From Büchi to Borel Structures.

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Automata

An automaton M is given by

- a finite *alphabet* Σ
- a finite set of *states* S
- a nonempty subset $F \subseteq S$, the set of *accepting states*
- a start state $s_0 \in S$
- a *transition table* $\Delta \subseteq S \times \Sigma \times S$.

A *run on a string* $x \in \Sigma^{<\omega}$ is a string $\rho \in S^{|x|}$ s.t. $\rho(0) = s_0$ and $\forall n$
$$(\rho(n), x(n), \rho(n+1)) \in \Delta.$$

M *accepts* x if there exists a run ρ on x that ends in F.

$L(M)$ is the set of strings accepted by M.

$L \subseteq \Sigma^{<\omega}$ is *regular* if $L = L(M)$ for some automata M.

Automatic Structures

An automatic presentation of a (relational) structure $\mathcal{A}$ is $(D, R_1, \ldots, R_n)$ where

- the domain $D \subseteq \Sigma^{<\omega}$ is regular and
- the relations $R_i \subseteq (\Sigma^{<\omega})^n$ are regular.

such that $\mathcal{A}$ is isomorphic to $(D, R_1, \ldots, R_n)$.

Note that an n-tuple of strings in $\Sigma^{<\omega}$ can be viewed as one sting in $(\Sigma^n)^{<\omega}$ with the use of extra blank symbols.

Examples

Examples:

- $(\mathbb{N}, +)$,
- $(\mathbb{Q}, \leq)$,
- the interval Boolean algebra of ω,
- every finitely generated abelian group, etc...

FA-presentable structures have nice properties:

- They are decidable.
- They are closed under interpretations with first-order formulas.

Open question: Is $(\mathbb{Q}, +)$ automatic?

Büchi structures

A *Büchi presentation* of $\mathcal{A}$ is a tuple of relations

$$\mathcal{S} = (D, E, R_1, \ldots, R_n),$$

where $\mathcal{A} \cong \mathcal{S}/E$ and

- $D \subseteq 2^\omega$ is a set (the domain),
- E is an equivalence relation on D,
- $R_1, \ldots R_n$ are relations compatible with E.
- all, D, E and the R_i are recognizable by Büchi automata.

The presentation $\mathcal{S}$ is injective if E is the identity on D.

Büchi languages

An Büchi automaton M is given by

- a finite *alphabet* Σ
- a finite set of *states* S
- a nonempty subset $F \subseteq S$, the set of *accepting states*
- a start state $s_0 \in S$
- a *transition table* $\Delta \subseteq S \times \Sigma \times S$.

A *run on* $x \in \Sigma^\omega$ is a map $\rho : \mathbb{N} \to S$ such that $\rho(0) = s_0$ and $\forall n$
$$(\rho(n), x(n), \rho(n+1)) \in \Delta.$$

M accepts $x \in \Sigma^\omega$ if $\exists$ run ρ on x s.t. $\exists^\infty n\ \rho(n) \in F$.

$L(M)$ is the set of strings accepted by M.
$L \subseteq \Sigma^\omega$ is *Büchi* if $L = L(M)$ for some Büchi automata M.

Examples - Properties

Examples:

- $(\mathbb{R}, +, \leq)$,
- $(2^\omega, \leq_{lex})$,
- Monadic second order logic $(\mathbb{N}, \mathcal{P}(\mathbb{N}); Succ, \in)$,
- $(\mathcal{P}(\mathbb{N})/\equiv^*, \subseteq^*, \cup, \cap, \neg)$ using equivalence

Büchi-presentable structures also have decidable theories.

- Do we have a Lowenheim-Skolem theorem?
- Does $(\mathcal{P}(\mathbb{N})/\equiv^*, \subseteq^*)$ have an injective presentation? (open)
- Does every Büchi structure have an injective presentation?
- Given to isomorphic Büchi structures,
  how complicated can the isomorphism be? (open)
- How hard is it to tell if two Büchi structures are isomorphic?
  (Open)

Loops

Definition (KHMN)

A Σ-*loop* is a tuple $G = (V, v_0, E, l)$, where

- V is a finite set,
- $v_0 \in V$ is the *initial vertex*,
- E is the *edge function* $E\colon V \to V$, and
- l is the *labeling function* $l\colon V \to \Sigma$.

$$\begin{tikzcd}
v_0 \arrow[r] & v_1 \arrow[r] & v_2 \arrow[r] & v_3 \arrow[dr, bend left] & \\
& v_7 \arrow[ur, bend left] & & & v_4 \arrow[dl, bend left] \\
& & v_6 \arrow[ul, bend left] & v_5 \arrow[l] &
\end{tikzcd}$$

Loops represent infinite eventually periodic words.

