The jump of a structure.

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Study the complexity of relations within a given structure.

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In this talk:

- Propose a framework for this analysis.
- Describe the jump of a relation and of a structure.
- Examples.
- Recent results.

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Def: The *spectrum* of the isomorphism type of $\mathcal{A}$:

$$Sp(\mathcal{A}) = \{X \subseteq \mathbb{N} : X \text{ computes a copy of } \mathcal{A}\}.$$

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Def: $R \subseteq A^n$ is *r.i.c.e. (relatively intrinsically computably enumerable)* if for every presentation $(\mathcal{B}, R^{\mathcal{B}})$ of $(\mathcal{A}, R)$, $R^{\mathcal{B}}$ is c.e. in $\mathcal{B}$.

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Def: $R \subseteq A^n$ is *r.i.computable (relatively intrinsically computable)* if R and $(A^n \setminus R)$ are both r.i.c.e.

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$R \subseteq A^n$. The following are equivalent:

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r.i.c.e. relations on $\mathcal{A}$ are the analog of c.e. subsets of $\mathbb{N}$.

We now want a *complete* r.i.c.e. relation.

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Ex: Let $\mathcal{V}$ be a $\mathbb{Q}$-vector space. Then $\vec{LD} = (LD_1, LD_2, ...)$, given by $LD_i = \{(v_1, ..., v_i) : v_1, ..., v_i$ are linearly dependent$\}$, is r.i.c.e.

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Example: Let $\mathcal{A}$ be a ring. Then $\vec{R} = (R_1, R_2, ...)$, given by
$R_i = \{(a_0, ..., a_i) : a_i x^i + ... + a_1 x + a_0$ is reducible polynomial$\}$, is r.i.c.e.

The upper-semi lattice of sequences of relations – à la Soskov's structure-degrees

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Def: Let $\vec{R} \oplus \vec{Q}$ be the sequence $(R_0, Q_0, R_1, Q_1, ...)$.

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I.e. If $\vec{Q}$ is r.i.c.e., there is $\bar{a} \in A^{<\omega}$ and a computable $f : \mathbb{N} \to \mathbb{N}$ s.t.

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Note: We can also define $\vec{Q}''^{\mathcal{A}}$ as $\vec{K}^{(\mathcal{A}, \vec{Q}'^{\mathcal{A}})}$.

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Recall: ${\emptyset'}^{\mathcal{A}} = \vec{K}^{\mathcal{A}} = (K_0, K_1, ...)$ where $\mathcal{A} \models \bar{x} \in K_i(\bar{x}) \iff \varphi_i(\bar{x})$.
Notice: $\overrightarrow{0'}$ is the sequence of trivial relations that codes $0' \subseteq \mathbb{N}$.

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Ex: Let $\mathcal{A}$ be a *linear ordering* with endpoints. Then
$\emptyset''^{\mathcal{A}} \equiv_s^{\mathcal{A}} limleft(x) \oplus limright(x) \oplus \bigoplus_n D_n(x, y) \oplus \overrightarrow{0''}$
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Ex: Let $\mathcal{A} = (A, \equiv)$ where $\equiv$ is an *equivalence relation*. Then
$\emptyset'^{\mathcal{A}} \equiv_s^{\mathcal{A}} (E_k(x) : k \in \mathbb{N}) \oplus \vec{R} \oplus \overrightarrow{0'}$,
where $E_k(x) \iff$ there are $\geq k$ elements equivalent to x,
and $R = \{\langle n, k\rangle \in \mathbb{N}^2 :$ there are $\geq n$ equivalence classes with $\geq k$ elements$\}$.

Theorem[Vatev][Stukachev][M] For every $\vec{Q}$, $\quad \vec{Q} <_s^{\mathcal{A}} \vec{Q}'^{\mathcal{A}}$.

No fixed point for the jump of relations

Theorem[Vatev][Stukachev][M] For every $\vec{Q}$, $\quad \vec{Q} <_s^{\mathcal{A}} \vec{Q}'^{\mathcal{A}}$.

