On the back-and-forth relation on Boolean Algebras.

Antonio Montalbán.
U. of Chicago

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Joint work with Kenneth Harris (University of Michigan).

Boolean Algebras

Definition A *Boolean algebra, BA,* is a structure $\mathcal{B} = (B, \leq, 0, 1, \vee, \wedge, \neg)$, where

- $(B, \leq)$ is a partial ordering,
- 0 is the least element and 1 the greatest,
- $x \vee y$ is the least upper bound of x and y,
- $x \wedge y$ is the greatest lower bound of x and y,
- $\neg x \vee x = 1$ and $\neg x \wedge x = 0$

Example: $(\mathcal{P}(X), \subseteq, \emptyset, X, \cup, \cap, X \setminus \cdot)$

We will only consider countable BAs and assume $B \subseteq \omega$.

A BA $\mathcal{B}$ is *X-computable* if

X can compute B and all the operations in $\mathcal{B}$.

Low Boolean Algebras

Theorem: [Downey, Jockusch 94]

Every low Boolean Algebra has a computable copy.

i.e. If X is low and $\mathcal{B}$ is X-computable, then there is a computable BA isomorphic to $\mathcal{B}$.

Theorem: [Thurber 95]

Every low_2 Boolean Algebra has a computable copy.

Theorem: [Knight, Stob 00]

Every low_4 Boolean Algebra has a computable copy.

Open Question:

Does every low_n Boolean Algebra have a computable copy?

Boolean Algebra Predicates

- 1-predicates
  - atom(x)
- 2-predicates
  - atomless(x)
  - infinite(x)
- 3-predicates
  - atomic(x)
  - 1-atom(x)
  - atominf(x)
- 4-predicates
  - $\sim$-inf(x)
  - $I(\omega+\eta)(x)$
  - infatomicless(x)
  - 1-atomless(x)
  - nomaxatomless(x)

n-predicates have n alternations of quantifiers

Definition

For $n = 0, 1, 2, 3, 4$, a BA $\mathcal{B}$ is *n-approximable* if
$0^{(n)}$ can compute $\mathcal{B}$ and all its m-predicates for $m \leq n$.

Note: $\mathcal{B}$ is 0-approximable $\iff$ $\mathcal{B}$ is computable.

Note: $\mathcal{B}$ is low_n $\implies$ $\mathcal{B}$ is n-approximable.

Lemma:[Downey, Jockusch 94; Thurber 95; Knight, Stob 00]
For $n = 0, 1, 2, 3$,
every $(n+1)$-approximable BA has an n-approximable copy.

So:
$\mathcal{B}$ low_4 $\implies$ 4-approx $\implies$ 3-approx copy $\implies$ 2-approx copy $\implies$ 1-approx copy $\implies$ 0-approx copy $\implies$ computable copy.

General Idea

$\mathcal{A}$ and $\mathcal{B}$ are *n*-equivalent iff $0^{(n)}$ cannot distinguish them.

Def:
Let $\mathcal{A} \leq_n \mathcal{B} \iff$ given $\mathcal{C}$ that's isomorphic to either $\mathcal{A}$ or $\mathcal{B}$, deciding whether $\mathcal{C} \cong \mathcal{A}$ is $\mathbf{\Sigma}^0_n$-hard.

We will write $\mathcal{A} \equiv_n \mathcal{B}$ iff both $\mathcal{A} \leq_n \mathcal{B}$ and $\mathcal{B} \leq_n \mathcal{A}$.

Back-and-Forth relations

Notation: $a_1, ..., a_k$ is a partition of a BA $\mathcal{B}$ if

$$a_0 \vee ... \vee a_k = 1 \text{ and } \forall i \neq j \ (a_i \wedge a_j = 0).$$

We write $\mathcal{B} \restriction a$ for the BA whose domain is $\{x \in \mathcal{B} : x \leq a\}$.

Theorem[Ash, Knight] TFAE

1. $\mathcal{A} \leq_n \mathcal{B}$.
2. All the infinitary Σ_n sentences true in $\mathcal{B}$ are true in $\mathcal{A}$.
3. for every partition $(b_i)_{i \leq k}$ of $\mathcal{B}$,

   there is a partition $(a_i)_{i \leq k}$ of $\mathcal{A}$ such that $\forall i \leq k$

   $$\mathcal{B} \restriction b_i \leq_{n-1} \mathcal{A} \restriction a_i \ .$$

The bf-types

Obs: $\equiv_n$ is an equivalence relation on the class of BAs.

We call the equivalence classes *n-bf-types*.

We study the following family of *ordered monoids*

$$(BAs/\equiv_n\ ,\ \leq_n\ ,\ \oplus)$$

where $\mathcal{A} \oplus \mathcal{B}$ is the product BA with coordinatewise operations,
together with the projections $(\cdot)_{n-1} : BAs/\equiv_n \rightarrow BAs/\equiv_{n-1}$.

