Veblen Functions for Computability Theorists.

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We'll use *Computability Theory* to exhibit the properties that make these functions so interesting to Proof Theorists.

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Obs: ϵ_0 can be represented by a prim. rec. relation $\leqslant_{\epsilon_0}$ on ω.

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That means that,

- Finitiary methods + Trans-Ind. up to $\epsilon_{\epsilon_{\epsilon_{\dots}}} \vdash \mathrm{Cons}(\ \mathsf{ACA}_0^+)$
- For $\alpha < \epsilon_{\epsilon_{\epsilon_{\dots}}}$, $\mathsf{ACA}_0^+ \vdash \alpha$ is an ordinal.

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Def: $\Gamma_0 = \sup\{\varphi_0(0), \varphi_{\varphi_0(0)}(0), \varphi_{\varphi_{\varphi_0(0)}(0)}(0), \ldots\}$
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Γ_0 is the proof theoretic ordinal of ATR_0.

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Thm [Friedman]: TFAE over RCA_0.

- For every well ordering $\mathcal{X}$, $\varphi_{\mathcal{X}}(0)$ is also a well-ordering.
- ATR_0 (Arithmetic Transfinite Recursion).

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WO($\mathcal{F}$): If $\mathcal{X}$ is well-ordering, so is $\mathcal{F}(\mathcal{X})$.

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Q: What is the computability theoretic complexity of WO($\mathcal{F}$)?

Given a LO $\mathcal{X}$ and a descending sequence in $\mathcal{F}(\mathcal{X})$,
how difficult is it to find a descending sequence in $\mathcal{X}$?

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Pf. We have $a_0 > a_1 > .... > a_n > ... \in \omega^{\mathcal{X}}$ where

$$a_0 = \omega^{x_{0,0}} + \omega^{x_{0,1}} + ... + \omega^{x_{0,k_0}}$$

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- We continue like this. If we never succeed this way, we get

$$x_{n_0,0} \geqslant x_{n_1,1} \geqslant x_{n_2,2} \geqslant ... \text{ computable in } 0' \text{ doesn't stabilize.}$$

Thm [Hirst]:
There exists a comp. LO $\mathcal{X}$ s.t. $\omega^{\mathcal{X}}$ has a comp. desc. sequence but all descending sequences in $\mathcal{X}$ compute $0'$.

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Using previous proof n_0 times to $(a_n)_{n\in\mathbb{N}} \subseteq \omega^{\omega^{\cdots\omega^{\epsilon_{x_0+1}}}}$,
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... Continue like this and build $x_0 > x_1 > ... \in \mathcal{X}$ computable in $0^{(\omega)}$.

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Corollary

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Ashfari and Rathjen [2009] found a purely proof-theoretic proof of this corollary, using different logic systems, cut elimination, etc..

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Corollary (MM)

TFAE over RCA_0.

- *If $\mathcal{X}$ is well ordered, then so is $\varphi_\alpha(\mathcal{X})$.*
- Π^0_α*-CA_0* $\equiv$ *$RCA_0 + \forall X, X^{(\omega^\alpha)}$ exists.*

Corollary 2 [Friedman]: TFAE over RCA_0.

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Rathjen and Weiermann [2009] found a purely proof theoretic proof of this corollary, using different logic systems, cut elimination, etc..

System	p.t.o.	$\mathbf{F}(\mathcal{X})$	references
ACA_0	ϵ_0	$\boldsymbol{\omega}^{\mathcal{X}}$	Girard; Hirst.
ACA_0^+	$\varphi_2(0)$	$\boldsymbol{\epsilon}_{\mathcal{X}}$	[MM]; Afshari-Rathjen
$\Pi^0_{\omega^\alpha}\text{-}CA_0$	$\varphi_{\alpha+1}(0)$	$\boldsymbol{\varphi}(\alpha, \mathcal{X})$	[MM].
ATR_0	Γ_0	$\boldsymbol{\varphi}(\mathcal{X}, 0)$	Friedman; Rathjen-Weiermann; [MM].

where:
p.t.o. is the proof theoretic ordinal of the system;
$\mathcal{F}$ is such that $RCA_0 \vdash$ system $\Leftrightarrow$ $WOP(\mathcal{F})$;
references are in historical order.

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- Find a sort of fixed point of the operator h.

$$\begin{array}{lll} \text{given } g\colon \mathcal{J}^\omega(T) \to \mathcal{X} & & (\subset, >_{\mathcal{X}})\text{-monotone} \\ \text{returns } h_g^\omega\colon T \to \epsilon_{\mathcal{X}} & & (\subset, >_{\epsilon_{\mathcal{X}}})\text{-monotone.} \end{array}$$

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- Contradiction.

Claim:[Rathjen] Statements of the form $\mathrm{WOP}(\mathcal{F})$ are equivalent to statements of the form

"Every X belongs to a countably coded ω-model of T".

Thm:[Rathjen] $WOP(\mathcal{X} \mapsto \Gamma_{\mathcal{X}})$ is equivalent to

"Every X belongs to a countably coded ω-model of ATR_0".

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Conjecture: [Rathjen]
Statements saying that

operators $(LO \to LO) \to (LO \to LO)$ preserve WOP,

are equivalent to statements saying that

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Conjecture: [M]
TFAE over RCA_0

- Π^1_1-CA_0
- $\mathrm{WOPP}(f \mapsto \vartheta(f(\Omega+1)))$.
- $\mathrm{WOP}(f) \Rightarrow \exists \alpha \in \mathrm{WO}\ \big(\alpha <_1 f(\alpha+1)\big)$.
- $\mathrm{WOP}(f) \Rightarrow \mathrm{WQO}(T(f(REC)))$