The jump of a structure.

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Singapore – August 2011
General program:
Study the complexity of relations within a given structure.
Goal

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Study the complexity of relations within a given structure.

In this talk:

- Propose a framework for this analysis.
- Describe the jump of a relation and of a structure.
- Examples.
- Recent results.
Def: By *structure* we mean a tuple $\mathcal{A} = (A; P_0, P_1, \ldots, f_0, f_1, \ldots)$ where $P_i \subseteq A^{n_i}$, and $f_i: A^{m_i} \rightarrow A$. 

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We will code the functions as relations, so $\mathcal{A} = (A; P_0, P_1, ..., ....)$.
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Def: The *spectrum* of the isomorphism type of $\mathcal{A}$:

$Sp(\mathcal{A}) = \{ X \subseteq \mathbb{N} : X \text{ computes a copy of } \mathcal{A} \}$. 
Let \( \mathcal{A} \) be a structure.

**Def:** \( R \subseteq A^n \) is *r.i.c.e.* *(relatively intrinsically computably enumerable)* if for every presentation \((B, R^B)\) of \((\mathcal{A}, R)\), \( R^B \) is c.e. in \( B \).

Example: Let \( \mathcal{L} \) be a linear ordering. Then \( \neg \text{succ} = \{ (x, y) \in \mathcal{L}^2 : \exists z (x < z < y) \} \) is r.i.c.e.

Example: Let \( \mathcal{V} \) be a vector space. Then \( \text{LD}_3 = \{ (u, v, w) \in \mathcal{V}^3 : u, v \text{ and } w \text{ are not L.I.} \} \) is r.i.c.e.
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R.I.C.E. Relations

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**Def:** \( R \subseteq A^n \) is \textit{r.i.computable} (relatively intrinsically computable) if \( R \) and \((A^n \setminus R)\) are both r.i.c.e.
Thm: [Ash, Knight, Manasse, Slaman; Chisholm]

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R.I.C.E. – a frequently re-discovered concept

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- $R$ is defined by a c.e. disjunction of $\exists$-formulas, i.e. by a *computably infinitary $\Sigma_1$-formula* (à la Ash)

We now want a complete r.i.c.e. relation.
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And, when the language is finite:

- \( R \) is defined by an \( \exists \)-formula in \( \mathbb{HF}(A) \).  
  (à la Ershov)

  (\( \mathbb{HF}(A) \) is the hereditarily finite extension of \( A \))
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- $R$ is semi-search computable. (à la Moschovakis).
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r.i.c.e. relations on $A$ are the analog of c.e. subsets of $\mathbb{N}$.

We now want a *complete* r.i.c.e. relation.
We consider infinite sequences of relations $\bar{R} = (R_0, R_1, ...)$, (where $R_i \subseteq A^{a_i}$, and the arity function is always primitive computable)
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**Def:** $\vec{R}$ is *r.i.c.e.* in $A$ if
for every presentation $(\mathcal{B}, \vec{R}^\mathcal{B})$ of $(A, \vec{R})$, $\vec{R}^\mathcal{B}$ is uniformly c.e. in $\mathcal{B}$.
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**Ex:** Let \( \mathcal{V} \) be a \( \mathbb{Q} \)-vector space. Then \( LD = (LD_1, LD_2, \ldots) \), given by \( LD_i = \{ (v_1, \ldots, v_i) : v_1, \ldots, v_i \text{ are linearly dependent} \} \), is r.i.c.e.
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**Def:** $\vec{R}$ is r.i.c.e. in $A$ if for every presentation $(B, \vec{R}_B)$ of $(A, \vec{R})$, $\vec{R}_B$ is uniformly c.e. in $B$.

**Ex:** Let $V$ be a $\mathbb{Q}$-vector space. Then $\vec{LD} = (LD_1, LD_2, ...)$, given by $LD_i = \{ (v_1, ..., v_i) : v_1, ..., v_i \text{ are linearly dependent} \}$, is r.i.c.e.