Proof [M]: *Diagonalization:*
Let $K_{i,j}(\bar{x}) \equiv \psi_{i,j}(\bar{x})$ where $\psi_{i,j}$ is the ith Σ_1^c formula with arity j.
Suppose, toward a contradiction, that K is co-r.i.c.e.

Let $R_{e,j}(\bar{x}) \equiv \neg K_{\{e\}(e,j),2j}(\bar{x}, \bar{x})$.

Since R is r.i.c.e., there is $\bar{a} \in A^n$ and computable function $\{k\}$ s.t.

$$R_{e,j}(\bar{x}) \equiv K_{\{k\}(e,j),n+j}(\bar{a}, \bar{x}).$$

Diagonalize: $K_{k,2n}(\bar{a}, \bar{a}) \iff \neg K_{k,2n}(\bar{a}, \bar{a})$.

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Corollary:[Knight] Let $X \subseteq \omega$. TFAE:

- X is c.e. in every copy of $\mathcal{A}$.
- X is e-reducible to $\Sigma_1\text{-tp}_{\mathcal{A}}(\bar{a})$ for some $\bar{a} \in A^{<\omega}$.

Thm: [Ash, Knight, Manasse, Slaman; Chisholm]
Let $\vec{R} = (R_0, R_1, ...)$ be a sequences of relations in $\mathcal{A}$. TFAE:

- $\vec{R}$ is r.i.c.e.
- There is a $\bar{a} \in A^{<\omega}$ and a comp. list $\varphi_0, \varphi_1, ...$ of Σ_1^c-formulas such that $\bar{b} \in R_i \iff \varphi_i(\bar{a}, \bar{b})$.

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Corollary: [Selman] Let $A, B \subseteq \omega$. TFAE:

- Every enumeration of B computes an enumeration of A.
- There is a Turing operator that maps enumeration of B into enumerations of A.

Jump of a structure

Recall: $\emptyset'^{\mathcal{A}} = \vec{K}^{\mathcal{A}} = (K_0, K_1, \ldots)$ where $\mathcal{A} \models \bar{x} \in K_i \iff \varphi_i(\bar{x})$.

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There were various independent definitions of the jump of a structure $\mathcal{A}'$:

- Baleva.
  - domain: Moschovakis extension of $\mathcal{A} \times \mathbb{N}$.
  - relation: add a universal computably infinitary Σ_1 relation.
- I. Soskov.
  - domain: Moschovakis extension of $\mathcal{A}$.
  - relation: add a predicate for forcing Π_1 formulas.
- Stukachev. considered arbitrary cardinality, and Σ-reducibility
  - domain: Hereditarily finite extension of $\mathcal{A}$, $\mathbb{HF}(\mathcal{A})$.
  - relation: add a universal finitary Σ_1 relation.
- Montalbán. The definition above.

Computational-reductions between structures

Let $\mathcal{A}$ and $\mathcal{B}$ be structures.
Recall: $Sp(\mathcal{A}) = \{X \subseteq \mathbb{N} : X \text{ computes a copy of } \mathcal{A}\}$.

Def: $\mathcal{A}$ is *Muchnik-reducible* to $\mathcal{B}$:
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Obs: $\mathcal{A} \leq_I \mathcal{B} \implies \mathcal{A} \leq_\Sigma B \implies \mathcal{A} \leq_w \mathcal{B}$.

Three main theorems about the jump

- 1st Jump inversion theorem.
- 2nd Jump inversion theorem.
- Fixed point theorem.

First Jump Inversion Theorem

Theorem (1st Jump inversion Theorem)

If $\overrightarrow{0'}$ is r.i.computable in $\mathcal{A}$,
there exists a structure $\mathcal{B}$ such that $\mathcal{B}'$ is equivalent to $\mathcal{A}$.

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Q: Which structures are $\equiv_I$-equivalent to the jump of a structure?

First Jump Inversion Theorem – applications

Theorem (1st Jump inversion Theorem - α-iteration)

If $\overrightarrow{0^{(\alpha)}}$ is r.i.computable in $\mathcal{A}$,
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[Greenberg, M, Slaman] used to build a structure whose spectrum is non-HYP

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Cor: If R is r.i.Σ^0_2 in $\mathcal{A}$, then R is r.i.c.e. in $\mathcal{A}'$.
It follows that r.i.Σ^0_n relations are Σ^c_n-definable.

