Reverse Mathematics.

Antonio Montalbán.
University of Chicago.

September 2011

Reverse Mathematics refers to the program, whose motivating question is

> "What set-existence axioms are necessary to do mathematics?"

asked in the setting of second-order arithmetic.

The Main Question

What axioms are necessary to do mathematics?

The Main Question

What axioms are necessary to do mathematics?

This is an old question.

The greek mathematicians were already asking whether the fifth postulate necessary for Euclidean geometry.

The Main Question

What axioms are necessary to do mathematics?

This is an old question.

The greek mathematicians were already asking whether the fifth postulate necessary for Euclidean geometry.

Motivations:

- Philosophy – Understand the *foundations* that support today's mathematics.

The Main Question

What axioms are necessary to do mathematics?

This is an old question.

The greek mathematicians were already asking whether the fifth postulate necessary for Euclidean geometry.

Motivations:

- Philosophy – Understand the *foundations* that support today's mathematics.
- Pure Math – Study the *complexity* of the mathematical objects and constructions we use today.

Foundations of Mathematics

- In mathematics we search for *true statements* about abstract, but very concrete, objects.

Foundations of Mathematics

- In mathematics we search for *true statements* about abstract, but very concrete, objects.
- These statements are called *theorems*. To know they are true, we prove them.

Foundations of Mathematics

- In mathematics we search for *true statements* about abstract, but very concrete, objects.
- These statements are called *theorems*. To know they are true, we prove them.
- These proofs are based on other statements, called *axioms*, that we assume true without proof.

- In mathematics we search for *true statements* about abstract, but very concrete, objects.
- These statements are called *theorems*. To know they are true, we prove them.
- These proofs are based on other statements, called *axioms*, that we assume true without proof.

Example: Goldbach's *conjecture*:
Every even number $n > 2$ is equal to the sum of two prime numbers.

Foundations of Mathematics

- In mathematics we search for *true statements* about abstract, but very concrete, objects.
- These statements are called *theorems*. To know they are true, we prove them.
- These proofs are based on other statements, called *axioms*, that we assume true without proof.

Example: Goldbach's *conjecture*:
Every even number $n > 2$ is equal to the sum of two prime numbers.

Example: Pythagoras' *theorem*: In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Foundations of Mathematics

- In mathematics we search for *true statements* about abstract, but very concrete, objects.
- These statements are called *theorems*. To know they are true, we prove them.
- These proofs are based on other statements, called *axioms*, that we assume true without proof.

Example: Goldbach's *conjecture*:
Every even number $n > 2$ is equal to the sum of two prime numbers.

Example: Pythagoras' *theorem*: In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

The *axioms* of Euclidean geometry are needed to prove Pythagoras' theorem.

Set-existence axioms

Example:

Halting set = the set of all computer programs that eventually halt.

Set-existence axioms

Example:

Halting set = the set of all computer programs that eventually halt.

Theorem: There is no algorithm to decide whether a program eventually halts or continues running for ever.

Example:
Halting set = the set of all computer programs that eventually halt.

Theorem: There is no algorithm to decide whether a program eventually halts or continues running for ever.

Set-existence axioms refers to the axioms that allow us to define sets according to certain rules.

Set-existence axioms

Example:
Halting set = the set of all computer programs that eventually halt.

Theorem: There is no algorithm to decide whether a program eventually halts or continues running for ever.

Set-existence axioms refers to the axioms that allow us to define sets according to certain rules.

Recursive Comprehension Axiom (RCA) states that if we have a computer program p that always halts, then there exists a set X such that

$$X = \{n : p(n) = \mathtt{yes}\}.$$

Second-Order Arithmetic (SOA).

The Setting of Reverse Mathematics

Second-Order Arithmetic (SOA).

- *SOA* is much weaker than Set Theory (ZFC).

The Setting of Reverse Mathematics

Second-Order Arithmetic (SOA).

- *SOA* is much weaker than Set Theory (ZFC).
- *SOA* it is strong enough to develop most of mathematics.

Second-Order Arithmetic (SOA).

- *SOA* is much weaker than Set Theory (ZFC).
- *SOA* it is strong enough to develop most of mathematics.
- *SOA* is weak enough to better calibrate the theorems of mathematics.

Second-Order Arithmetic (SOA).

- *SOA* is much weaker than Set Theory (ZFC).
- *SOA* it is strong enough to develop most of mathematics.
- *SOA* is weak enough to better calibrate the theorems of mathematics.

In this setting, we give a concrete meaning to

"theorem A is *stronger than* theorem B"

and to

"theorem A is *equivalent* theorem B".

Examples

Theorem

The following are equivalent over RCA_0*:*

- *Weak König's lemma (WKL)*
- *Every continuous function on* $[0,1]$ *can be approximated by polynomials.*
- *Every continuous function on* $[0,1]$ *is Riemann integrable.*

Examples

Theorem

The following are equivalent over RCA_0:

- *Weak König's lemma (WKL)*
- *Every continuous function on $[0, 1]$ can be approximated by polynomials.*
- *Every continuous function on $[0, 1]$ is Riemann integrable.*

Theorem

The following are equivalent over RCA_0:

- *Arithmetic Comprehension Axiom (ACA)*
- *Every bounded sequence of real numbers has a convergent subsequence.*
- *The halting set exists relative to any oracle.*
- *Every countable vector space has a basis.*

The "big five" phenomenon

After a few decades, and many researchers working in this program, the following phenomenon was discovered:

There are *5* axioms systems such that *most* theorems in mathematics are equivalent to one of them.

Π^1_1-CA_0

|

ATR_0

|

ACA_0

|

WKL_0

|

RCA_0

The "big five" phenomenon

After a few decades, and many researchers working in this program, the following phenomenon was discovered:

There are *5* axioms systems such that *most* theorems in mathematics are equivalent to one of them.

Π^1_1-CA_0
|
ATR_0
|
ACA_0
|
WKL_0
|
RCA_0

Q: *What makes these 5 axiom systems so important?*

Robust systems

We say that an axiom system is *robust* if it is equivalent to any small perturbation of itself.

Robust systems

We say that an axiom system is *robust* if it is equivalent to any small perturbation of itself.

This is not a fomal definition

We say that an axiom system is *robust* if it is equivalent to any small perturbation of itself.

This is not a fomal definition

- All big five systems are robust
- No other system appears to be robust.

Robust systems

We say that an axiom system is *robust* if it is equivalent to any small perturbation of itself.

This is not a fomal definition

- All big five systems are robust
- No other system appears to be robust.

Project's goal: understand the notion of robust system.