**Example:** Let $A$ be a ring. Then $\vec{R} = (R_1, R_2, ....)$, given by $R_i = \{ (a_0, ..., a_i) : a_i x^i + ... + a_1 x + a_0 \text{ is reducible polynomial} \}$, is r.i.c.e.
Def: Let $\mathcal{SR}(A)$ be the set of all sequences of relations in $A$ with primitive recursive arity functions.

Let $\vec{R}, \vec{Q} \in \mathcal{SR}(A)$. 
The upper-semi lattice of sequences of relations – à la Soskov’s structure-degrees

**Def:** Let $\mathcal{SR}(\mathcal{A})$ be the set of all *sequences of relations* in $\mathcal{A}$ with primitive recursive arity functions. Let $\bar{R}, \bar{Q} \in \mathcal{SR}(\mathcal{A})$.

**Def:** Let $\bar{R} \leq_{\mathcal{A}} \bar{Q} \iff \bar{R}$ is r.i.-computable in $(\mathcal{A}, \bar{Q})$. 
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Def: Let $\vec{R} \leq^A \vec{Q} \iff \vec{R}$ is r.i.computable in $(\mathcal{A}, \vec{Q})$.

Def: Let $\vec{R} \oplus \vec{Q}$ be the sequence $(R_0, Q_0, R_1, Q_1, \ldots)$. 
Let $\varphi_0, \varphi_1, \ldots$ be an effective listing of all c.e.-disjunctions of $\exists$-formulas about $A$, ...
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**Definition**

Let $\vec{K}_A = (K_0, K_1, \ldots)$ be such that $A|\bar{x} \iff \varphi_i(\bar{x})$. 

Obs: $\vec{K}_A$ is complete among r.i.c.e. sequences of relations in $A$. I.e. If $\vec{Q}$ is r.i.c.e., there is $\bar{a} \in A^{<\omega}$ and a computable $f: \mathbb{N} \to \mathbb{N}$ s.t. $\forall \bar{b} \forall i (\bar{b} \in Q_i \iff (\bar{a}, \bar{b}) \in K_f(i))$. 

**Definition**

Given $\vec{Q}$, let the jump of $\vec{Q}$ in $A$ be $\vec{K}((A, \vec{Q}))$. We denote it by $\vec{Q}'_A$. 

Note: $\vec{K}_A = \emptyset'_{\vec{A}}$. 

Note: We can also define $\vec{Q}''_A$ as $\vec{K}((A, \vec{Q}'_A))$. 

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The jump of a relation

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Given \( \vec{Q} \), let the jump of \( \vec{Q} \) in \( \mathcal{A} \) be \( \vec{K}^{(\mathcal{A}, \vec{Q})} \). We denote it by \( \vec{Q}'^\mathcal{A} \).
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**Def:** Given $X \subseteq \mathbb{N}$, let $\vec{X} = (X_0, X_1, \ldots)$ where $X_i = \begin{cases} A & \text{if } i \in X \\ \emptyset & \text{if } i \notin X \end{cases}$
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**Obs:** Then, if $X$ is c.e. $\Rightarrow \vec{X}$ is r.i.c.e. in $\mathcal{A}$.
**Def:** Given $X \subseteq \mathbb{N}$, let $\vec{X} = (X_0, X_1, ..)$ where $X_i = \begin{cases} A & \text{if } i \in X \\ \emptyset & \text{if } i \notin X \end{cases}$

**Obs:** Then, if $X$ is c.e. $\implies \vec{X}$ is r.i.c.e. in $A$.

**Obs:** $X \leq_T Y \implies \vec{X} \leq_s A \vec{Y}$. 
**Def:** Given $X \subseteq \mathbb{N}$, let $\vec{X} = (X_0, X_1, ..)$ where $X_i = \begin{cases} A & \text{if } i \in X \\ \emptyset & \text{if } i \not\in X \end{cases}$

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**Recall:** $\emptyset^A = \vec{K}^A = (K_0, K_1, ...)$ where $A \models \bar{x} \in K_i(\bar{x}) \iff \varphi_i(\bar{x})$.

