On the proof-theoretic strength of Jullien's results.

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Definition:

- A *linear ordering* is a poset (partial ordered set) $(L, \leq_L)$ such that $\forall x, y \in L\ (x \leq_L y \vee y \leq_L x)$.
- A *linearization* of a poset $\mathcal{P} = (P, \leq_P)$ is a linear ordering $(P, \leq_L)$ such that
$$\forall x, y \in P(x \leq_P y \Rightarrow x \leq_L y).$$

Theorem: (RCA_0)
Every poset has a linearization.

(The non-effective version is due to [Szpilrajn 30].)

Proof: Given $\mathcal{P} = (\{p_0, p_1, p_2, ...\}, \leq_P)$, we define $\leq_L$ by stages. At stage $s+1$ we define $\leq_{L,s+1}$ on $\{p_0, ..., p_s\}$ extending $\leq_P$ and $\leq_{L,s}$. Everything works out fine.

We ask about linearizations that preserve certain properties of the poset, as for example well-foundedness.

Lemma 1 Every well-founded poset has a well-ordered linearization.

Proof:(ATR$_0$) Consider $\mathcal{P} = (\{p_0, p_1, p_2, ...\}, \leq_P)$ a well-founded poset and the rank function rank: $\mathcal{P} \to \alpha$, where $\alpha = \text{rank}(P)$.

Define $\leq_L$ as follows:

$$p_i \leq_{\mathcal{L}} p_j \iff \text{rank}(p_i) < \text{rank}(p_j) \text{ or } \text{rank}(p_i) = \text{rank}(p_j) \ \& \ i \leq j.$$

$(P, \leq_L)$ is well-founded cause it's a subordering of $\alpha \times \omega$.

Theorem: [Rosenstein, Kierstead] Every computable well-founded poset has a computable well-founded linearization.

Theorem: [Rosenstein, Statman] There is a computable poset without computable descending sequences which has no computable linearization without computable descending sequences.

Corollary: RCA$_0$ **doesn't** prove Lemma 1.

Theorem:(Downey, Hirschfeldt, Lempp, Solomon [DHLS'03])
Over RCA$_0$: WKL$_0$ $\subsetneq$ Lemma 1 $\subseteq$ ACA$_0$.

Extendability

Definition: A linear ordering $\mathcal{L}$ is *extendible* if every poset which does not embed $\mathcal{L}$ has a linearization which does not embed $\mathcal{L}$ either.

Example: ω^*, ω, $\mathbb{Z}$, $\mathbb{Q}$, and ω^α are extendible.
$\mathbf{1} + \mathbf{1}$, and $\omega + \omega^*$ are **not** extendible.

Pierre Jullien gave a characterization of the countable extendible linear orderings in 1969.

Question:[Downey, Remmel '00] What is the proof-theoretic strength of Jullien's Thm?

Extendability of $\mathbb{Z}$ and $\mathbb{Q}$.

Theorem:[DHLS'03] The extendibility of $\mathbb{Z}$ is equivalent to ATR_0 over RCA_0.

Theorem:(Becker [DHLS'03])
The extendibility of $\mathbb{Q}$ follows from Π^1_1-CA_0, and is not provable in WKL_0.

Theorem:(J. Miler) The extendibility of $\mathbb{Q}$ implies WKL_0 over RCA_0, and
implies ATR_0 over Σ^1_1-$Choice_0$.

Theorem: The extendibility of $\mathbb{Q}$ follows from $ATR_0+\Sigma^1_1$-IND.

Corollary: The extendibility of $\mathbb{Q}$ is equivalent to ATR_0 over Σ^1_1-$Choice_0$ + Σ^1_1-IND.

Definitions.

Let $\mathcal{L}$ be a linear ordering.

- $\mathcal{L}$ is *scattered* if $\mathbb{Q} \not\preceq \mathcal{L}$.
- $\mathcal{L}$ is *indecomposable to the right* if for every non-trivial cut $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have $\mathcal{L} \preceq \mathcal{B}$.
- $\mathcal{L}$ is *indecomposable to the left* if for every non-trivial cut $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have $\mathcal{L} \preceq \mathcal{A}$.

Examples: ω and ω^ω are indecomposable to the right.
ω^* is indecomposable to the left.

