Embeddability and Decidability in the Turing Degrees

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Jump upper semilattice embeddings
- Background
- JUSL Embeddings
- Other Embeddability results

Local Structures
- High/Low Hierarchy
- Ordering of the classes
- Fragments of the theory
Given sets $A, B \subseteq \mathbb{N}$ we say that $A$ is computable in $B$, and we write $A \leq_T B$, if there is a computable procedure that can tell whether an element is in $A$ or not using $B$ as an oracle. (Note: Instead of $\mathbb{N}$ we could’ve chosen $2^{<\omega}$, $\omega^{<\omega}$, or $V(\omega),...$)

This defines a quasi-ordering on $\mathcal{P}(\mathbb{N})$.

We say that $A$ is Turing equivalent to $B$, and we write $A \equiv_T B$ if $A \leq_T B$ and $B \leq_T A$.

[Kleene Post 54] We let $\mathcal{D} = (\mathcal{P}(\mathcal{D})/ \equiv_T)$, and $\mathcal{D} = (\mathcal{D}, \leq_T)$.

**Question**: How does $\mathcal{D}$ look like?
Some simple observations about $\mathcal{D}$

- There is a least degree $0$. The degree of the computable sets.

- $\mathcal{D}$ has the *countable predecessor property*, i.e., every element has at countably many elements below it. Because there are countably many programs one can write.

- Each Turing degree contains countably many sets.

- So, $\mathcal{D}$ has size $2^{\aleph_0}$. Because $\mathcal{P}(\mathbb{N})$ has size $2^{\aleph_0}$, and each equivalence class is countable.
Operations on $\mathcal{D}$

**Turing Join**
Every pair of elements $a, b$ of $\mathcal{D}$ has a least upper bound (or *join*), that we denote by $a \cup b$. So, $\mathcal{D}$ is an upper semilattice.

Given $A, B \subseteq \mathbb{N}$, we let $A \oplus B = \{2n : n \in A\} \cup \{2n + 1 : n \in B\}$. Clearly $A \leq_T A \oplus B$ and $B \leq_T A \oplus B$, and if both $A \leq_T C$ and $B \leq_T C$ then $A \oplus B \leq_T C$.

**Turing Jump**
Given $A \subseteq \mathbb{N}$, we let $A'$ be the *Turing jump of $A$*, that is, $A' = \{\text{programs that HALT, when run with oracle $A$}\}$. For $a \in \mathcal{D}$, let $a'$ be the degree of the Turing jump of any set in $a$

- $a <_T a'$
- If $a \leq_T b$ then $a' \leq_T b'$.
A jump upper semilattice (JUSL) is structure \((A, \leq, \lor, j)\) such that

1. \((A, \leq)\) is a partial ordering.
2. For every \(x, y \in A\), \(x \lor y\) is the l.u.b. of \(x\) and \(y\),
3. \(x < j(x)\), and
4. if \(x \leq y\), then \(j(x) \leq j(y)\).

\[ \mathcal{D} = (D, \leq_T, \lor, ') \text{ is a JUSL.} \]
Questions one may ask

- Are there incomparable degrees?  
  YES

- Are there infinitely many degrees such that none of them can be computed from all the other ones together?  
  YES

- What about $\aleph_1$ many?  
  YES

- Is there a descending sequence of degrees $a_0 \geq_T a_1 \geq_T \ldots$?  
  YES

- Could we also get such a sequence with $a'_{n+1} = a_n$?  
  YES

A more general question:
Which structures can be embedded into $\mathcal{D}$?
Theorem: The following structures can be embedded into the Turing degrees.

- Every countable upper semilattice. [Kleene, Post '54]
- Every partial ordering of size $\aleph_1$ with the countable predecessor property (c.p.p.). [Sacks '61]
  (It’s open whether this is true for size $2^{\aleph_0}$.)
- Every upper semilattice of size $\aleph_1$ with the c.p.p. Moreover, the embedding can be onto an initial segment. [Abraham, Shore '86]
- Every ctble. jump partial ordering $(A, \leq, \lor, \prime)$. [Hinman, Slaman '91]

Theorem (M.)

Every ctble. jump upper semilattice $(A, \leq, \lor, \prime)$ is embeddable in $\mathcal{D}$. 
Idea of the proof

**Definition:** A JUSL \( \mathcal{J} \) is \textit{h-embeddable} if there is a map
\( H: J \to \mathcal{P}(\mathbb{N}) \) s.t., for all \( x, y \in P \),
- if \( x <_{\mathcal{J}} y \) then \( H(x)' \leq_T H(y) \).
- uniformity condition: \( \mathcal{J} \leq_T H(y) \), and \( \bigoplus_{x \leq_{\mathcal{J}} y} H(x) \leq_T H(y) \);

**Obs:** Every well-founded JUSL is h-embeddable, by taking \( x \mapsto 0^{rk(x)} \).

**Theorem**

\textit{Every ctle JUSL which is h-embeddable, is embeddable into} \( \mathcal{D} \).

