Up to equimorphism, hyperarithmetic is computable.

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Spector's Theorem.

Theorem:[Spector '55] Every hyperarithmetic well ordering is isomorphic to a computable one.

Definition:

- The *Turing degree* of a linear ordering $\mathcal{X} = \langle X, \leq_X \rangle$, with $X \subseteq \omega$, is
$$\text{deg}(X) \oplus \text{deg}(\leq_X) .$$
- A *computable (hyperarithmetic) linear ordering* is a linear ordering of computable (hyperarithmetic) degree.
- A *computable (hyperarithmetic) well ordering* is a well ordering $\langle X, \leq_X \rangle$ that is computable (hyperarithmetic) as a linear ordering.
- The order type of a computable well ordering is a *computable ordinal*.
- ω_1^{CK} is the least non-computable ordinal.

Hyperarithmetic sets.

Proposition: [Suslin-Kleene, Ash]

For a set $X \subseteq \omega$, the following are equivalent:

- X is $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$.
- X is computable in $0^{(\alpha)}$ for some $\alpha < \omega_1^{CK}$.

  ($0^{(\alpha)}$ is the αth Turing jump of 0.)
- $X = \{x : \varphi(x)\}$, where φ is a computable infinitary formula.

  (*Computable infinitary formulas* are 1st order formulas which may contain infinite computable disjunctions or conjunctions.)

A set satisfying the conditions above is said to be *hyperarithmetic*.

In particular, every computable, Δ^0_2, and arithmetic set is hyperarithmetic.

Theorem:[Spector 1955] Every hyperarithmetic well ordering is isomorphic to a computable one.

Spector's theorem.

Spector's theorem doesn't directly extend to linear orderings:
Not every hyperarithmetic linear ordering is isomorphic to a computable one.

Theorem: There is a linear ordering of Turing degree $\mathbf{a}$ which does not have a computable copy if

- $\mathbf{a}'' >_T 0''$; [Lerman '81]
- $\mathbf{a}$ is c.e. and $\mathbf{a} \not\equiv_T 0$; [Jockusch, Soare '91]
- $0 <_T \mathbf{a} \leq 0'$; [Downey '98][Seetapun]
- $\mathbf{a} \not\equiv_T 0$. [Knight '2000]

But this is not the only way we could extend Spector's theorem to linear orderings.

Theorem:[Spector 1955] Every hyperarithmetic well ordering is isomorphic to a computable one.

Our main result

Definition:

- Given linear orderings $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ *embeds in* $\mathcal{B}$ if there is a strictly increasing map $f\colon \mathcal{A} \hookrightarrow \mathcal{B}$. We write $\mathcal{A} \preccurlyeq \mathcal{B}$.
- $\mathcal{A}$ and $\mathcal{B}$ are *equimorphic* if $\mathcal{A} \preccurlyeq \mathcal{B}$ and $\mathcal{B} \preccurlyeq \mathcal{A}$. We write $\mathcal{A} \sim \mathcal{B}$.

Example:

$$\omega + \omega^* + \omega + \omega^* + \cdots \quad \sim \quad \omega^* + \omega + \omega^* + \omega + \cdots$$

Observation: If α is an ordinal and $\mathcal{L} \sim \alpha$, then $\mathcal{L}$ is isomorphic to α.

Proof: $\mathcal{L} \preccurlyeq \alpha \implies \mathcal{L}$ is an ordinal and $\mathcal{L} \leq \alpha$.
$\alpha \preccurlyeq \mathcal{L} \implies \alpha \leq \mathcal{L}$ and hence $\mathcal{L} \cong \alpha$. $\square$

Theorem: Every hyperarithmetic linear ordering is equimorphic to a recursive one.

Definition:

- Given a l.o. $\mathcal{L}$, we define another l.o. $\mathcal{L}'$ by identifying the elements of $\mathcal{L}$ which have finitely many elements in between.
- Then we define $\mathcal{L}^0 = \mathcal{L}$, $\mathcal{L}^{\alpha+1} = (\mathcal{L}^\alpha)'$, and take direct limits when α is a limit ordinal.
- $\mathrm{rk}(\mathcal{L})$, the *Hausdorff rank* of $\mathcal{L}$, is the least α such that $\mathcal{L}^\alpha$ is finite.

Examples:
$$\mathrm{rk}(\omega) = \mathrm{rk}(\mathbb{Z}) = 1, \qquad \mathrm{rk}(\omega^\alpha) = \alpha,$$
$$\mathrm{rk}(\mathbb{Z} + \mathbb{Z} + \mathbb{Z} + \cdots) = 2, \quad \mathrm{rk}(\mathbb{Q}) = \infty$$
(where $\mathbb{Z}$ and $\mathbb{Q}$ are the integers and the rationals)

Observation: If $\mathcal{A} \preccurlyeq \mathcal{B}$, then $\mathrm{rk}(\mathcal{A}) \leq \mathrm{rk}(\mathcal{B})$. Therefore, $\mathcal{A} \sim \mathcal{B} \implies \mathrm{rk}(\mathcal{A}) = \mathrm{rk}(\mathcal{B})$

Proposition:[Cantor, Hausdorff] For a countable l.o. $\mathcal{L}$, the following are equivalent

- $\mathbb{Q} \not\preccurlyeq \mathcal{L}$,
- $\mathcal{L}$ is not equimorphic to $\mathbb{Q}$,
- $\mathrm{rk}(\mathcal{L}) < \omega_1$.

