On the Strength of Fraïssés conjecture..

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The embeddability relation on Linear Orderings

A linear ordering (a.k.a. total ordering) is a structure $\mathcal{L} = (L, \leqslant)$, where $\leqslant$ is a is transitive, reflexive, antisymmetric and $\forall x, y(x \leqslant y \vee y \leqslant x)$.

A linear ordering $\mathcal{A}$ embeds into another linear ordering $\mathcal{B}$ if $\mathcal{A}$ is isomorphic to a subset of $\mathcal{B}$. We write $\mathcal{A} \preccurlyeq \mathcal{B}$.

$\mathcal{A}$ and $\mathcal{B}$ are equimorphic if $\mathcal{A} \preccurlyeq \mathcal{B}$ and $\mathcal{B} \preccurlyeq \mathcal{A}$.
We denote this by $\mathcal{A} \sim \mathcal{B}$.

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1. Equimorphism types of Linear Orderings
2. Computable Mathematics
3. Reverse Mathematics
4. Lengths of WQOs

Hausdorff rank

Definition:

- Given a l.o. $\mathcal{L}$, we define another l.o. $\mathcal{L}'$ by identifying the elements of $\mathcal{L}$ which have finitely many elements in between.
- Then we define $\mathcal{L}^0 = \mathcal{L}$, $\mathcal{L}^{\alpha+1} = (\mathcal{L}^\alpha)'$, and take direct limits when α is a limit ordinal.
- rk$(\mathcal{L})$, the Hausdorff rank of $\mathcal{L}$, is the least α such that $\mathcal{L}^\alpha$ is finite.

Examples: rk$(\mathbb{N})$ = rk$(\mathbb{Z}) = 1$, rk$(\mathbb{Z}+\mathbb{Z}+\mathbb{Z}+\cdots) = 2$, rk$(\omega^\alpha) = \alpha$, rk$(\mathbb{Q}) = \infty$.

If $\mathcal{A} \preccurlyeq \mathcal{B}$, then rk$(\mathcal{A}) \leqslant$ rk$(\mathcal{B})$. So, $\mathcal{A} \sim \mathcal{B} \Rightarrow$ rk$(\mathcal{A})$ = rk$(\mathcal{B})$

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Scattered and Indecomposable linear orderings

Two other properties are preserved under equimorphism:
Definition: $\mathcal{L}$ is scattered if $\mathbb{Q} \not\preccurlyeq \mathcal{L}$.

Observation: A linear ordering $\mathcal{L}$ is scattered
$\Leftrightarrow$ for some α, $\mathcal{L}^\alpha$ is finite
$\Leftrightarrow$ $\mathrm{rk}(\mathcal{L}) \neq \infty$.

Definition: $\mathcal{L}$ is indecomposable if whenever
$\mathcal{L} \preccurlyeq \mathcal{A} + \mathcal{B}$, either $\mathcal{L} \preccurlyeq \mathcal{A}$ or $\mathcal{L} \preccurlyeq B$.

Example: ω, ω^*, ω^2 are indecomposable. $\mathbb{Z}$ is not.

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The structure of the scattered linear orderings

Theorem: [Laver '71] Every scattered linear ordering can be written as a finite sum of indecomposable ones.

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Every ctble. indecomposable linear ordering can be written as either an ω-sum or an ω^*-sum of indecomposable l.o. of smaller rank.

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Up to equimorphism, hyperarithmetic is computable.

Obs: If α is an ordinal and $\mathcal{L} \sim \alpha$, then $\mathcal{L}$ is isomorphic to α.
Proof: $\mathcal{L} \preccurlyeq \alpha \Rightarrow \mathcal{L}$ is an ordinal and $\mathcal{L} \leqslant \alpha$.
$\alpha \preccurlyeq \mathcal{L} \Rightarrow \alpha \leqslant \mathcal{L}$ and hence $\mathcal{L} \cong \alpha$.

Theorem

Every hyperarithmetic linear ordering is equimorphic to a computable one.

Lemma

- *Every hyperarithmetic scattered l.o. has rank* $< \omega_1^{CK}$.
- *If* $\mathrm{rk}(\mathcal{L}) < \omega_1^{CK}$ *then* $\mathcal{L}$ *is equimorphic to a computable l.o.*

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Equimorphism types

Definition: Let $\mathbb{L}$ be the partial ordering of equimorphism types of countable linear orderings, ordered by embeddablity.

Let $\mathbb{L}_\alpha$ be the restriction of $\mathbb{L}$ to the linear orderings of rank $< \alpha$.

Theorem

For every ordinal α,
$\mathbb{L}_\alpha$ is computably presentable $\Leftrightarrow$ $\alpha < \omega_1^{CK}$.

Furthermore, a primitive recursive presentation of $\mathbb{L}_\alpha$ can be computed uniformly form $\alpha < \omega_1^{CK}$.

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Fraïssé's Conjecture

Theorem [Fraïssé's Conjecture '48; Laver '71]
FRA: The countable linear orderings form a WQO with respect to embeddablity.
(i.e., there are no infinite descending sequences and no infinite antichains.)

Obs: Π^1_2-$CA_0 \vdash$FRA. By Laver's original proof.

Obs: FRA$\not\Rightarrow$ Π^1_2-CA_0. Because no true Π^1_2 statement does.

