Embedding Jump Upper Semilattices into the Turing Degrees.

Antonio Montalbán.
Cornell University.

Definition: A *partial jump upper semilattice* (PJUSL) is a structure

$$\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \cup, \mathrm{j} \rangle$$

- $\langle J, \leq_{\mathcal{J}} \rangle$ is a partial ordering.
- $x \cup y$ is the least upper bound of x and y.
- $x <_{\mathcal{J}} \mathrm{j}(x)$.
- $x \leq_{\mathcal{J}} y \implies \mathrm{j}(x) \leq_{\mathcal{J}} \mathrm{j}(y)$.

A *jump upper semilattice* (JUSL) is a PJUSL where j and $\cup$ are total operations.
A *jump partial ordering* (JPO) is a PJUSL where j is total but $\cup$ is undefined.

Example: The structure of Turing Degrees.

$$\mathcal{D} = \langle \mathbf{D}, \leq_T, \vee, ' \rangle.$$

Known Results.

Question: Which PJUSLs can be embedded in $\mathcal{D}$?

Theorem[Kleene-Post, 54]: Every finite upper semilattice can be embedded in $\mathcal{D}$.

Theorem[Sacks, 61]: Every partial ordering, of size $\aleph_1$ with the countable predecessor property can be embedded in $\mathcal{D}$.

Theorem[Abraham-Shore, 86]: Every upper semilattice of size $\aleph_1$, with the countable predecessor property, can be embedded in $\mathcal{D}$ as an initial segment.

Theorem[Hinman-Slaman, 91]: Every countable JPO, $\langle P, \leq, \mathrm{j} \rangle$, can be embedded in $\mathcal{D}$.

Known Results.

Question: Which fragments of $Th(\mathbf{D}, \leq_T, \vee, ')$ are decidable?

- [Kleene-Post, 54]

  $\exists - Th(\mathbf{D}, \leq_T)$ is decidable.
- [Lachlan, 68]

  $Th(\mathbf{D}, \leq_T)$ is undecidable.

- [Jockusch-Slaman, 93]

  $\forall\exists - Th(\mathbf{D}, \leq_T, \vee)$ is decidable.
- [Shmerl]

  $\exists\forall\exists - Th(\mathbf{D}, \leq_T)$ is undecidable.

- [Hinman-Slaman, 91]

  $\exists - Th(\mathbf{D}, \leq_T, ')$ is decidable.

Theorem: Every countable PJUSL, $\langle J, \leq_{\mathcal{J}}, \vee, \mathbf{j} \rangle$, can be embedded into $\mathcal{D}$.

Corollary: $\exists - \mathit{Th}(\mathbf{D}, \leq_T, \vee, ')$ is decidable.

Proof: Essentially, for an $\exists$-formula φ,

$\langle \mathbf{D}, \leq_T, \vee, ' \rangle \models \varphi \iff \varphi$ is not obviously false.

i.e. It does not contradict the axioms of PJUSL.
$\square$

Theorem[Shore-Slaman, to appear]:
$\forall\exists - \mathit{Th}(\mathbf{D}, \leq_T, \vee, ')$ is undecidable.

Every countable PJUSL, $\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \vee, \mathrm{j} \rangle$, is embeddable in $\mathcal{D}$.

Outline of the proof:

Definition: A *Jump Hierarchy* (JH) over $\mathcal{J}$ is a map $H\colon J \to \omega^\omega$ s.t., for all $x, y \in P$,

- $\mathcal{J} \leq_T H(x)$;
- if $x <_{\mathcal{J}} y$ then $H(x)' \leq_T H(y)$.
- $\bigoplus_{x \leq_{\mathcal{J}} y} H(x) \leq_T H(y)$;

Theorem: Every countable PJUSL which supports a JH can be embedded in $\mathcal{D}$.

Proof: Forcing Construction. $\square$

Example: [Harrison, 68]
There is a recursive linear ordering

$$\mathcal{L} \cong \omega_1^{CK} \cdot (1 + \eta),$$

which supports a JH, $H_{\mathcal{L}} \colon \mathcal{L} \to \omega^\omega$.

Observation: If there is a strictly monotone map $\mathrm{lev} \colon \mathcal{J} \to \mathcal{L}$, s.t. the pair $\langle \mathcal{J}, \mathrm{lev} \rangle$ is HYP, then $\mathcal{J}$ supports a JH.
(Essentially, compose $\mathrm{lev} \colon \mathcal{J} \to \mathcal{L}$ with $H_{\mathcal{L}} \colon \mathcal{L} \to \omega^\omega$.)

