The jump of a structure.

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Basic definitions in computability

For $A, B \subseteq \mathbb{N}$, A is *B-computable* ($A \leq_T B$) if
there is a computable procedure that answers "$n \in A$?",
using B as an *oracle*.

We impose no restriction on time or space.

A is *Turing-equivalent* to B ($A \equiv_T B$) if $A \leq_T B$ and $B \leq_T A$.

Def: A is *B-computably enumerable* (*B-c.e.*) if
there is a B-computable procedure that lists the elements of A.

Obs: A is B-computable $\iff$ A and $(\mathbb{N} \setminus A)$ are both B-c.e.

The join

For $A, B \subseteq \mathbb{N}$, let $A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\}$.

For $A_0, A_1, A_2, ... \subseteq \mathbb{N}$, let $\bigoplus_{n \in \mathbb{N}} A_n = \{\langle n, i \rangle : n \in \mathbb{N}, i \in A_n\} \subseteq \mathbb{N}^2$.

Via an effective bijection $\mathbb{N} \leftrightarrow \mathbb{N}^2$, we view $\bigoplus_n A_n$ as $\subseteq \mathbb{N}$.

The Turing Jump

We define the *jump* of a set A:

$A' = \{\ulcorner p \urcorner : p$ is a program that halts with oracle $A\}$

$\equiv_T \{\ulcorner \varphi \urcorner : \varphi$ is a quantifier-free formula s.t. $(\mathbb{N}, A) \models \exists x \varphi(x)\}$

$\equiv_T \bigoplus_e W_e^A$. (where $W_0^A, W_1^A, \ldots$ is an effective list of al A-c.e. sets)

A' is *A-c.e.-complete*.

Properties: For all $A \subseteq \mathbb{N}$,

- $A \leq_T B$ then $A' \leq_T B'$,
- $A \leq_T A'$, but $A' \not\equiv_T A$.

Thm: (Jump inversion theorem, [Friedberg 57])
If $A \geq_T 0'$, then there exists B such that $B' \equiv_T A$.

Computable Mathematics

Study

1. how effective are constructions in mathematics?
2. how complex is it to represent mathematical structures?
3. how complex are the relations within a structure?

Various areas have been studied,

1. Combinatorics,
2. Algebra,
3. Analysis,
4. Model Theory

In many cases one needs to develop a better understanding of the mathematical structures to be able to get the computable analysis.

Example: effectiveness of constructions.

Theorem: Every $\mathbb{Q}$-vector space has a basis.

Note: A countable $\mathbb{Q}$-vector space $\mathcal{V} = (V, 0, +_v, \cdot_v)$ can be encoded by three sets: $V \subseteq \mathbb{N}$, $+_v \subseteq \mathbb{N}^3$ and $\cdot_v \subseteq \mathbb{Q} \times \mathbb{N}^2$.

We say that $\mathcal{V}$ is *computable* if V, $+_v$ and $\cdot_v$ are computable.

Theorem:
Not every computable $\mathbb{Q}$-vector space has a computable basis.
However, basis can be found computable in $\mathbf{0}'$.
Moreover, $\exists$ comp. vector sp., all whose basis compute $\mathbf{0}'$.

Representing Structures

Def: By *structure* we mean a tuple $\mathcal{A} = (A; P_0, P_1, ..., f_0, f_1, ..)$
where $P_i \subseteq A^{n_i}$, and $f_i \colon A^{m_i} \to A$.
The arity functions n_i and m_i are always computable.
We will code the functions as relations, so $\mathcal{A} = (A; P_0, P_1, ..., ...)$.
An isomorphic copy of $\mathcal{A}$ where $A \subseteq \mathbb{N}$ is called a *presentation* of $\mathcal{A}$.

Def: The presentation $\mathcal{A}$ is *X-computable* if
A and $\bigoplus_i P_i$ are X-computable.

Def: X is *computable in* the presentation $\mathcal{A}$ if $X \leq A \oplus \bigoplus_i P_i$.