**Notice:** $0'$ is the sequence of trivial relations that codes $0' \subseteq \mathbb{N}$. 

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Obs: $0' \leq^A \emptyset'$. 
**Examples of Jump of Structure**

**Ex:** Let $\mathcal{A}$ be a $\mathbb{Q}$-vector space. Then

$$\emptyset^\mathcal{A} \equiv^s \mathcal{A} \overset{L\overline{D}}{\oplus} 0'.$$
Examples of Jump of Structure

**Ex:** Let \( \mathcal{A} \) be a \( \mathbb{Q} \)-vector space. Then
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\emptyset'_{\mathcal{A}} \equiv_{s}^{\mathcal{A}} L \hat{D} \oplus \vec{0}'.
\]

**Ex:** Let \( \mathcal{A} \) be a linear ordering. Then
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**Ex:** Let $\mathcal{A}$ be a linear ordering. Then
\[ \emptyset'_{\mathcal{A}} \equiv^A_s \text{succ}(x, y) \oplus 0'. \]

**Ex:** Let $\mathcal{A}$ be a linear ordering with endpoints. Then
\[ \emptyset''_{\mathcal{A}} \equiv^A_s \text{limleft}(x) \oplus \text{limright}(x) \oplus \bigoplus_n D_n(x, y) \oplus 0'' \]
where $D_n(x, y) \equiv \text{“exists } n\text{-string of succ in between } x \text{ and } y\text{.”}$
Examples of Jump of Structure

**Ex:** Let $\mathcal{A}$ be a $\mathbb{Q}$-vector space. Then
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**Ex:** Let $\mathcal{A}$ be a *linear ordering* with endpoints. Then
$$\emptyset''^A \equiv^s_A \limleft(x) \oplus \limright(x) \oplus \bigoplus_n D_n(x, y) \oplus 0''$$
where $D_n(x, y) \equiv \text{“exists } n\text{-string of succ in between } x \text{ and } y.\text{”}\n$

**Ex:** Let $\mathcal{A} = (A, \equiv)$ where $\equiv$ is an *equivalence relation*. Then
$$\emptyset'^A \equiv^s_A (E_k(x) : k \in \mathbb{N}) \oplus \overrightarrow{R} \oplus 0',$$
where $E_k(x) \iff \text{there are } \geq k \text{ elements equivalent to } x,$
and $R = \{\langle n, k \rangle \in \mathbb{N}^2 : \text{there are } \geq n \text{ equivalence classes with } \geq k \text{ elements}\}$.
Theorem [Vatev][Stukachev][M] For every $\bar{Q}$, $\bar{Q} \prec^A_{s} \bar{Q}'^A$. 
No fixed point for the jump of relations

**Theorem** [Vatev][Stukachev][M] For every \( \bar{Q}, \bar{Q}' \), \( \bar{Q} \preceq^A \bar{Q}'^A \).

**Proof** [M]: *Diagonalization:* Let \( K_{i,j}(\bar{x}) \equiv \psi_{i,j}(\bar{x}) \) where \( \psi_{i,j} \) is the \( i \)th \( \Sigma_1^c \) formula with arity \( j \).

Suppose, toward a contradiction, that \( K \) is co-r.i.c.e.

Let \( R_{e,j}(\bar{x}) \equiv \neg K_{\{e\}(e,j),2j}(\bar{x}, \bar{x}) \).

Since \( R \) is r.i.c.e., there is \( \bar{a} \in A^n \) and computable function \( \{k\} \) s.t.

\[ R_{e,j}(\bar{x}) \equiv K_{\{k\}(e,j),n+j}(\bar{a}, \bar{x}). \]

Diagonalize: \( K_{k,2n}(\bar{a}, \bar{a}) \iff \neg K_{k,2n}(\bar{a}, \bar{a}). \)
3 at the price of 1.

**Thm:** [Ash, Knight, Manasse, Slaman; Chisholm]

Let $\vec{R} = (R_0, R_1, \ldots)$ be a sequences of relations in $\mathcal{A}$. TFAE:

- $\vec{R}$ is r.i.c.e.

- There is a $\bar{a} \in A^{<\omega}$ and a comp. list $\varphi_0, \varphi_1, \ldots$ of $\Sigma_1^c$-formulas such that $\bar{b} \in R_i \iff \varphi_i(\bar{a}, \bar{b})$.