Loop automata

Let $\mathbf{M} = (\Sigma, S, F, s_0, \Delta)$ be an automata as definition before.

A *run* of $\mathbf{M}$ on a Σ-loop G is a sequence
$\mathbf{r} = (v_0, s_0), (v_1, s_1), ..., (v_k, s_k) \in (V \times S)^{<\omega}$,

$$v_{i+1} = E(v_i) \quad \text{and} \quad (s_i, l(v_i), s_{i+1}) \in \Delta,$$

and there is a unique i so that $(v_i, s_i) = (v_k, s_k)$.

M accepts $x \in \Sigma^\omega$ if $\exists$ run ρ on x s.t. $\exists n \; (i \leq n \leq k \& \rho(n) \in F)$.

$L(M)$ is the set of strings accepted by M.
$L \subseteq \Sigma^\omega$ is *Loop automatic* if $L = L(M)$ for some Loop automata M.

Loop-automatic structures

Examples:

- all automatic structures,
- $(\mathbb{Q}, +, \leq)$, not known to be automatic
- the atomless Boolean algebra, known to be NOT automatic

Loop automatic structures also have nice properties:

- They are decidable.
- They are closed under interpretations with first-order formulas.

Lowenheim-Skolem

Theorem (KHMN: Löwenheim-Skolem for Büchi structures)

Every Büchi presentable structure has a loop-automatic elementary substructure.

Idea of Pf: The set of eventually periodic words of a Büchi structure is an elementary substructure. This uses the fact that every non-empty Büchi set contains an eventually periodic word.

Borel sets

The class of *Borel subsets of* $\mathbb{R}^n$ is the smallest σ-algebra containing all the open sets.

i.e., it is the class sets is generated from the open sets
using countable unions,
countable intersections and
complements.

Borel structures

A *Borel presentation* of $\mathcal{A}$ is a tuple of relations

$$\mathcal{S} = (D, E, R_1, \ldots, R_n),$$

where $\mathcal{A} \cong \mathcal{S}/E$ and

- $D \subseteq 2^\omega$ is a set (the domain),
- E is an equivalence relation on D,
- $R_1, \ldots R_n$ are relations compatible with E.
- all, D, E and the R_i are Borel.

The presentation $\mathcal{S}$ is injective if E is the identity on D.

Borel structucres

Examples:

- All Büchi structures.
- $(\mathbb{R}, +, \times, exp, \leq)$, $(\mathbb{C}, +, \times)$
- The Torus $S^1 \times S^1$ with its two rotations,
- Second order arithmetic $(\mathbb{N}, \mathcal{P}(\mathbb{N}); +, \times, \leq, \in)$,
- The Boolean algebra $(\mathcal{P}(\mathbb{N})/\equiv^*, \subseteq^*, \cup, \cap, \neg)$ using equivalence

In contrast,
$(\omega_1, \leq)$ is not Borel presentable.

Friedman's work

The first one who studied Borel Structures was H. Friedman 1978

Theorem [F78]: Every countable theory with infinite models has an uncountable injective Borel model.

He also looked at the language extended with quantifiers

- For all but countably many x ...
- For every x in a co-meager set ...
- For every x in a full-meassure set ...

and studied axiomatizations, completeness, decidability, etc...
(See papers by Steinhorn.)

Borel Linear orderings

Write $\mathcal{A} \preccurlyeq \mathcal{B}$ if the LO $\mathcal{A}$ embeds in $\mathcal{B}$.

Theorem [Harrington Shelah]: For every Borel LO $\mathcal{A}$,
there exists $\xi < \omega_1$ such that $\mathcal{A} \preccurlyeq 2^\xi$.

Corollary: No Borel LO contains a copy of ω_1 nor of ω_1^*.

Theorem [Louveau 89; Marker]: For every Borel LO $\mathcal{A}$ and $\xi < \omega_1$
either $\mathcal{A} \preccurlyeq 2^{\omega \cdot \xi}$ or $2^{\omega \cdot \xi + 1} \preccurlyeq \mathcal{A}$.

Theorem [Louveau Saint-Raymond 90]: (hyperproj. determinancy)
The Borel suborderings of $\mathbb{R}^\omega$ are well-quasi-ordered under $\preccurlyeq$.

There is also work done on Borel partial orderings
[Harrington, Marker, Shelah 88] [Kanovei 98], etc...

The Basic Lemma

Lemma (Basic Lemma)

Let $\mathcal{B} = (\mathcal{P}(\mathbb{N}), \subseteq)$. Each isomorphism between two Borel presentations of $\mathcal{B}$ has a Borel graph.

Proof.