Def: The *spectrum* of the isomorphism type of $\mathcal{A}$:
$$Sp(\mathcal{A}) = \{X \subseteq \mathbb{N} : X \text{ computes a copy of } \mathcal{A}\}.$$

R.I.C.E. Relations

Let $\mathcal{A}$ be a structure.

Def: $R \subseteq A^n$ is *r.i.c.e. (relatively intrinsically computably enumerable)* if for every presentation $(\mathcal{B}, R^{\mathcal{B}})$ of $(\mathcal{A}, R)$, $R^{\mathcal{B}}$ is c.e. in $\mathcal{B}$.

Example: Let $\mathcal{L}$ is a linear ordering. Then $\neg succ = \{(x, y) \in L^2 : \exists z(x < z < y)\}$ is r.i.c.e.

Example: Let $\mathcal{V}$ be a vector space.
Then $LD_3 = \{(u, v, w) \in V^3 : u,\ v \text{ and } w \text{ are not L.I.}\}$ is r.i.c.e.

Def: $R \subseteq A^n$ is *r.i.computable (relatively intrinsically computable)* if R and $(A^n \setminus R)$ are both r.i.c.e.

R.I.C.E. – a frequently re-discovered concept

Thm: [Ash, Knight, Manasse, Slaman; Chishholm][Vaĭtsenavichyus][Gordon]
$R \subseteq A^n$. The following are equivalent:

- R is r.i.c.e.
- R is defined by a c.e. disjunction of $\exists$-formulas. (à la Ash)
- R is defined by an $\exists$-formula in $\mathbb{HF}(\mathcal{A})$. (à la Ershov)
  ($\mathbb{HF}(\mathcal{A})$ is the hereditarily finite extension of $\mathcal{A}$)
- R is semi-search computable. (à la Moschovakis).

r.i.c.e. relations on $\mathcal{A}$ are the analog of c.e. subsets of $\mathbb{N}$.

We now want a *complete* r.i.c.e. relation.

Sequences of relations

We consider infinite sequences of relations $\vec{R} = (R_0, R_1, ...)$,
(where $R_i \subseteq A^{a_i}$, and the arity function is always primitive computable)

Def: $\vec{R}$ is *r.i.c.e.* in $\mathcal{A}$ if
for every presentation $(\mathcal{B}, \vec{R}^{\mathcal{B}})$ of $(\mathcal{A}, \vec{R})$, $\vec{R}^{\mathcal{B}}$ is uniformly c.e. in $\mathcal{B}$.

Example: Let $\mathcal{V}$ be a $\mathbb{Q}$-vector space. Then $\vec{LD} = (LD_1, LD_2, ...)$, given by $LD_i = \{(v_1, ..., v_i) : v_1, ..., v_i \text{ are lin. dependent}\}$, is r.i.c.e.

Example: Given $X \subseteq \mathbb{N}$, let $\vec{X} = (X_0, X_1, ..)$ where $X_i = \begin{cases} A & \text{if } i \in X \\ \emptyset & \text{if } i \notin X \end{cases}$

Then, if X is c.e. $\implies$ $\vec{X}$ is r.i.c.e. in $\mathcal{A}$.
Example: In particular $\overrightarrow{0'}$ is r.i.c.e. in $\mathcal{A}$.

The upper-semi lattice of sequences of relations – à la Soskov's structure-degrees

Let $\vec{R}$ and $\vec{Q}$ be sequences of relations in $\mathcal{A}$.

Def: Let $\vec{R} \leq_s^{\mathcal{A}} \vec{Q} \iff \vec{R}$ is r.i.computable in $(\mathcal{A}, \vec{Q})$.

Def: Let $\vec{R} \oplus \vec{Q}$ be the sequence $(R_0, Q_0, R_1, Q_1, ...)$.