- A *finite decomposition* of $\mathcal{L}$ is a tuple $(\mathcal{A}_0, ..., \mathcal{A}_k)$ such that $\mathcal{L} = \mathcal{A}_0 + ... + \mathcal{A}_k$, and each $\mathcal{A}_i$ is either indecomposable or $\mathbf{1}$.
- $\mathcal{L}$ has *signature* $\sigma \in \{\mathbf{1}, \leftarrow, \rightarrow\}^{<\omega}$ if $\mathcal{L}$ has a decomposition of minimal length, $\mathcal{L} = \sum_{i<|\sigma|} \mathcal{A}_i$ such that
  - if $\sigma(i) = \mathbf{1}$, then $\mathcal{A}_i = \mathbf{1}$,
  - if $\sigma(i) = \leftarrow$, then $\mathcal{A}_i$ is indec. to the left,
  - if $\sigma(i) = \rightarrow$, then $\mathcal{A}_i$ is indec. to the right.

Examples:

- $\omega^2 + \omega^* + \omega + \mathbf{1}$ has signature $(\rightarrow, \leftarrow, \rightarrow, \mathbf{1})$.
- $\mathbb{Z}$ has signature $(\leftarrow, \rightarrow)$.

Theorem:[Jul69] Every scattered linear ordering $\mathcal{L}$ has a unique signature σ and it is extendible iff for no i we have

$$\text{either } \sigma(i) = \sigma(i+1) = \mathbf{1},$$
$$\text{or } \sigma(i) = \rightarrow \text{ and } \sigma(i+1) = \leftarrow.$$

Proving that every scattered linear ordering has a signature is already too hard.

Theorem: The following are equivalent over RCA_0.

- Every scattered l.o. has a signature.
- Every scattered l.o. has a unique signature.
- Fraïssé's Conjecture.

So, in weak systems, this version of Jullien's theorem does not work as a characterization of the extendible linear orderings.

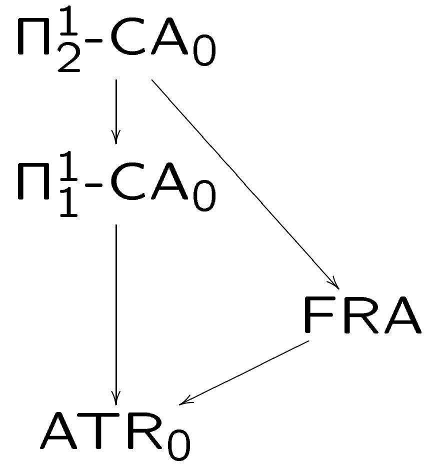
Fraïssé's Conjecture

Theorem: [Fraïsé's Conjecture '48; Laver '71]

FRA: The countable linear orderings form a WQO with respect to embeddablity.
(i.e., there are no infinite descending sequences and no infinite antichains.)

Theorem:[Shore '93]
FRA implies ATR_0 over RCA_0.

Conjecture:[Clote '90] [Simpson '99][Marcone]
FRA is equivalent to ATR_0 over RCA_0.



Another formulation of Jullien's theorem.

Definition: • Let $\mathcal{L} = \mathcal{A} + \mathcal{B} + \mathcal{C}$.
$\mathcal{B}$ is an essential segment of $\mathcal{L}$ if
whenever $\mathcal{L} \preceq \mathcal{A} + \mathcal{B}' + \mathcal{C}$, $\mathcal{B} \preceq \mathcal{B}'$.

• A linear ordering $\mathcal{B}$ is *bad* if either $\mathcal{B} = \mathbf{1} + \mathbf{1}$ or $\mathcal{B}$ has signature $(\rightarrow, \leftarrow)$.

Theorem:[Jullien '69]
JUL: $\mathcal{L}$ is extendible iff
it has no bad essential segments.

Theorem: The following statements are equivalent over $\mathsf{RCA}_0 + \Sigma^1_1$-IND.

(1) JUL
(2) FRA
(3) Every scattered l.o. has a signature.

RCA_0 alone can prove (1) $\Rightarrow$ (2) $\iff$ (3).

Definition: The class of *h-indecomposable* linear orderings is defined inductively:

- 1 is h-indecomposable and
- if $\mathcal{L}_0, \mathcal{L}_1, ...$ are h-indecomposable, so are
  - $\mathcal{L}_0 + (\mathcal{L}_0 + \mathcal{L}_1) + (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2) + \cdots$ and
  - $\cdots + (\mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0) + (\mathcal{L}_1 + \mathcal{L}_0) + \mathcal{L}_0$.

Two linear orderings are *equimorphic* if each can be embedded into the other.

Theorem:[Laver '71] Every scattered countable linear ordering is equimorphic to a finite sum of h-indecomposables.

Theorem: Laver's thm. is equivalent to FRA over RCA_0.