**Corollary:** [Knight]

Let $X \subseteq \omega$. TFAE:

- $X$ is c.e. in every copy of $A$.

- $X$ is e-reducible to $\Sigma_1^{\mathcal{A}}(\bar{a})$ for some $\bar{a} \in A^{<\omega}$.

**Corollary:** [Selman]

Let $A, B \subseteq \omega$. TFAE:

- Every enumeration of $B$ computes an enumeration of $A$.

- There is a Turing operator that maps enumeration of $B$ into enumerations of $A$. 

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Let \( \vec{R} = (R_0, R_1, \ldots) \) be a sequences of relations in \( \mathcal{A} \). TFAE:
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Corollary:[Knight] Let \( X \subseteq \omega \). TFAE:
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COROLLARY: [Knight] Let $X \subseteq \omega$. TFAE:

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COROLLARY: [Selman] Let $A, B \subseteq \omega$. TFAE:

- Every enumeration of $B$ computes an enumeration of $A$.
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Recall: \( \overline{A}' = (K_0, K_1, \ldots) \) where \( A| = \overline{x} \in K_i \iff \phi_i(\overline{x}) \).

Definition

Let \( A' \) be the structure \( (A, \overrightarrow{K}_A) \).

(i.e. add infinitely many relations to the language interpreting the \( K_i \)’s)

There were various independent definitions of the jump of a structure \( A' \):

- Baleva. domain: Moschovakis extension of \( A \times \mathbb{N} \).
  relation: add a universal computably infinitary \( \Sigma_1 \) relation.

- I. Soskov. domain: Moschovakis extension of \( A \).
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- Stukachev. considered arbitrary cardinality, and \( \Sigma \)-reducibility
  domain: Hereditarily finite extension of \( A \), \( \text{HF}(A) \).
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- Montalbán. The definition above.
Recall: $\emptyset^A = \vec{K}^A = (K_0, K_1, \ldots)$ where $\mathcal{A} \models \bar{x} \in K_i \iff \varphi_i(\bar{x})$. 

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Let $\mathcal{A}$ and $\mathcal{B}$ be structures.

Recall: $Sp(\mathcal{A}) = \{X \subseteq \mathbb{N} : X$ computes a copy of $\mathcal{A}\}$.

**Def:** $\mathcal{A}$ is *Muchnik-reducible* to $\mathcal{B}$:

$\mathcal{A} \leq_w \mathcal{B} \iff Sp(\mathcal{A}) \supseteq Sp(\mathcal{B})$.
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$\mathcal{A} \leq_I \mathcal{B} \iff$ there is an interpretation of $\mathcal{A}$ in $\mathcal{B}$, where
the domain of $\mathcal{A}$ is interpreted in $\mathcal{B}$ by an $n$-ary r.i.c.e. relation,
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$\mathcal{A} \leq_{\Sigma} \mathcal{B} \iff \mathcal{A} \leq_I HF(\mathcal{B})$.

**Obs:** $\mathcal{A} \leq_I \mathcal{B} \implies \mathcal{A} \leq_{\Sigma} \mathcal{B} \implies \mathcal{A} \leq_w \mathcal{B}$.
Three main theorems about the jump

1. 1st Jump inversion theorem.
2. 2nd Jump inversion theorem.
3. Fixed point theorem.
First Jump Inversion Theorem

Theorem (1st Jump inversion Theorem)

If $\vec{0}'$ is r.i. computable in $A$, there exists a structure $B$ such that $B'$ is equivalent to $A$. 

[Antonio Montalbán. U. of Chicago]

The jump of a structure.
First Jump Inversion Theorem

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Q: Which structures are $\equiv_I$-equivalent to the jump of a structure?
Theorem (1st Jump inversion Theorem - $\alpha$-iteration)

If $0^{(\alpha)}$ is r.i. computable in $A$, there exists a structure $B$ such that $B^{(\alpha)}$ is equivalent to $A$.
Theorem (1st Jump inversion Theorem - $\alpha$-iteration)

If $0^{(\alpha)}$ is r.i. computable in $\mathcal{A}$,
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[Greenberg, M, Slaman] used to build a structure whose spectrum is non-HYP
Theorem (2nd Jump Inversion Theorem)

If $Y$ can compute a copy of $A'$, then there exists $X$ that computes a copy of $A$ and $X' \equiv_T Y$. 