Recall: Given $X \subseteq \mathbb{N}$, let $\vec{X} = (X_0, X_1, ..)$ where $X_i = \begin{cases} A & \text{if } i \in X \\ \emptyset & \text{if } i \notin X \end{cases}$

Obs: $X \leq_T Y \implies \vec{X} \leq_s^{\mathcal{A}} \vec{Y}$.

The jump of a relation

Let $\varphi_0, \varphi_1, \ldots$ be an effective listing of
all c.e.-disjunctions of $\exists$-formulas about $\mathcal{A}$.

Definition

Let $\vec{K}^{\mathcal{A}} = (K_0, K_1, \ldots)$ be such that $\mathcal{A} \models \bar{x} \in K_i \iff \varphi_i(\bar{x})$.

Obs: $\vec{K}^{\mathcal{A}}$ is *complete among r.i.c.e. sequences* of relations in $\mathcal{A}$.
I.e. If $\vec{Q}$ is r.i.c.e., there is $\bar{a} \in A^{<\omega}$ and a computable $f\colon \mathbb{N} \to \mathbb{N}$ s.t.
$$\forall \bar{b} \forall i \; (\bar{b} \in Q_i \iff (\bar{a}, \bar{b}) \in K_{f(i)})$$

Definition

Given $\vec{Q}$, let $\vec{Q}'^{\mathcal{A}}$ be $\vec{K}^{(\mathcal{A}, \vec{Q})}$.

Note: $\vec{K}^{\mathcal{A}} = \emptyset'^{\mathcal{A}}$.
Note: We can also define $\vec{Q}''^{\mathcal{A}}$ as $\vec{K}^{(\mathcal{A}, \vec{Q}'^{\mathcal{A}})}$.

Examples of Jump of Structure

Recall: $\emptyset'^{\mathcal{A}} = \vec{K}^{\mathcal{A}} = (K_0, K_1, ...)$ where $\mathcal{A} \models \bar{x} \in K_i(\bar{x}) \iff \varphi_i(\bar{x})$.
Recall that $\overrightarrow{0'}$ is the seq. of rel. that codes $0' \subseteq \mathbb{N}$, and NOT $\emptyset'^{\mathcal{A}}$.

Ex: Let $\mathcal{A}$ be a *$\mathbb{Q}$-vector space*. Then
$\emptyset'^{\mathcal{A}} \equiv_s^{\mathcal{A}} \vec{LD} \oplus \overrightarrow{0'}$.

Ex: Let $\mathcal{A}$ be a *linear ordering*. Then
$\emptyset'^{\mathcal{A}} \equiv_s^{\mathcal{A}} succ(x, y) \oplus \overrightarrow{0'}$.

Ex: Let $\mathcal{A}$ be a *linear ordering* with endpoints. Then
$\emptyset''^{\mathcal{A}} \equiv_s^{\mathcal{A}} limleft(x) \oplus limright(x) \oplus \bigoplus_n D_n(x, y) \oplus \overrightarrow{0''}$
where $D_n(x, y) \equiv$ "exists n-string of succ in between x and y."

Ex: Let $\mathcal{A} = (A, \equiv)$ where $\equiv$ is an *equivalence relation*. Then
$\emptyset'^{\mathcal{A}} \equiv_s^{\mathcal{A}} (E_k(x) : k \in \mathbb{N}) \oplus \vec{R} \oplus \overrightarrow{0'}$,
where $E_k(x) \iff$ there are $\geq k$ elements equivalent to x,
and $R = \{\langle n, k\rangle \in \mathbb{N}^2$: there are $\geq n$ equivalence classes with $\geq k$ elements$\}$.

Jump of a structure

Recall: $\emptyset'^{\mathcal{A}} = \vec{K}^{\mathcal{A}} = (K_0, K_1, ...)$ where $\mathcal{A} \models \bar{x} \in K_i \iff \varphi_i(\bar{x})$.