The invariants

For each n we define a set $\mathbf{INV}_n$ of finite objects, and an invariant map $T_n\colon BAs \to \mathbf{INV}_n$ such that

$$\mathcal{A} \equiv_n \mathcal{B} \quad\Longleftrightarrow\quad T_n(\mathcal{A}) = T_n(\mathcal{B})$$

Moreover, on $\mathbf{INV}_n$ we define $\leq_n$ and $+$ so that

$$(BAs/\equiv_n, \leq_n, \oplus) \cong (\mathbf{INV}_n, \leq_n, +),$$

Indecomposable Boolean Algebras

Definition

A BA $\mathcal{A}$ is *n-indecomposable* if for every partition $a_1, ..., a_k$ of $\mathcal{A}$, there is an $i \leq k$ such that $\mathcal{A} \equiv_n \mathcal{A} \restriction a_i$.

Theorem

1. *Every BA is a finite product of n-indecomposable BAs.*
2. *There are finitely many $\equiv_n$-equivalence classes among the n-indecomposable BAs.*

Let $\mathbf{BF}_n = \{T_n(\mathcal{B}) : \mathcal{B} \text{ is } n\text{-indecomposable}\} \subset \mathbf{INV}_n$.

$\mathbf{BF}_n$ is a finite generator of $(\mathbf{INV}_n, \leq_n, +)$.

n	1	2	3	4	5	6	...
$\lvert\mathbf{BF}_n\rvert$	2	3	5	9	27	1578	...

Definition

For each $\alpha \in \mathbf{BF}_n$ we define a relation $\mathrm{R}_\alpha(\cdot)$ on $\mathcal{B}$:

$$\mathrm{R}_\alpha(x) \iff T_n(\mathcal{B} \restriction x) \geq_n \alpha.$$

Observation For $n = 0, 1, 2, 3, 4$, the $(\leq n)$-predicates are boolean combinations of the R_α for $\alpha \in \mathbf{BF}_{\leq n}$, and vice versa.

Lemma

The relations R_α for $\alpha \in \mathbf{BF}_n$ can be defined by computable infinitary Π_n formulas of BAs.

bf-relations for 1- and 2-indecomposable bf-types

$$\begin{array}{cccc} \underline{\text{projection}} & \underline{a_0} & \underline{b_0} & \underline{b_1} \\ & & & \\ & b_0 & c_0 & c_2 \\ & | & & | \\ & b_1 & & c_1 \end{array}$$

bf-relations for 3-indecomposable bf-types

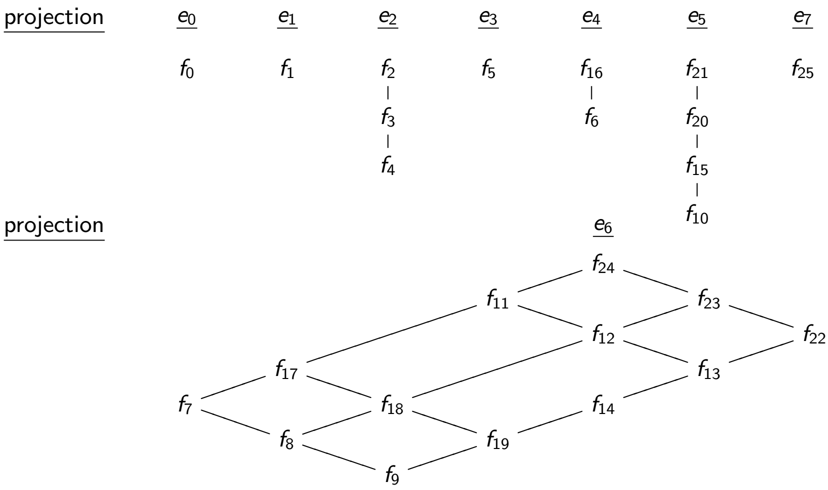
$$\begin{array}{cccc} \underline{\text{projection}} & \underline{c_0} & \underline{c_1} & \underline{c_2} \\ & & & \\ & d_0 & d_1 & d_4 \\ & & | & \\ & & d_2 & \\ & & | & \\ & & d_3 & \end{array}$$

bf-relations for 4-indecomposable bf-types

projection	d_0	d_1	d_2	d_3	d_4
	e_0	e_1	e_3	e_7	e_8
			$\vert$	$\vert$	
			e_2	e_4	
				$\vert$	
				e_5	
				$\vert$	
				e_6	

Picture - Level 5

bf-relations for 5-indecomposable bf-types



Quantifier Elimination.