First proved by [I. Soskov], and then, independently, by [Montalbán], using their respective notions of jump, but similar proofs.
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Cor: $Sp(A') = \{ x' : x \in Sp(A) \}$
Theorem (2nd Jump Inversion Theorem)

If $Y$ can compute a copy of $\mathcal{A}'$, then there exists $X$ that computes a copy of $\mathcal{A}$ and $X' \equiv_T Y$.

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**Cor:** [Frolov] If $0'$ computes a copy of $(\mathcal{L}, \text{succ})$, $\mathcal{L}$ has a low copy.
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Cor: If \( R \) is r.i.\( \Sigma^0_2 \) in \( \mathcal{A} \), then \( R \) is r.i.c.e. in \( \mathcal{A}' \). It follows that r.i.\( \Sigma^0_n \) relations are \( \Sigma^c_n \)-definable.

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**Theorem (2nd Jump Inversion Theorem)**

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It follows that r.i.$\Sigma^0_n$ relations are $\Sigma^c_n$-definable.

[Ash, Knight, Manasse, Slaman; Chisholm]

**Cor:** [M] Given $A$, the following are equivalent:

- Low property: If $X \in Sp(A)$ and $X' \equiv_T Y'$ then $Y \in Sp(A)$.
- Strong jump inversion: If $X' \in Sp(A')$ then $X \in Sp(A)$.
Recall: For $A \subseteq \mathbb{N}$, $A \not\equiv_T A'$.
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**Theorem ([M])**

The existence of $A$ with $\text{Sp}(A) = \text{Sp}(A')$, is not provable in full nth-order arithmetic for any $n$. 

The jump of a structure.
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**Note:** Almost all of classical mathematics can be proved in $n$th-order arithmetic for some $n$, (except for set theory or model theory).
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Theorem ([M] using $0\#$; [Puzarenko; S.Friedman, Welch] in ZFC)

There is a structure $A$ such that $A \equiv_I A'$.
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**Theorem ([M] using 0#; [Puzarenko; S.Friedman, Welch] in ZFC)**

There is a structure $A$ such that $A \equiv_I A'$.

**Idea of proof:** Build $A$ as a non-well-founded $\omega$-model of $V = L$ such that for some $\alpha \in A$, $A \cong L^A_\alpha$. 
Question:
For which $A$ and $n$ is there a nice description of $A^{(n)}$?
Complete sets of $\Sigma^c_n$ relations

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For which $\mathcal{A}$ and $n$ is there a nice description of $\mathcal{A}^{(n)}$?

**Definition (M.)**

$P_0, \ldots, P_k, \ldots$ are a **complete sequence of $\Sigma^c_n$ relations on $\mathcal{A}$** if they are uniformly $\Sigma^c_n$ and

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Examples of Jump of Structure

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**Examples of Jump of Structure**

**Ex:** Let $\mathcal{A}$ be a *Boolean algebra*. Then

$$\emptyset^\mathcal{A} \equiv^s_\mathcal{A} \text{atom} \oplus 0'.$$

(4)

These relations were used by Thurber [95], Knight and Stob [00].

Theorem (K.Harris – M. 08) On Boolean algebras, $\forall n \in \mathbb{N}$, there is a finite sequence $P_0, \ldots, P_k_n$ of $\Sigma_c$ formulas such that for all $\mathcal{A}$

$$\emptyset^\mathcal{A}_n \equiv^s_\mathcal{A} P_0(\mathcal{A}) \oplus \ldots \oplus P_k_n(\mathcal{A}) \oplus -\rightarrow 0.$$
Ex: Let $\mathcal{A}$ be a *Boolean algebra*. Then

\[ \emptyset''^\mathcal{A} \equiv _s^\mathcal{A} atom(x) \oplus atomless(x) \oplus finite(x) \oplus 0'' \]
Ex: Let $A$ be a *Boolean algebra*. Then

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Examples of Jump of Structure