Definition

Let $\mathcal{A}'$ be the structure $(\mathcal{A}, \vec{K}^{\mathcal{A}})$.
(i.e. add infinitely many relations to the language interpreting the K_i's)

There were various independent definitions of the jump of a structure $\mathcal{A}'$:

- Baleva.
  - domain: Moschovakis extension of $\mathcal{A} \times \mathbb{N}$.
  - relation: add a universal computably infinitary Σ_1 relation.
- I. Soskov.
  - domain: Moschovakis extension of $\mathcal{A}$.
  - relation: add a predicate for forcing Π_1 formulas.
- Stukachev. considered arbitrary cardinality, and Σ-reducibility
  - domain: Hereditarily finite extension of $\mathcal{A}$, $\mathbb{HF}(\mathcal{A})$.
  - relation: add a universal finitary Σ_1 relation.
- Montalbán. The definition above.

Computational-reductions between structures

Let $\mathcal{A}$ and $\mathcal{B}$ be structures.
Recall: $Sp(\mathcal{A}) = \{X \subseteq \mathbb{N} : X \text{ computes a copy of } \mathcal{A}\}$.

Def: $\mathcal{A}$ is *Muchnik-reducible* to $\mathcal{B}$:
$\mathcal{A} \leq_w \mathcal{B} \iff Sp(\mathcal{A}) \supseteq Sp(\mathcal{B})$.

Def: $\mathcal{A}$ is *effectively interpretable* in $\mathcal{B}$:
$\mathcal{A} \leq_I \mathcal{B} \iff$ there is an interpretation of $\mathcal{A}$ in $\mathcal{B}$, where
the domain of $\mathcal{A}$ is interpreted in $\mathcal{B}$ by an n-ary r.i.c.e. relation,
and equality and the predicates of $\mathcal{A}$ by r.i.computable relations.

Def: $\mathcal{A}$ is *Σ-reducible* to $\mathcal{B}$: [Khisamiev, Stukachev]
$A \leq_\Sigma \mathcal{B} \iff \mathcal{A} \leq_I \mathbb{HF}(\mathcal{B})$.

Obs: $\mathcal{A} \leq_I \mathcal{B} \implies \mathcal{A} \leq_\Sigma B \implies \mathcal{A} \leq_w \mathcal{B}$.

Three main theorems about the jump

- 1st Jump inversion theorem.
- 2nd Jump inversion theorem.
- Fixed point theorem.

First Jump Inversion Theorem

Theorem (1st Jump inversion Theorem)

If $\overrightarrow{0'}$ is r.i.computable in $\mathcal{A}$,
there exists a structure $\mathcal{B}$ such that $\mathcal{B}'$ is equivalent to $\mathcal{A}$.

for $\equiv_w$. [Goncharov, Harizanov, Knight, McCoy, R. Miller and Solomon]

for $\equiv_w$. [A. Soskova]
independently, different proof, and relative to any structure.

for $\equiv_\Sigma$. [Stukachev]
for arbitrary size structures.

Question:
Which structures are $\equiv_I$-equivalent to the jump of a structure?

First Jump Inversion Theorem – applications

Theorem (1st Jump inversion Theorem - α-iteration)

If $\overrightarrow{0^{(\alpha)}}$ is r.i.computable in $\mathcal{A}$,
there exists a structure $\mathcal{B}$ such that $\mathcal{B}^{(\alpha)}$ is equivalent to $\mathcal{A}$.

[Goncharov, Harizanov, Knight, McCoy, R. Miller and Solomon] used it to build a structure that is Δ_α-categorical but not relatively so.

[Greenberg, M, Slaman] used to build a structure whose spectrum is non-HYP

Second Jump Inversion Theorem

Theorem (2nd Jump Inversion Theorem)

If Y can compute a copy of $\mathcal{A}'$, then there exists X that computes a copy of $\mathcal{A}$ and $X' \equiv_T Y$.

First proved by [I. Soskov], and then, independently, by [Montalbán], using their respective notions of jump, but similar proofs.