**Proof:** Forcing Construction.

**Lemma**

\textit{Every ctle JUSL embeds into one which is h-embeddable.}

**Proof:** Uses Fraïssé limits and non-standard ordinals.
Emedabbility results are usually related to the decidability of existential theories.

**Corollary**

\[ \exists - \text{Th}(D, \leq_T, \lor, ') \text{ is decidable.} \]

**Note:** \( \exists - \text{Th}(D, \leq_T, \lor, ') \) is the set of existential formulas, in the language of JUSL, true about \( D \)

**Proof:** An \( \exists \)-formula about \( (D, \leq_T, \lor, ') \) is true iff it does not contradict the axioms of jump upper semilattice.
History of Decidability Results.

- $\text{Th}(D, \leq_T)$ is undecidable. \cite{Lachlan '68}
- $\exists - \text{Th}(D, \leq_T)$ is decidable. \cite{Kleene, Post '54}

**Question:** Which fragments of $\text{Th}(D, \leq_T, \lor, \forall)$ are decidable?

- $\exists \forall \exists - \text{Th}(D, \leq_T)$ is undecidable. \cite{Shmerl}
- $\forall \exists - \text{Th}(D, \leq_T, \lor)$ is decidable. \cite{Jockusch, Slaman '93}
- $\exists - \text{Th}(D, \leq_T, \forall)$ is decidable. \cite{Hinman, Slaman '91}
- $\exists - \text{Th}(D, \leq_T, \lor, \forall)$ is decidable. \cite{M. 03}
- $\forall \exists - \text{Th}(D, \leq_T, \lor, \forall)$ is undecidable. \cite{Slaman, Shore '05}.
- $\exists - \text{Th}(D, \leq_T, \lor, \forall, 0)$ is decidable. \cite{Lerman, in preparation}

**Question:** Is $\forall \exists - \text{Th}(D, \leq_T, \forall)$ decidable?
Definition: A *jump upper semilattice with 0* (JUSL w/0) is a structure $\mathcal{J} = \langle J, \leq, \cup, j, 0 \rangle$ such that

- $\langle J, \leq, \cup, j \rangle$ is a JUSL, and
- $0$ is the least element of $\langle J, \leq \rangle$.

Q: Which JUSL w/0 can be embedded into $\mathcal{D}$?

Q: What about among the ones which have only finitely many generators?
Theorem (M. 03)

Not every JUSLw/0 even with one generator is embeddable in D.

Proof: There are \(2^{\aleph_0}\) JUSLw/0 with a generator \(x\) satisfying \(x \leq 0''\).

Theorem (Hinman, Slaman 91; M.03)

Every JPOw/0 with one generator is realized in D.

Question: What about JPOw/0 and with two generators?
Let $\kappa$ be a cardinal, $\aleph_0 < \kappa \leq 2^{\aleph_0}$.

**Q:** Is every JUSL with the c.p.p. and size $\kappa$ embeddable in $\mathcal{D}$?

**Proposition**

If $\kappa = 2^{\aleph_0}$, then the answer is **NO**.

**Proposition**

If Martin’s axiom holds at $\kappa$, the answer is **YES**.

**Corollary**

For $\kappa = \aleph_1$, it is independent of ZFC.

**Proof:** It is FALSE under CH, but TRUE under MA($\aleph_1$).
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2 Local Structures
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Limit lemma: Let $\mathcal{A} \subseteq \mathbb{N}$. The following are equivalent.

- $A \leq_T 0'$,
- $A$ is $\Delta^0_2 = \Sigma^0_2 \cap \Pi^0_2$,
- there is a computable func. $f : \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ such that $\forall n$
  - $n \in A \iff \lim_{s \to \infty} f(n, s) = 1 \iff (\exists m)(\forall s > m) \ f(n, s) = 1$
  - $n \notin A \iff \lim_{s \to \infty} f(n, s) = 0 \iff (\exists m)(\forall s > m) \ f(n, s) = 0$

Notation: $D(\leq 0') = \{x \in D : x \leq_T 0'\}$. 