Theorem[Shore '93] FRA $\Rightarrow$ ATR_0 over RCA_0.
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Conjecture:[Clote '90][Simpson '99][Marcone]
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$$\begin{tikzcd}
\Pi^1_2\text{-}\mathsf{CA}_0 \arrow[d] \arrow[ddr] & \\
\Pi^1_1\text{-}\mathsf{CA}_0 \arrow[dd] & \\
 & \mathsf{FRA} \arrow[dl] \\
\mathsf{ATR}_0 \arrow[d] & \\
\mathsf{ACA}_0 \arrow[d] & \\
\mathsf{WKL}_0 \arrow[d] & \\
\mathsf{RCA}_0 &
\end{tikzcd}$$

Fraïssé's conjecture again.

Claim

RCA_0+FRA is the least system where it is possible to develop a reasonable theory of equimorphism types of linear orderings.

Theorem

The following are equivalent over RCA_0

- *FRA;*
- *Every scattered lin. ord. is a finite sum of indecomposables;*
- *Every indecomposable lin. ord. is either an ω-sum or an ω^*-sum of indecomposable l.o. of smaller rank.*
- *Jullien's characterization of extendible linear orderings*

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A Partition theorem

Theorem:[Folklore] If we color $\mathbb{Q}$ with finitely many colors, there exists an embedding $\mathbb{Q} \to \mathbb{Q}$ whose image has only one color.

Theorem:[Laver '72]
For every ctble $\mathcal{L}$, there exists $n_{\mathcal{L}} \in \mathbb{N}$, such that:
if $\mathcal{L}$ is colored with finitely many colors, there is an embedding $\mathcal{L} \to \mathcal{L}$ whose image has at most $n_{\mathcal{L}}$ many colors.

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FRA is implied by Laver's Theorem above over RCA_0.

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Robust Systems

FRA is a *Robust* system, as the big five, in the sense that small modifications of it are equivalent to it.

Better quasi orderings

Thm:[Laver 71] The scattered linear orderings form a *Better quasi ordering* under embeddability.

The notion of *Better-quasi-ordering* is stronger than WQO, and enjoys more closer properties.

Marcone studied the reverse mathematics of FRA though the study of Better-quasi-orderings.

For instance he showed that

if $\mathsf{ATR}_0 \vdash \mathsf{FRA}$, it would need a completely new proof,

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Well-quasi-orderings

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Example: The following sets are WQO under an embeddability relation:

- finite strings over a finite alphabet [Higman 52];
- finite trees [Kruskal 60],
- labeled transfinite sequences with finite labels [Nash-Williams 65];
- scattered linear orderings [Laver 71];
- finite graphs [Robertson, Seymour 04].

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Length

Obs: Every linearization of a wpo is well-ordered.
(A *linearization* of $(P, \leqslant_P)$ is a linear ordering $\leqslant_L$ of P such that $x \leqslant_P y \Rightarrow x \leqslant_L y$.)

Definition: The *length* of $\mathcal{W} = (W, \leqslant_W)$ is
$o(\mathcal{W}) = \sup\{\mathrm{ordTy}(W, \leqslant_L) :$ where $\leqslant_L$ is a linearization of $\mathcal{W}\}$.

Def: $\mathbb{B}\mathrm{ad}(\mathcal{W}) = \{\langle x_0, ..., x_{n-1}\rangle \in W^{<\omega} : \forall i < j < n\ (x_i \not\leqslant_W x_j)\}$,
Note: $\mathcal{W}$ is a wpo $\Leftrightarrow$ $\mathbb{B}\mathrm{ad}(\mathcal{W})$ is well-founded.

Theorem: [De Jongh, Parikh 77] $o(\mathcal{W}) + 1 = \mathrm{rk}(\mathbb{B}\mathrm{ad}(\mathcal{W}))$

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Friedman's result

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Every wpo $\mathcal{W}$ has a linearization of order type $o(\mathcal{W})$.

We call such a linearization, a *maximal linearization* of $\mathcal{W}$.

This is why $o(\mathcal{W})$ if often called the *maximal order type* of $\mathcal{W}$.

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Since $\mathbb{B}\text{ad}(\mathcal{W})$ is computable and well-founded, it has rank $< \omega_1^{CK}$. So, $o(\mathcal{W})$ is a computable ordinal.

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Let **a** *be a Turing degree. TFAE:*

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Back to FRA

Def: Let $\mathbb{L}_\alpha$ be the partial ordering of linear orderings of Hausdorff rank $< \alpha$, modulo equimorphism.

For countable α, $\mathbb{L}_\alpha$ is countable
For computable α, $(\mathbb{L}_\alpha, \preccurlyeq)$ is computably presentable

Obs: FRA is equivalent to "$\forall$ ordinal α $(\mathbb{L}_\alpha$ is WQO)".

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Finite Hausdorff rank

Theorem ([Marcone, M 08])

The length of $\mathbb{L}_\omega$ *is* $\epsilon_{\epsilon_{\epsilon_{\dots}}}$*,*
the first fixed point of the function $\alpha \mapsto \epsilon_\alpha$

Def: ACA^+ is the system $\mathsf{RCA}_0 + \forall X (X^{(\omega)}$ *exists*$)$.

Note: $\epsilon_{\epsilon_{\epsilon_{\dots}}}$ is the proof-theoretic ordinal of ACA^+.
(So $\epsilon_{\epsilon_{\epsilon_{\dots}}}$ is the least ordinal that ACA^+ ca**n't** prove is well-ordered.)

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A Conjecture

Conjecture:

The following are equivalent:

- $\mathrm{ATR}_0 \nvdash \mathrm{FRA}$
- There exists $\alpha < \Gamma_0$, s.t. $\mathrm{length}(\mathbb{L}_\alpha) \geqslant \Gamma_0$.