- Suppose Φ: $(\mathcal{P}(\mathbb{N}), \subseteq) \cong (A, E, \leq)/E$.
- Let $[a_n]_E = \Phi(\{n\})$.
- Then $\Phi(X) = [b]_E \Leftrightarrow \forall n\,(n \in X \leftrightarrow a_n \leq b)$.
- This is a ctble intersection of Borel relations, hence Borel too.

Theorem

There is a Büchi structure $\mathcal{A}$ with no injective Borel presentation.

Proof. The signature consists of binary predicates $\leq, R$ and unary predicate U.

Recall that $\mathcal{B} = (\mathcal{P}(\mathbb{N}), \subseteq)$ and $\mathcal{B}^* = (\mathcal{P}(\mathbb{N})/ =^*, \leq)$. Let $\mathcal{A}$ be the disjoint sum of $\mathcal{B}$ and $\mathcal{B}^*$ as partial orders.
Let $U^{\mathcal{A}}$ the left side and let $R^{\mathcal{A}}$ be the projection $\mathcal{B} \mapsto \mathcal{B}^*$.

The Büchi presentation is $(\mathcal{B}_0 \sqcup \mathcal{B}_1, E, \leq, \mathcal{B}_0, S)$, where

- E is identity on the left and $=^*$ on the right,
- S is the natural bijection between the two copies of $\mathcal{B}$.

To see that $\mathcal{A}$ has no injective Borel representation, we need a fact from descriptive set theory.

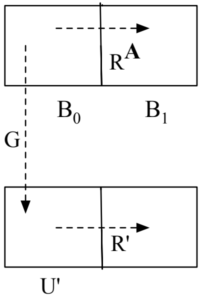
Fact: There is no Borel function $F\colon 2^\omega \to \mathbb{R}$ such that for each $X, Y \subseteq \omega$

$$X =^* Y \Leftrightarrow F(X) = F(Y).$$

Now assume that $\mathcal{S} = (D, \leq', U', R')$ is an injective Borel presentation of $\mathcal{A}$. Let Φ be an isomorphism and G be the restriction of Φ to $\mathcal{B}_0$, which is Borel by the Basic Lemma. Then,

$$\begin{aligned} R'(G(X)) = R'(G(Y)) &\Leftrightarrow \Phi(R^{\mathcal{A}}(X)) = \Phi(R^{\mathcal{A}}(Y)) \\ &\Leftrightarrow R^{\mathcal{A}}(X) = R^{\mathcal{A}}(Y) \\ &\Leftrightarrow X =^* Y. \end{aligned}$$

contrary to the fact from descriptive set theory.



Borel categoricity

A Borel structure is *Borel categorical* if any two Borel presentations of it are Borel isomorphic.

Example

- The Boolean algebra $(\mathcal{P}(\mathbb{N}), \subseteq)$
- The linear ordering $(\mathbb{R}, \leq)$.

Theorem

There are two Büchi presentations of $(\mathbb{R}, +)$ that are not Borel isomorphic.

Borel Categoricity

Theorem

There are two Büchi presentations of $(\mathbb{R}, +)$ that are not Borel isomorphic.

Proof: Consider $(\mathbb{R}, +)$ and $(\mathbb{R}, +) \times (\mathbb{R}, +)$.

They are isomorphic because they are both $\mathbb{Q}$-vector spaces of dimension $2^{\aleph_0}$.

They are NOT Borel isomorphic because any Borel isomorphism between Polish groups must be a homeomorphism.

Corollary

The isomorphism problem for Büchi structures is not a provably in ZF Σ^1_2 relation on numbers

proof: $(\mathbb{R}, +)$ and $(\mathbb{R}, +) \times (\mathbb{R}, +)$ are isomorphic in any model of ZFC.
But they're NOT isom. in Shelah generic extension with sets all measurable.
Provably Σ^1_2 relations are absolute for these types of models.

Uncountable Languages

Theorem

There is an (unctble) Borel theory with no Borel completion.

Idea of the Pf: The language has a constant c_A for each $A \subseteq \omega$, and a unary predicate U.
The axioms say that the set

$$\{A \subseteq \omega : U(c_A) \text{ holds}\}$$

is a non-principal ultrafilter in ω.
We know there are no Borel non-pricipal ultrafilters.

Uncountable Languages

Theorem

There is a complete (unctble) Borel theory with no Borel model.

Open questions

Does $(\mathcal{P}(\mathbb{N})/\equiv^*, \subseteq^*)$ have an injective Borel presentation?

What are the possible Borel dimensions?
The Borel dimension of a Borel structure is the number of non-Borel-isomorphic Borel presentations of the structure.

What is the complexity of the isomorphism problem for Büchi and for Borel structures?