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Order-theoretic Properties of $0'$

There is a history of results showing that $D(\leq 0')$ has special properties. To cite a few:

- Every ctable poset can be embedded below $0'$ [Kleene-Post ’54].
- There are minimal degrees below $0'$ [Sacks 61].
- Every degree below $0'$ joins up to $0'$ [Robison, Posner 72, 81]
- There are 1-generic degrees below $0'$

\[
(\forall b \leq_T 0')(\exists c <_T 0') 0' = b \lor c
\]

What is the relation between the computational complexity of a
High/Low Hierarchy.

**Definition:** A Turing degree $a \leq_T 0'$ is

- **low** if $a' = 0'$.
- **high** if $a' = 0''$.

**Definition [Soare ’74][Cooper ’74]** A Turing degree $a \leq_T 0'$ is

- **low$_n$** ($L_n$) if $a^{(n)} = 0^{(n)}$.
- **high$_n$** ($H_n$) if $a^{(n)} = 0^{(n+1)}$.
- **intermediate** ($I$) if $\forall n \ (0^{(n)} \leq_T a^{(n)} \leq_T 0^{(n+1)})$. 
Properties of $\mathbf{D}(\leq a)$

- Any ctbly poset embeds below any $a \not\in L_2$. [Jockusch-Posner 78]
- There are minimal degrees below $a \in H_1$. [Cooper 73]
- Every degree below $a \in H_1$ joins up to $a$. [Posner 77]
- There are 1-generic degrees below $a \not\in L_2$. [Jockusch-Posner 78]
定义：[Jockusch, Posner ’78] 一个图灵度 a
- 是 generalized low_n (GL_n) 如果 a^{(n)} = (a ∪ 0')(n−1).
- 是 generalized high_n (GH_n) 如果 a^{(n)} = (a ∪ 0')(n).
- 是 generalized intermediate (GI) 如果
  ∀n ((a ∪ 0')(n−1) <_T a^{(n)} <_T (a ∪ 0')(n)).

这个层次结构与位于 0' 之下的高/低层次结构一致。

问题：是否它实际上按其复杂性分类这些度？
Properties of $\mathbf{D}(\leq a)$

- Any ctbble poset embeds below any non-$GL_2$. \[JP 78\]
- There are minimal degrees below any $a \in GH_1$. \[Jockusch 77\]
- Every degree below $a \in GH_1$ joins up to $a$. \[Posner 77\]
- There are 1-generic deg. below any $a \notin GL_2$. \[JP 78\]
Definition: We say that a degree \( a \) has the **complementation property** if\( \forall b \leq_T a \exists c \leq_T a \) \( b \lor c = a \) \& \( b \land c = 0 \).

Theorem: \( 0' \) has the complementation property. History:

- Every \( b \in L_2 \) has a complement below \( 0' \). \[\text{[Robinson 72]}\]
- Every \( b \in H_1 \) has a complement below \( 0' \). \[\text{[Posner 77]}\]
- Every c.e. degree \( 0 \) has a complement below \( 0' \). \[\text{[Epstein 75]}\]
- Every \( b \not\in L_2 \) has a complement below \( 0' \). \[\text{[Posner 81]}\]
- The complement can be found uniformly, and can be choosen to be a 1-generic degree. \[\text{[Slaman-Steel 89]}\]
- The complement can be choosen a minimal degree. \[\text{[Lewis 03]}\]

Q: Does every \( GH_1 \) have the complementation property? \[\text{[Posner 81]}\]
- Yes, it does. \[\text{[Greenberg-M.-Shore 04]}\]

Q: Can the complement be found uniformly?
Ordering of the High/Low Hierarchy

Definition:
- $L_n^* = L_n \setminus L_{n-1}$
- $L_1^* = L_1$
- $H_n^* = H_n \setminus H_{n-1}$
- $H_1^* = H_1$
- $I^* = I$

This induces a partition of $\mathcal{D}(\leq_T 0')$:

$$C^* = \{L_1^*, L_2^*, \ldots\} \cup \{I^*\} \cup \{H_1^*, H_2^*, \ldots\}.$$ 

On $C^*$ we define a linear ordering:

$$L_1^* < L_2^* < \cdots < I^* < \cdots < H_2^* < H_1^*.$$ 

Observation: For all $x \in X \in C^*$ and $y \in Y \in C^*$

$$x \leq_T y \Rightarrow X \preceq Y. \quad (*)$$

Theorem:[Lerman ’85] Every finite partial ordering labeled with elements of $C^*$ satisfying ($*$) can be embedded into $\mathcal{D}(\leq_T 0')$

(of course, preserving labels).

Corollary:[Lerman ’85]

$$\exists \neg \text{Th}(\mathcal{D}(\leq_T 0'), \leq_T, 0, 0', L_1, L_2, \ldots, I, \ldots, H_1)$$ is decidable.
Non-ordering of the Generalized High/Low Hierarchy.