[Ash, Knight, Manasse, Slaman; Chisholm]

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Cor:[M] Given $\mathcal{A}$, the following are equivalent:

- Low property: If $X \in Sp(\mathcal{A})$ and $X' \equiv_T Y'$ then $Y \in Sp(\mathcal{A})$.
- Strong jump inversion: If $X' \in Sp(\mathcal{A}')$ then $X \in Sp(\mathcal{A})$.

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Idea of proof: Build $\mathcal{A}$ as a non-well-founded ω-model of $V = L$ such that for some $\alpha \in \mathcal{A}$, $\mathcal{A} \cong L_\alpha^{\mathcal{A}}$.

Question:
For which $\mathcal{A}$ and n is there a nice description of $\mathcal{A}^{(n)}$?

Complete sets of Σ_n^c relations

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Definition (M.)

$P_0, ..., P_k, ...$ are a *complete sequence of Σ_n^c relations on $\mathcal{A}$* if they are uniformly Σ_n^c and $\bigoplus_k P_k \oplus \overrightarrow{0^{(n)}} \equiv_s^{\mathcal{A}} \emptyset^{(n)^{\mathcal{A}}}$.

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These relations were used by Thurber [95], Knight and Stob [00].

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Theorem (K.Harris – M. 08)

On Boolean algebras, $\forall n \in \mathbb{N}$, there is a finite sequence $P_0, .., P_{k_n}$, of Σ^c_n formulas such that for all $\mathcal{A}$
$$\emptyset^{(n)\mathcal{A}} \equiv_s^{\mathcal{A}} P_0^{\mathcal{A}}(x) \oplus \ldots \oplus P_{k_n}^{\mathcal{A}}(x) \oplus \overrightarrow{0^{(n)}}.$$

Let $\mathcal{L}$ be a linear ordering.

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Thm: [M.] There is no relativizable (and hence nice) set of Σ^c_3 relations that work for all linear orderings simultaneously.

Let $\mathcal{V}$ be an infinite dimensional $\mathbb{Q}$-vector space.

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Let $\mathcal{V}$ be an infinite dimensional $\mathbb{Q}$-vector space.

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where $\vec{LD} = (LD_1, LD_2, ...)$, and $LD_i = \{(v_1, ..., v_i) : v_1, ..., v_i$ are lin. dep.$\}$

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Thm:[Knight-R. Miller-M.-Soskov-Soskova-Soskova-VanDendreissche-Vatev]
No finite set of relations is Σ^c_1 complete in $\mathcal{V}$.

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Suppose that $\mathcal{A}$ has infinitely many classes of each size.

Thm:[Knight-R. Miller-M.-Soskov-Soskova-Soskova-VanDendreissche-Vatev]

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Nice description of jump VS coding information.

Theorem ([M])

Let $\mathbb{K}$ be an axiomatizable class of structures.
Exactly one of the following holds:

(relative to any sufficiently large oracle)

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2. *Every set can be coded in $\mathcal{A}^{(n-1)}$:*

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   - *No set can be coded by the $(n$-$1)$st jump of any $\mathcal{A} \in \mathbb{K}$.*
   - *There are countably many n-back-and-forth equivalence classes*
2. *Every set can be coded in $\mathcal{A}^{(n-1)}$:*

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1. *There is a nice characterization of $\mathcal{A}^{(n)}$:*
   - *There is a uniform, rel, countable complete sets of Σ^c_n rels.*
   - *No set can be coded by the (n-1)st jump of any $\mathcal{A} \in \mathbb{K}$.*
   - *There are countably many n-back-and-forth equivalence classes*
2. *Every set can be coded in $\mathcal{A}^{(n-1)}$:*
   - *There is no uniform, rel, countable complete sets of Σ^c_n rels.*
   - *$\forall X \subseteq \omega$, there is a $\mathcal{A} \in \mathbb{K}$ s.t. X is a r.i.c.e. real in $\mathcal{A}^{(n-1)}$,*
   - *$\exists$ Continuum many n-back-and-forth equivalence classes*