Second Jump Inversion Theorem – applications

Theorem (2nd Jump Inversion Theorem)

If Y can compute a copy of $\mathcal{A}'$, then there exists X that computes a copy of $\mathcal{A}$ and $X' \equiv_T Y$.

Cor: $Sp(\mathcal{A}') = \{x' : x \in Sp(\mathcal{A})\}$

Cor: [Frolov] If $0'$ computes a copy of $(\mathcal{L}, succ)$, $\mathcal{L}$ has a low copy.

Cor: If R is r.i.Σ^0_2 in $\mathcal{A}$, then R is r.i.c.e. in $\mathcal{A}'$.
It follows that r.i.Σ^0_n relations are Σ^c_n-definable.

[Ash, Knight, Manasse, Slaman; Chisholm]

Cor:[M] Given $\mathcal{A}$, the following are equivalent:

- Low property: If $X \in Sp(\mathcal{A})$ and $X' \equiv_T Y'$ then $Y \in Sp(\mathcal{A})$.
- Strong jump inversion: If $X' \in Sp(\mathcal{A}')$ then $X \in Sp(\mathcal{A})$.

Fixed point theorem

Recall: For $A \subseteq \mathbb{N}$, $A \not\equiv_T A'$.

Theorem ([M])

The existence of $\mathcal{A}$ with $Sp(\mathcal{A}) = Sp(\mathcal{A}')$, is not provable in full nth-order arithmetic for any n.

Note: Almost all of classical mathematics can be proved in nth-order arithmetic for some n, (except for set theory or model theory).

Theorem ([M] using $0^{\#}$; [S.Friedman, Welch] in ZFC)

There is a structure $\mathcal{A}$ such that $\mathcal{A} \equiv_I \mathcal{A}'$.

Idea of proof: Build $\mathcal{A}$ as a non-well-founded ω-model of $V = L$ such that for some $\alpha \in \mathcal{A}$, $\mathcal{A} \cong L_{\alpha}^{\mathcal{A}}$.

Complete sets of Σ^c_n relations

Definition (M.)

$P_0, ..., P_k, ...$ are a *complete set of Σ^c_n relations on $\mathcal{A}$* if they are uniformly Σ^c_n and $\bigoplus_k P_k \oplus \overrightarrow{0^{(n)}} \equiv^{\mathcal{A}}_s \emptyset^{(n)\mathcal{A}}$.

Question:
For which $\mathcal{A}$ and n, is there a finite complete sets of Σ^c_n relations?

Question:
For which $\mathcal{A}$ and n, is there a nice complete sets of Σ^c_n relations?

Examples of Jump of Structure

Ex: Let $\mathcal{A}$ be a *Boolean algebra*. Then

$$\emptyset'^{\mathcal{A}} \equiv_s^{\mathcal{A}} atom \oplus \overrightarrow{0'}.$$

$$\emptyset''^{\mathcal{A}} \equiv_s^{\mathcal{A}} atom(x) \oplus atomless(x) \oplus finite(x) \oplus \overrightarrow{0''}$$

$$\emptyset'''^{\mathcal{A}} \equiv_s^{\mathcal{A}} atom \oplus atomless \oplus finite \oplus atomic \oplus 1\text{-}atom \oplus atominf \oplus \overrightarrow{0'''}$$

$$\emptyset^{(4)\mathcal{A}} \equiv_s^{\mathcal{A}} atom \oplus atomless \oplus finite \oplus atomic \oplus 1\text{-}atom \oplus atominf \oplus \\ \sim\text{-}inf \oplus Int(\omega+\eta) \oplus infatomicless \oplus 1\text{-}atomless \oplus nomaxatomless \oplus \overrightarrow{0^{(4)}}$$

These relations were used by Thurber [95], Knight and Stob [00].

Theorem (K.Harris – M. 08)

On Boolean algebras, $\forall n \in \mathbb{N}$, there is a finite sequence $P_0, .., P_{k_n}$, s.t. for all $\mathcal{A}$

$$\emptyset^{(n)\mathcal{A}} \equiv_s^{\mathcal{A}} P_0(x) \oplus \ldots \oplus P_{k_n}(x) \oplus \overrightarrow{0^{(n)}}.$$

Examples: Nice complete sets of Σ^c_n relations.