**Question:** [Lerman '85]
Can this be proved for the generalized high/low hierarchy?

The generalized high/low hierarchy induces a partition of $D$:
$$G^* = \{GL_1^*, GL_2^*, \ldots\} \cup \{GI^*\} \cup \{GH_1^*, GH_2^*, \ldots\}.$$  

**Theorem (M.)**

> Every finite partial ordering labeled with elements of $G^*$ can be embedded into $D$.

Note that there is no restriction at all on the labels.

**Corollary**

$$\exists \rightarrow Th(D, \leq_T, 0, GL_1, GL_2, \ldots, GI, \ldots, GH_1) \text{ is decidable.}$$
Idea of the proof

Lemma (M.)
There exists sets $e_i$ and $x_i$ as in the picture.

Lerman’s bounding lemma:
Given $x \leq_T y$, $x \in \text{GL}_1$, $y \in \text{GH}_1$, and $X \in \mathcal{G}^*$, there exists $z \in X$ with $x \leq_T z \leq_T y$.

$y \in \text{GH}_1$

$z \in X$

$x \in \text{GL}_1$
Complexity of $\text{Th}(D(\leq a'), \leq)$.

**Question:**
How does the complexity of $a$ relate to the complexity of $\text{Th}(D(\leq a'), \leq)$?
Complexities of the Theories

Obs: $\text{Th}(D, \leq_T) \leq_1 \text{Th}^2(N, +, \times)$.

**Theorem:** [Simpson 77]
$\text{Th}(D, \leq_T) \equiv_1 \text{Th}^2(N, +, \times)$.

Obs: $\text{Th}(D(\leq 0'), \leq_T) \leq_1 \text{Th}(N, +, \times) \equiv_1 0^{(\omega)}$.

**Theorem:** [Shore 81]
$\text{Th}(D(\leq 0'), \leq_T) \equiv_1 \text{Th}(N, +, \times) \equiv_1 0^{(\omega)}$.

**Theorem:** [Harrington, Slaman, Woodin]
$\text{Th}(R, \leq_T) \equiv_1 \text{Th}(N, +, \times) \equiv_1 0^{(\omega)}$. 
Upper bound of $Th(D(\leq a'), \leq_T)$

- $Th(D(\leq a), \leq_T) \leq_1 a^{(\omega)}$.

- $(D(\leq a), \leq)$ has a presentation $\Sigma_3^0(a)$

**Theorem:** [Lachlan - Lerman - Abraham, Shore]
Every countable upper semilattice can be embedded as an initial segment of $D$.

- there are degrees $a$ such that $Th(D(\leq a), \leq_T)$ is decidable. (Lerman’s method only produces $L_2$ such degrees.)

- there are degrees $a$ such that $Th(D(\leq a), \leq_T) \geq_1 0^{(\omega)}$
Local Theories

**Theorem:** [Shore 81] $\text{Th}(D(\leq a), \leq_T) \geq_1 0^{(\omega)}$ whenever $a$ is either

- $\geq 0'$,
- computable enumerable,
- or high.

**Proof:**

- Find a way of defining models of arithmetic embedded in $D(\leq a)$ using only finitely many parameters.
- Find a way to recognize when the finitely many parameters are coding the standard model of arithmetic.
- Translate formulas.
Local theory below a 1-generic

**Theorem**

[Greenberg, M.] \( \text{Th}(\mathcal{D}(\leq a), \leq_T) \geq_1 0^{(\omega)} \) whenever \( a \) is either

- 1-generic and \( \leq 0' \),
- 2-generic,
- \( n \)-REA

Recall that a set \( G \in 2^\mathbb{N} \) is 1-generic if for every \( \Sigma_1^0 \) formula \( \varphi \),

\[ \exists p \subset G (p \models \varphi) \lor (p \models \neg \varphi). \]
Let $\mathcal{J}$ be an antichain of Turing degrees. There are degrees $c$, $g_0$, and $g_1$ such that
the elements of $\mathcal{J}$ are the minimal solutions below $c$ of the following inequality in $x$:
\[(g_0 \lor x) \cap (g_1 \lor x) \neq x.\]
Moreover, this degrees $c$, $g_0$, and $g_1$ can be found below any 2-generic over $\mathcal{J}$.
[Odifreddi, Shore 91] The can also be found below $0'$ if $\mathcal{J} \subseteq \mathcal{D}(\leq 0')$.

**Lemma (Greenberg, M.)**

1-genericity is enough to find the parameters $c$, $g_0$, and $g_1$. 