Theorem

Every infinitary Σ_{n+1} formula is equivalent to an infinitary Σ_1 formula over the predicates R_α for $\alpha \in \mathbf{BF}_n$.

Quantifier Elimination.

Notation: Given $\bar{\alpha} = \langle \alpha_1, ..., \alpha_m \rangle$ and $\bar{\beta} = \langle \beta_1, ..., \beta_k \rangle \in \mathbf{BF}_n^{<\omega}$ let

$$\begin{aligned} \mathrm{R}_{\bar{\alpha},\bar{\beta}}(x) \iff & \exists y_1 \dot\vee \ldots \dot\vee y_m = x \left(\mathrm{R}_{\alpha_1}(y_1) \,\&\ldots\&\, \mathrm{R}_{\alpha_m}(y_m) \right) \,\& \\ & \exists z_1 \dot\vee \ldots \dot\vee z_k = \neg x \left(\mathrm{R}_{\beta_1}(z_1) \,\&\ldots\&\, \mathrm{R}_{\beta_k}(z_k) \right) \end{aligned}$$

where $\exists y_1 \dot\vee \ldots \dot\vee y_m = x$ is short for
"there is a partition $y_1, ..., y_m$ of x such that..."

Theorem

Let $\mathcal{B}$ be a BA, and $R \subseteq B$. TFAE

1. *If $\mathcal{A} \cong \mathcal{B}$ and $(\mathcal{A}, Q) \cong (\mathcal{B}, R)$ then Q is $\Sigma^{0,\mathcal{A}}_{n+1}$.*
2. *R can be defined in $\mathcal{B}$ by a comp infinitary Σ^c_{n+1} formula.*
3. *There is a $0^{(n)}$-comp seq $\{(\bar{\alpha}_i, \bar{\beta}_i)\}_{i\in\omega} \subseteq \mathbf{BF}_n^{<\omega}$ such that*
$$x \in R \iff \bigvee_{i\in\omega} \mathrm{R}_{\bar{\alpha}_i,\bar{\beta}_i}(x)$$

The equivalence between (1) and (2) is due to Ash, Knight, Manasse, Slaman; Chisholm.

n-approximable Boolean Algebras

Theorem

Let $\mathcal{B}$ be a presentation of a Boolean algebra. TFAE.

1. *The Σ^c_{n+1}-diagram of $\mathcal{B}$ is Σ^0_{n+1};*
2. *The relations $\mathrm{R}_\alpha(\mathcal{B})$ for $\alpha \in \mathbf{BF}_n$ are computable in $0^{(n)}$.*

Definition

If a BA satisfies these conditions, we say it's *n-approximable*.

Question: Does every $n+1$-approximable BA have an n-approximable copy?

Difficulties at level 5

Definition

$\alpha \in \mathbf{BF}_n$ is a *isomorphism type* if
whenever $T_n(\mathcal{A}) = T_n(\mathcal{B}) = \alpha$, $\mathcal{A} \cong \mathcal{B}$.

$\alpha \in \mathbf{BF}_n$ is an *exclusive type* if whenever $T_n(\mathcal{A}) = \alpha$ and $a \in \mathcal{A}$
either $\mathcal{A} \restriction a \equiv_n \mathcal{A}$ or $\mathcal{A} \restriction (\neg a) \equiv_n \mathcal{A}$, but not both.

Observation: For $n \leq 4$, and $\alpha \in \mathbf{BF}_n$,
α is an exclusive type $\implies$ α is an isomorphism type.

This is not true for $n = 5$.

bf-relations for 1- and 2-indecomposable bf-types

$$\begin{array}{cccc} \underline{\text{projection}} & \underline{a_0} & \underline{b_0} & \underline{b_1} \\ & b_0 & c_0 & c_2 \\ & | & & | \\ & b_1 & & c_1 \end{array}$$

1-indecomposable bf-types

Name	R_u	Example
b_0	atom	atom
b_1	non-zero	infinite

2-indecomposable bf-types

Name	$(\cdot)_1$	R_u	Example
c_0	b_0	atom	atom
c_1	b_1	infinite	inf-atoms
c_2	b_1	atomless	atomless

Picture - Level 3

bf-relations for 3-indecomposable bf-types

$$\begin{array}{cccc} \underline{\text{projection}} & \underline{c_0} & \underline{c_1} & \underline{c_2} \\ & & & \\ & d_0 & d_1 & d_4 \\ & & | & \\ & & d_2 & \\ & & | & \\ & & d_3 & \end{array}$$

Name	$(\cdot)_2$	R_u	Example
d_0	c_0	atom	atom
d_1	c_1	1-atom	1-atom
d_2	c_1	atomic & infinite	2-atom, 1-atomless
d_3	c_1	atominf	$Int(\omega + \eta)$
d_4	c_2	atomless	atomless