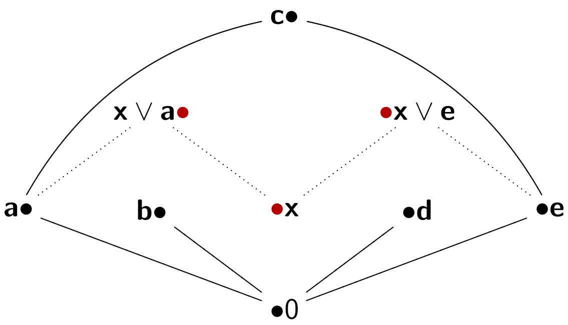
Theorem

$(\mathcal{P}, \mathcal{P}[\mathbf{x}]) \in \mathbb{E}^{c.e.} \implies (\mathcal{P}, \mathcal{P}[\mathbf{x}]) \models$ *anti-cupping condition.*

The A, B, C, D, E theorem

Theorem

There exist c.e. sets $\mathbf{a}, \mathbf{b}, \mathbf{d}, \mathbf{e}$ *all incomparable and* $\leq_T \mathbf{c}$ *c.e., such that* $\forall \mathbf{x} \leq \mathbf{c}\ (\mathbf{x} \vee \mathbf{a} \geq \mathbf{b} \Longrightarrow \mathbf{x} \vee \mathbf{e} \geq \mathbf{d})$.



Multi-1-generic

Theorem

Let $\mathcal{P}$ be any usl.
Let $\mathcal{Q} = \mathcal{P}[\mathbf{x}]$ be such that $\forall \mathbf{a}, \mathbf{b} \in \mathcal{P}$ $(\mathbf{x} \vee \mathbf{a} \geq \mathbf{b} \iff \mathbf{a} \geq \mathbf{b})$.
Then $(\mathcal{P}, \mathcal{Q}) \in \mathbb{E}^{c.e.}$.

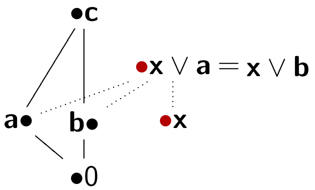
Lemma

Let C be c.e. and $A_0, \ldots, A_k <_T C$.
There exists $G \leq_T C$ that is 1-generic relative to all A_i.

No-least-join Theorem

Theorem

Let $\mathbf{c}$ *be c.e.,* $\mathbf{a}, \mathbf{b} <_T \mathbf{c}$ *and* $\mathbf{a} \not\leq_T \mathbf{b}$.
Then, there exists $\mathbf{x} \leq \mathbf{c}$, *such that* $\mathbf{x} \vee \mathbf{a} \geq \mathbf{b}$ *and* $\mathbf{x} | \mathbf{b}$.



The difference spectrum

Definition

$\mathbf{b} - \mathbf{a} = \{\mathbf{x} \in \mathbf{D} : \mathbf{x} \vee \mathbf{a} \geq_T \mathbf{b}\}$.

- $\mathbf{b} - \mathbf{a}$ is never an upper cone unless $\mathbf{a} = 0$.
- $\mathbf{b} - \mathbf{a}$ contains minimal degrees, minimal pairs, 1-generics.
- [JS] $\mathbf{b} - \mathbf{a} \subseteq \mathbf{d} - \mathbf{e} \iff \mathbf{e} \geq \mathbf{a}$ & $\mathbf{d} \geq \mathbf{e} \vee \mathbf{b}$ or $\mathbf{e} \geq \mathbf{d}$

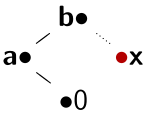
Definition

$\mathbf{b} -_{\mathbf{c}} \mathbf{a} = \{\mathbf{x} \leq_T \mathbf{c} : \mathbf{x} \vee \mathbf{a} \geq_T \mathbf{b}\}$.

- $\exists \mathbf{a} < \mathbf{b} < \mathbf{c}$ all c.e. s.t. $\mathbf{b} -_{\mathbf{c}} \mathbf{a}$ is the upper cone above $\mathbf{b}$.
- If $\mathbf{a}, \mathbf{b} < \mathbf{c}$, $\mathbf{c}$, c.e. and $\mathbf{a}|\mathbf{b}$, then $\mathbf{b} -_{\mathbf{c}} \mathbf{a}$ is never an upper cone.
- $\exists \mathbf{c}$ c.e. s.t. $\mathbf{b} -_{\mathbf{c}} \mathbf{a} \subseteq \mathbf{d} -_{\mathbf{c}} \mathbf{e} \quad \not\Rightarrow \mathbf{e} \geq \mathbf{a}$ & $\mathbf{d} \geq \mathbf{e} \vee \mathbf{b}$ or $\mathbf{e} \geq \mathbf{d}$

Non-low$_2$ cupping

Thm:[Posner 77] Let $0 <_T \mathbf{a} <_T \mathbf{b}$ where $\mathbf{b}$ is High.
There exists $\mathbf{x} <_T \mathbf{b}$, $\mathbf{x} \vee \mathbf{a} \equiv_T \mathbf{b}$

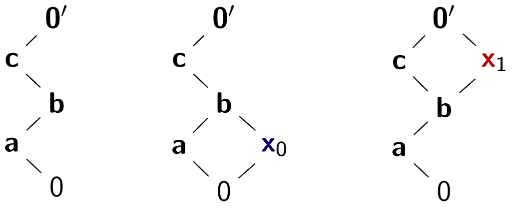


Theorem

Let $0 <_T \mathbf{a} <_T \mathbf{b}$ where $\mathbf{b}$ is non-low$_2$.
There exists $\mathbf{x} <_T \mathbf{b}$, $\mathbf{x} \vee \mathbf{a} \equiv_T \mathbf{b}$

$\forall\exists$ theory is hard

Let $\mathcal{P} = \{0 < \mathbf{a} < \mathbf{b} < \mathbf{c} < \mathbf{0}'\} \subset \mathcal{D}_{(\leq 0')}$.
Let $\mathcal{Q}_0 = \mathcal{P} \cup \{\mathbf{x}_0\}$ where $0 < \mathbf{x}_0 < \mathbf{b}$ and $\mathbf{a} \vee \mathbf{x}_0 = \mathbf{b}$.
Let $\mathcal{Q}_1 = \mathcal{P} \cup \{\mathbf{x}_0\}$ where $\mathbf{b} < \mathbf{x}_1 < \mathbf{0}'$ and $\mathbf{a} \vee \mathbf{x}_1 = \mathbf{0}'$.



Obs: $(\mathcal{P}, \mathcal{Q}_0) \notin \mathbb{E}^{jump}$ and $(\mathcal{P}, \mathcal{Q}_1) \notin \mathbb{E}^{jump}$
But, every embedding of $\mathcal{P}$, either extends to $\mathcal{Q}_0 \hookrightarrow \mathcal{D}$
or to $\mathcal{Q}_1 \hookrightarrow \mathcal{D}$.
Because, either $\mathbf{b}$ is non-low$_2$, or $\mathbf{0}'$ is non-low$_2$ over $\mathbf{b}$.