Reverse Mathematics refers to the program, whose motivating question is

“What set-existence axioms are necessary to do mathematics?”

asked in the setting of second-order arithmetic.
What axioms are necessary to do mathematics?
The Main Question

What axioms are necessary to do mathematics?

This is an old question.
The greek mathematicians were already asking whether the fifth postulate necessary for Euclidean geometry.
What axioms are necessary to do mathematics?

This is an old question. The Greek mathematicians were already asking whether the fifth postulate necessary for Euclidean geometry.

Motivations:
- Philosophy – Understand the *foundations* that support today’s mathematics.
What axioms are necessary to do mathematics?

This is an old question. The Greek mathematicians were already asking whether the fifth postulate necessary for Euclidean geometry.

Motivations:

- **Philosophy** – Understand the *foundations* that support today’s mathematics.
- **Pure Math** – Study the *complexity* of the mathematical objects and constructions we use today.
In mathematics we search for *true statements* about abstract, but very concrete, objects.
In mathematics we search for *true statements* about abstract, but very concrete, objects.

These statements are called *theorems*. To know they are true, we prove them.
In mathematics we search for *true statements* about abstract, but very concrete, objects.

These statements are called *theorems*. To know they are true, we prove them.

These proofs are based on other statements, called *axioms*, that we assume true without proof.

---

Example: Goldbach's conjecture: Every even number $n > 2$ is equal to the sum of two prime numbers.

Example: Pythagoras' theorem: In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. The axioms of Euclidean geometry are needed to prove Pythagoras' theorem.
In mathematics we search for *true statements* about abstract, but very concrete, objects.

These statements are called *theorems*. To know they are true, we prove them.

These proofs are based on other statements, called *axioms*, that we assume true without proof.

**Example:** Goldbach’s *conjecture*:
Every even number $n > 2$ is equal to the sum of two prime numbers.
In mathematics we search for *true statements* about abstract, but very concrete, objects.

These statements are called *theorems*. To know they are true, we prove them.

These proofs are based on other statements, called *axioms*, that we assume true without proof.

**Example:** Goldbach’s *conjecture*: Every even number $n > 2$ is equal to the sum of two prime numbers.

**Example:** Pythagoras’ *theorem*: In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.
In mathematics we search for *true statements* about abstract, but very concrete, objects.

These statements are called *theorems*. To know they are true, we prove them.

These proofs are based on other statements, called *axioms*, that we assume true without proof.

**Example:** Goldbach’s *conjecture:* Every even number $n > 2$ is equal to the sum of two prime numbers.

**Example:** Pythagoras’ *theorem:* In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

The *axioms* of Euclidean geometry are needed to prove Pythagoras’ theorem.
Set-existence axioms

Example:
Halting set = the set of all computer programs that eventually halt.

Theorem:
There is no algorithm to decide whether a program eventually halts or continues running for ever.

Set-existence axioms refers to the axioms that allow us to define sets according to certain rules.

Recursive Comprehension Axiom (RCA) states that if we have a computer program $p$ that always halts, then there exists a set $X$ such that $X = \{n : p(n) = \text{yes}\}$.
Example:

*Halting set* = the set of all computer programs that eventually halt.
Example:

*Halting set* = the set of all computer programs that eventually halt.

**Theorem:** There is no algorithm to decide whether a program eventually halts or continues running for ever.
Example:

*Halting set* = the set of all computer programs that eventually halt.

**Theorem:** There is no algorithm to decide whether a program eventually halts or continues running for ever.

*Set-existence axioms* refers to the axioms that allow us to define sets according to certain rules.
**Example:**

*Halting set* = the set of all computer programs that eventually halt.

**Theorem:** There is no algorithm to decide whether a program eventually halts or continues running for ever.

*Set-existence axioms* refers to the axioms that allow us to define sets according to certain rules.

*Recursive Comprehension Axiom (RCA)* states that if we have a computer program $p$ that always halts, then there exists a set $X$ such that

$$X = \{ n : p(n) = \text{yes} \}.$$
Second-Order Arithmetic (SOA).
Second-Order Arithmetic (SOA).

- SOA is much weaker than Set Theory (ZFC).
Second-Order Arithmetic (SOA).

- SOA is much weaker than Set Theory (ZFC).
- SOA is strong enough to develop most of mathematics.
Second-Order Arithmetic (SOA).

- SOA is much weaker than Set Theory (ZFC).
- SOA is strong enough to develop most of mathematics.
- SOA is weak enough to better calibrate the theorems of mathematics.
The Setting of Reverse Mathematics

Second-Order Arithmetic (SOA).

- SOA is much weaker than Set Theory (ZFC).
- SOA is strong enough to develop most of mathematics.
- SOA is weak enough to better calibrate the theorems of mathematics.

In this setting, we give a concrete meaning to

"theorem A is stronger than theorem B"

and to

"theorem A is equivalent theorem B".
**Theorem**

The following are equivalent over RCA_0:

- **Weak König’s lemma (WKL)**
- **Every continuous function on [0, 1] can be approximated by polynomials.**
- **Every continuous function on [0, 1] is Riemann integrable.**
Examples

Theorem

The following are equivalent over RCA₀:

- **Weak König’s lemma (WKL)**
- *Every continuous function on [0, 1] can be approximated by polynomials.*
- *Every continuous function on [0, 1] is Riemann integrable.*

Theorem

The following are equivalent over RCA₀:

- **Arithmetic Comprehension Axiom (ACA)**
- *Every bounded sequence of real numbers has a convergent subsequence.*
- *The halting set exists relative to any oracle.*
- *Every countable vector space has a basis.*
The “big five” phenomenon

After a few decades, and many researchers working in this program, the following phenomenon was discovered:

There are 5 axioms systems such that *most* theorems in mathematics are equivalent to one of them.

\[ \Pi^1_1-CA_0 \]
\[ ATR_0 \]
\[ ACA_0 \]
\[ WKL_0 \]
\[ RCA_0 \]
The “big five” phenomenon

After a few decades, and many researchers working in this program, the following phenomenon was discovered:

There are 5 axioms systems such that *most* theorems in mathematics are equivalent to one of them.

\[ \Pi^1_1-\text{CA}_0 \]
\[ \text{ATR}_0 \]
\[ \text{ACA}_0 \]
\[ \text{WKL}_0 \]
\[ \text{RCA}_0 \]

Q: *What makes these 5 axiom systems so important?*
We say that an axiom system is *robust* if it is equivalent to any small perturbation of itself.
We say that an axiom system is *robust* if it is equivalent to any small perturbation of itself.

This is not a formal definition.
We say that an axiom system is \textit{robust} if it is equivalent to any small perturbation of itself.

This is not a formal definition.

- All big five systems are robust
- No other system appears to be robust.
We say that an axiom system is robust if it is equivalent to any small perturbation of itself.

This is not a formal definition

- All big five systems are robust
- No other system appears to be robust.

Project’s goal: understand the notion of robust system.