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\]

\[
\emptyset^{(4)}^\mathcal{A} \equiv_s^\mathcal{A} \text{atom} \oplus \text{atomless} \oplus \text{finite} \oplus \text{atomic} \oplus 1\text{-}atom \oplus \text{atominf} \oplus \\
\sim\text{-}inf \oplus \text{Int}(\omega + \eta) \oplus \text{infatomicless} \oplus 1\text{-}atomless \oplus \text{nomaxatomless} \oplus 0^{(4)}
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These relations were used by Thurber [95], Knight and Stob [00].
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These relations were used by Thurber [95], Knight and Stob [00].

Theorem (K.Harris – M. 08)

On Boolean algebras, $\forall n \in \mathbb{N}$, there is a *finite* sequence $P_0, ..., P_{k_n}$, of $\Sigma^c_n$ formulas such that for all $\mathcal{A}$

\[ \emptyset^{(n)}^A \equiv S^A P^A_0(x) \oplus ... \oplus P^A_{k_n}(x) \oplus 0^{(n)}. \]
Examples: Nice complete sets of $\Sigma^c_n$ relations.

Let $\mathcal{L}$ be a linear ordering.
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**Ex:** $\emptyset^L \equiv^L_s \text{limleft}(x) \oplus \text{limright}(x) \oplus \bigoplus_n D_n(x, y) \oplus \overrightarrow{0}''$

where $D_n(x, y) \equiv \text{“exists } n\text{-string of succ in between } x \text{ and } y\text{.”}$
Examples: Nice complete sets of $\Sigma^c_n$ relations.

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**Ex:** [Knight-R. Miller-M.-Soskov-Soskova-Soskova-VanDendreissche-Vatev] We don’t need infinitely many relations.
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**Ex:** [Knight-R. Miller-M.-Soskov-Soskova-Soskova-VanDendreissche-Vatev]

We don’t need infinitely many relations.

$$\emptyset''^{\mathcal{L}} \equiv_s^{\mathcal{L}} \text{limleft}(x) \oplus \text{limright}(x) \oplus P(x, y, z, w) \oplus \overrightarrow{0}$$

where $P(x, y, z, w) \equiv \bigvee_n (\text{succ}^n(y) = z \& D_{n+2}(x, w))$
Examples: Nice complete sets of $\Sigma^c_n$ relations.

Let $\mathcal{L}$ be a linear ordering.

**Ex:** Let $\mathcal{L}$ be a *linear ordering*. Then

$\emptyset^A \equiv^L_s \text{succ}(x, y) \oplus 0^j$.

**Ex:** $\emptyset''^L \equiv^L_s \limleft(x) \oplus \limright(x) \oplus \bigoplus_n D_n(x, y) \oplus 0''$

where $D_n(x, y) \equiv \text{“exists } n\text{-string of succ in between } x \text{ and } y\text{.”}$

**Ex:** [Knight-R. Miller-M.-Soskov-Soskova-Soskova-VanDendreissche-Vatev] We don’t need infinitely many relations.

$\emptyset''^L \equiv^L_s \limleft(x) \oplus \limright(x) \oplus P(x, y, z, w) \oplus 0''$

where $P(x, y, z, w) \equiv \bigvee_n (\text{succ}^n(y) = z \& D_{n+2}(x, w))$

**Thm:** [M.] There is no relativizable (and hence nice) set of $\Sigma^c_3$ relations that work for all linear orderings simultaneously.
Examples: Nice complete sets of $\Sigma_n^c$ relations.

Let $\mathcal{V}$ be an infinite dimensional $\mathbb{Q}$-vector space.
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Let $\mathcal{V}$ be an infinite dimensional $\mathbb{Q}$-vector space.

\[ \emptyset^A_s \equiv^A LD \oplus 0' \]

where $LD = (LD_1, LD_2, ...)$, and $LD_i = \{(v_1, ..., v_i) : v_1, ..., v_i \text{ are lin. dep.}\}$
Examples: Nice complete sets of $\Sigma_n^c$ relations.