Let $\mathcal{L}$ be a linear ordering.

Ex: Let $\mathcal{L}$ be a *linear ordering*. Then
$\emptyset'^{\mathcal{A}} \equiv^{\mathcal{L}}_s succ(x,y) \oplus \overrightarrow{0'}$.

Ex: $\emptyset''^{\mathcal{L}} \equiv^{\mathcal{L}}_s limleft(x) \oplus limright(x) \oplus \bigoplus_n D_n(x,y) \oplus \overrightarrow{0''}$
where $D_n(x,y) \equiv$"exists n-string of succ in between x and y."

Ex: [Knight-R. Miller-M.-Soskov-Soskova-Soskova-VanDendreissche-Vatev]
We don't need infinitely many relations.
$\emptyset''^{\mathcal{L}} \equiv^{\mathcal{L}}_s limleft(x) \oplus limright(x) \oplus P(x,y,z,w) \oplus \overrightarrow{0''}$
where $P(x,y,z,w) \equiv \exists n \; (succ^n(y) = z \;\&\; D_{n+2}(x,w))$

Thm: [M.] There is no relativizable (and hence nice) set of Σ^c_3 relations that work for all linear orderings simultaneously.

Examples: Nice complete sets of Σ^c_n relations.

Let $\mathcal{V}$ be an infinite dimensional $\mathbb{Q}$-vector space.

$\emptyset'^{\mathcal{A}} \equiv^{\mathcal{A}}_s \vec{LD} \oplus \overrightarrow{0'}$

where $\vec{LD} = (LD_1, LD_2, ...)$, and $LD_i = \{(v_1, ..., v_i) : v_1, ..., v_i \text{ are lin. dep.}\}$

Thm:[Knight-R. Miller-M.-Soskov-Soskova-Soskova-VanDendreissche-Vatev]
No finite set of relations is Σ^c_1 complete in $\mathcal{V}$.

Examples: Nice complete sets of Σ^c_n relations.

Let $\mathcal{A} = (A; \equiv)$ be an equivalence structure.

Ex: $\emptyset'^{\mathcal{A}} \equiv^{\mathcal{A}}_{s} (E_k(x) : k \in \mathbb{N}) \oplus \overrightarrow{R} \oplus \overrightarrow{0'}$,

where $E_k(x) \iff$ there are $\geq k$ elements equivalent to x,

and $R = \{\langle n, k\rangle \in \mathbb{N}^2 :$ there are $\geq n$ equivalence classes with $\geq k$ elements$\}$.

Suppose that $\mathcal{A}$ has infinitely many classes of each size.

Thm:[Knight-R. Miller-M.-Soskov-Soskova-Soskova-VanDendreissche-Vatev]

No finite set of relations is Σ^c_1 complete in $\mathcal{A}$.

Nice description of jump VS coding information.

Theorem ([M])

Let $\mathbb{K}$ be an axiomatizable class of structures.
Exactly one of the following holds:

(relative to any sufficiently large oracle)

1. *There is a nice characterization of $\mathcal{A}^{(n)}$:*
    - *There is a uniform, rel, countable complete sets of Σ^c_n rels.*
    - *No set can be coded by the $(n$-$1)$st jump of any $\mathcal{A} \in \mathbb{K}$.*
    - *There are countably many n-back-and-forth equivalence classes*
2. *Every set can be coded in $\mathcal{A}^{(n-1)}$:*
    - *$\forall X \subseteq \omega$, there is a $\mathcal{A} \in \mathbb{K}$ s.t. X is a r.i.c.e. real in $\mathcal{A}^{(n-1)}$,*
    - *There is no uniform, rel, countable complete sets of Σ^c_n rels.*
    - *$\exists$ Continuum many n-back-and-forth equivalence classes*