Statement. *JUL(w/signature):*
If $\mathcal{L}$ has a minimal decomposition $\mathcal{L} = \mathcal{F}_0 + ... + \mathcal{F}_n$, where each $\mathcal{F}_i$ is either h-indec. of 1, then
$\mathcal{L}$ is extendible iff for every $i < n$,
neither $\mathcal{F}_i = \mathcal{F}_{i+1} = 1$,
nor $(\mathcal{F}_i + \mathcal{F}_{i+1})$ is $(\rightarrow, \leftarrow)$.

Note that the original version of Jullien's thm, is equivalent to FRA together with JUL(w/signature).

Theorem: JUL(w/signature) is equivalent to ATR_0 over $\mathsf{RCA}_0 + \Sigma^1_1$-IND.

The implication $\Rightarrow$ follows from the fact that the extendibility of $\mathbb{Z}$ implies ATR_0.

Theorem: ($\mathsf{ATR}_0 + \Sigma^1_1$-IND) If $\mathcal{L}$ is as in the statement of JUL(w/signature) and $\mathcal{L} \not\preceq \mathcal{P}$, then $\mathcal{P}$ has a linearization hyperarithmetic in $\mathcal{L} \oplus \mathcal{P}$ which does not embed $\mathcal{L}$.

Use of Σ^1_1-induction.

Theorem: In ATR_0 we can prove:

- If $\mathcal{L} = \sum_{m\in\omega} \mathcal{L}_m$ and the $\mathcal{L}_m$'s are uniformly extendible, then $\mathcal{L}$ is extendible.
- If $\mathcal{A} + \mathbf{1}$ and $\mathbf{1} + \mathcal{B}$ are extendible, then $\mathcal{A} + \mathbf{1} + \mathcal{B}$ is extendible too.

We need Σ^1_1-induction to prove:

- Every h-indecomposable is extendible
- If $\mathcal{L} = \mathcal{F}_1 + \mathbf{1} + \mathcal{F}_2 + \mathbf{1} + ... + \mathbf{1} + \mathcal{F}_n$ where each $\mathcal{F}_i$ is h-indecomposable, then $\mathcal{L}$ is extendible.

Σ^1_1-induction wouldn't be enough for our proof if it wasn't for fact that we can get the linearizations to be hyperarithmetic. This allow as to simplify the complexity of the formulas we prove by induction.

We use that $\mathsf{ATR}_0+\Sigma^1_1$-IND proves that every h-indecomposable is extendible to prove:

Theorem:($\mathsf{ATR}_0+\Sigma^1_1$-IND) $\mathbb{Q}$ is extendible.

Definition: $\omega^{\mathcal{L}}$ is the linear ordering of formal sums of the form $\omega^{l_0}\cdot n_0+\omega^{l_1}\cdot n_1+...+\omega^{l_k}\cdot n_k$ where $n_i \in \mathbb{N}$ and $l_0 > l_1 > ... > l_k \in \mathcal{L}$.

Obs: (ACA_0) $\mathcal{L}$ is well ordered iff $\omega^{\mathcal{L}}$ is scattered.

Fix $\mathcal{P}$ such that $\mathbb{Q} \not\preceq \mathcal{P}$

Claim: There is an ordinal α such that $\omega^\alpha \not\preceq \mathcal{P}$.
Otherwise, we would have
$\{\mathcal{L} : \omega^{\mathcal{L}} \preceq \mathcal{P}\} = \{\mathcal{L} : \mathcal{L}$ is a well ordering$\}$.
$\nwarrow \Sigma^1_1$ class $\qquad\qquad \nwarrow$ Π^1_1-complete.

ω^α is extendible because it is h-indecomposable. Then $\mathcal{P}$ has a linearization $(P, \leq_{\mathcal{L}})$ which does not embed ω^α. But then $\mathbb{Q} \not\preceq (P, \leq_{\mathcal{L}})$.

Indecomposability.

- $\mathcal{L}$ is *scattered* if $\mathbb{Q} \not\preceq \mathcal{L}$.
- $\mathcal{L}$ is *indecomposable* if whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$, either $\mathcal{L} \preceq \mathcal{A}$ or $\mathcal{L} \preceq B$.
- $\mathcal{L}$ is *indecomposable to the right* if for every non-trivial cut $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have $\mathcal{L} \preceq \mathcal{B}$.
- $\mathcal{L}$ is *indecomposable to the left* if for every non-trivial cut $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have $\mathcal{L} \preceq \mathcal{A}$.

Theorem: *INDEC:* Every scattered indecomposable linear ordering is indecomposable either to the right or to the left.

Theorem: INDEC follows from Δ^1_1-CA_0.

Theorem: Every ω-model of RCA_0+INDEC is closed under hyperarithmetic reduction.