Let $\mathcal{V}$ be an infinite dimensional $\mathbb{Q}$-vector space.

$$\emptyset^A \equiv_s^A L\vec{D} \oplus \vec{0}
$$

where $L\vec{D} = (LD_1, LD_2, ...)$, and $LD_i = \{(v_1, ..., v_i) : v_1, ..., v_i \text{ are lin. dep.}\}$

**Thm:** [Knight-R. Miller-M.-Soskov-Soskova-Soskova-VanDendreissche-Vatev]

No finite set of relations is $\Sigma_1^c$ complete in $\mathcal{V}$. 
Let $\mathcal{A} = (A; \equiv)$ be an equivalence structure.
Examples: Nice complete sets of $\Sigma_n^c$ relations.

Let $\mathcal{A} = (A; \equiv)$ be an equivalence structure.

Ex: $\emptyset^A \equiv_s^A (E_k(x) : k \in \mathbb{N}) \oplus \vec{R} \oplus 0'$,
where $E_k(x) \iff$ there are $\geq k$ elements equivalent to $x$,
and $R = \{\langle n, k \rangle \in \mathbb{N}^2 : \text{there are } \geq n \text{ equivalence classes with } \geq k \text{ elements}\}$.
Examples: Nice complete sets of $\Sigma^c_n$ relations.

Let $\mathcal{A} = (A; \equiv)$ be an equivalence structure.

Ex: $\emptyset^A \equiv^A_s (E_k(x) : k \in \mathbb{N}) \oplus \overrightarrow{R} \oplus \overrightarrow{0}'$, where $E_k(x) \iff \text{there are } \geq k \text{ elements equivalent to } x$, and $R = \{\langle n, k \rangle \in \mathbb{N}^2 : \text{there are } \geq n \text{ equivalence classes with } \geq k \text{ elements}\}$.

Suppose that $\mathcal{A}$ has infinitely many classes of each size.

Thm: [Knight-R. Miller-M.-Soskov-Soskova-Soskova-VanDendreissche-Vatev]

No finite set of relations is $\Sigma^c_1$ complete in $\mathcal{A}$.
Examples: Nice complete sets of $\Sigma_n^c$ relations.

Let $\mathcal{A} = (A; \equiv)$ be an equivalence structure.

Ex: $\emptyset^\mathcal{A}_s \equiv (E_k(x) : k \in \mathbb{N}) \oplus \vec{R} \oplus \vec{0}$,

where $E_k(x) \iff$ there are $\geq k$ elements equivalent to $x$,

and $R = \{(n, k) \in \mathbb{N}^2 : \text{there are } \geq n \text{ equivalence classes with } \geq k \text{ elements}\}$.

Suppose that $\mathcal{A}$ has infinitely many classes of each size.

Thm: [Knight-R. Miller-M.-Soskov-Soskova-Soskova-VanDendreissche-Vatev]

No finite set of relations is $\Sigma_1^c$ complete in $\mathcal{A}$.

There is a finite set of relations is $\Sigma_2^c$ complete in $\mathcal{A}$. 

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The jump of a structure.
Theorem ([M])

Let $\mathbb{K}$ be an axiomatizable class of structures.

Exactly one of the following holds:

(relative to any sufficiently large oracle)

1. There is a nice characterization of $\mathcal{A}^{(n)}$:

2. Every set can be coded in $\mathcal{A}^{(n-1)}$:
Theorem ([M])

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1. There is a nice characterization of $\mathcal{A}^{(n)}$:
   - There is a uniform, rel, countable complete sets of $\Sigma^n_c$ rels.
   - No set can be coded by the $(n-1)^{st}$ jump of any $\mathcal{A} \in \mathcal{K}$.
   - There are countably many $n$-back-and-forth equivalence classes

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Theorem ([M])

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   - There are countably many $n$-back-and-forth equivalence classes

2. Every set can be coded in $A^{(n-1)}$:
   - There is no uniform, rel, countable complete sets of $\Sigma^c_n$ rels.
   - $\forall X \subseteq \omega$, there is a $A \in \mathbb{K}$ s.t. $X$ is a r.i.c.e. real in $A^{(n-1)}$,
   - $\exists$ Continuum many $n$-back-and-forth equivalence classes

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