Definition: A *partial jump upper semilattice with levels in* $\mathcal{L}$ is a pair $\langle \mathcal{J}, \mathrm{lev} \rangle$ where

- $\mathcal{J}$ is a PJUSL, and
- lev is a map, $\mathrm{lev} \colon \mathcal{J} \to \mathcal{L}$, s.t.

$$x <_{\mathcal{J}} y \implies \mathrm{lev}(x) < \mathrm{lev}(y).$$

Every countable PJUSL, $\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \vee, \mathrm{j}\rangle$, is embeddable in $\mathcal{D}$.

Outline of the proof:

Suppose that $\mathcal{J}$ is recursive.

Lemma: There is a level map lev: $\mathcal{J} \to \mathcal{L}$, an ordinal $\alpha < \omega_1^{CK}$, and a sequence, $\{\langle \mathcal{J}_n, l_n\rangle\}_n$, of finitely generated PJUSL w/ levels in $\mathcal{L}$, s.t.

$\langle \mathcal{J}_1, l_1\rangle \subseteq \langle \mathcal{J}_2, l_2\rangle \subseteq \langle \mathcal{J}_3, l_3\rangle \subseteq \cdots \subset \langle \mathcal{J}, \mathrm{lev}\rangle$,
$\langle \mathcal{J}, \mathrm{lev}\rangle = \bigcup_n \langle \mathcal{J}_n, l_n\rangle$,
and each $\langle \mathcal{J}_n, l_n\rangle$ is arithmetic in $0^{(\alpha)}$.

Definition: Let

$$\mathcal{K}_\alpha = \left\{ \langle \mathcal{F}, l\rangle : \begin{array}{l} \langle \mathcal{F}, l\rangle \text{ is a fin. generated PJUSL w/} \\ \text{levels in } \mathcal{L}\text{, which is arithmetic in } 0^{(\alpha)} \end{array} \right\}$$

Let $\mathcal{P}_\alpha = \langle \mathcal{Q}_\alpha, l_\alpha\rangle$, be the Fraïssé limit of $\mathcal{K}_\alpha$.

Properties: • $\mathcal{J}$ can be embedded in $\mathcal{Q}_\alpha$.
• $\mathcal{P}_\alpha$ has a presentation recursive in $0^{(\alpha+\omega)}$.
Therefore, $\mathcal{Q}_\alpha$ supports a JH, and hence it can be embedded in $\mathcal{D}$.

Other results.

Definition: A *partial jump upper semilattice with 0* (PJUSL w/0) is a structure

$$\mathcal{J} = \langle J, \leq_{\mathcal{J}}, \cup, \mathrm{j}, 0 \rangle$$

such that
- $\langle J, \leq_{\mathcal{J}}, \cup, \mathrm{j} \rangle$ is a PJUSL, and
- 0 is the least element of $\langle J, \leq_{\mathcal{J}} \rangle$.

Question: Which PJUSL w/0 can be embedded into $\mathcal{D}$?

Question: Which quantifier free types of PJUSL w/0 are realized in $\mathcal{D}$?

Note that realizing an q.f. n-type of PJUSL w/0 is equivalent to embedding an n-generated PJUSL w/0.

Other results.

A negative answer: Not every quantifier free 1-type of JUSL w/0 is realizable in $\mathcal{D}$.

Proof: There are $2^{\aleph_0}$ q.f. 1-types, $p(x)$, containing the formula $x \leq 0''$. $\square$

Corollary: Not every countable JUSL w/0 can be embedded in $\mathcal{D}$.

A positive answer: Every quantifier free 1-type of JPO w/0 is realized in $\mathcal{D}$.

Note: Hinman and Slaman proved this result for types containing a formula of the form $x \leq 0^{(n)}$.

Other results.

Let κ be a cardinal, $\aleph_0 < \kappa \leq 2^{\aleph_0}$.

Question: Is every PJUSL with the c.p.p. and size κ embeddable in $\mathcal{D}$?

Proposition:
If $\kappa = 2^{\aleph_0}$, then the answer is **NO**.

Proposition:
If MA(κ) holds, the answer is **YES**.

Corollary: For $\kappa = \aleph_1$, it is independent of ZFC.

Proof: It is FALSE under CH,
but TRUE under MA($\aleph_1$). $\square$