A l.o. $\mathcal{L}$ such that $\mathbb{Q} \not\preccurlyeq \mathcal{L}$ is said to be *scattered.*

Lemma: If $\mathcal{L}$ is a hyperarithmetic scattered linear ordering, then $\text{rk}(\mathcal{L}) < \omega_1^{CK}$.

Proof: A standard overspill argument.

Theorem: If $\mathcal{L}$ is scattered then
$\text{rk}(\mathcal{L}) < \omega_1^{CK} \iff \mathcal{L}$ is equimorphic to a computable linear ordering.

Proof of $\Leftarrow$: Use the lemma and the observation above.

Proof of our main theorem using the theorem above:

Let $\mathcal{L}$ be a hyperarithmetic linear ordering. If $\mathbb{Q} \preccurlyeq \mathcal{L}$, then $\mathcal{L} \sim \mathbb{Q}$. Otherwise, $\text{rk}(\mathcal{L}) < \omega_1^{CK}$, and hence $\mathcal{L}$ is equimorphic to a computable linear ordering.

Definition: Let $\mathbb{L}$ be the partial ordering of equimorphism types of countable linear orderings, ordered by embeddablity.

Theorem: [Fraïsé's Conjecture '48; Laver '71]
$\mathbb{L}$ is a *well partial ordering*.
(i.e., $\mathbb{L}$ has no infinite descending sequences and no infinite antichains.)
Also, for every scattered $x \in \mathbb{L}$,
$\{y \in \mathbb{L} : y \preccurlyeq x\}$ is countable.

Definition: Let $\mathbb{L}_\alpha$ be the restriction of $\mathbb{L}$ to the linear orderings of rank $\leq \alpha$.

Theorem: For every ordinal α,
$\alpha < \omega_1^{CK} \iff \mathbb{L}_\alpha$ is computably presentable.

Very General Idea of the proof

Definition: Given a countable subset $S = \{\mathcal{L}_0, \mathcal{L}_1, ...\} \subseteq \mathbb{L}$ let
$F(+, S) = \mathcal{L}_0 + (\mathcal{L}_0 + \mathcal{L}_1) + (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2) + \cdots$
and
$F(-, S) = \cdots + (\mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0) + (\mathcal{L}_1 + \mathcal{L}_0) + \mathcal{L}_0$.
(F is well defined on sets of equimorphism types.)

Definition: By transfinite recursion on α we define $\mathbb{H}_\alpha \subset \mathbb{L}$:

- $\mathbb{H}_0 = \{\mathbf{1}\}$,
- $\mathbb{H}_\alpha = \{F(+, S), F(-, S) : S \subseteq \bigcup_{\beta<\alpha} \mathbb{H}_\beta\} \cup \{\mathbf{1}\}$.

We let $\mathbb{H} = \bigcup \mathbb{H}_\alpha$ be the class of *h-indecomposables*.
Observe that $\mathbb{H}_\alpha = \mathbb{H} \cap \mathbb{L}_\alpha$.

Theorem:[Laver '71] Every scattered countable linear ordering is equimorphic to a finite sum of h-indecomposables.

(So, it is enough to prove that if $\mathcal{L}$ is h-indec. and rk$(\mathcal{L}) < \omega_1^{CK}$, then $\mathcal{L}$ is equimorphic to a computable l.o.)

By computable transfinite recursion we build:

- $\langle H_\alpha, \leq_\alpha \rangle$, a computable presentation of $\mathbb{H}_\alpha$;
- a computable family $\{\mathcal{L}^x : x \in H_\alpha\}$, such that $\mathcal{L}^x$ is a computable linear ordering in the equimorphism type corresponding to x.

Key points:

- If I is the downward closure of $S \subset \mathbb{L}$, then
$$F(+, S) \sim F(+, I).$$
(downward closed subset of $\mathbb{H}$ are called *ideals*)
- F induces a bijection between
$$\{+, -\} \times \{I : I \text{ ideal of } \mathbb{H}_\beta\}, \quad \text{and} \quad \mathbb{H}_{\beta+1}.$$
- Every ideal I of $\mathbb{H}_\beta$ is determined by the set of minimal elements of $\mathbb{H}_\beta \setminus I$, which is an antichain, and hence is <u>finite</u> because $\mathbb{H}_\beta$ is a well partial ordering.

Problem: Recognize $\mathbb{H}_\beta$ inside $\mathbb{H}_{\beta+1}$.
Solution: Deeper understanding of the structure of $\mathbb{H}_\beta$; Signed Trees.