Veblen Functions for Computability Theorists.

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The *Veblen functions* go : Ordinals $\longrightarrow$ Ordinals.

They are well-known and useful in *Proof theory*
to calculate the proof theoretic ordinals of predicative theories.

We'll use *Computability Theory* to exhibit the properties that make these functions so interesting to Proof Theorists.

Def: Let
$\epsilon_0 = \sup(\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots)$, the 1st fixed point of the func. $\gamma \mapsto \omega^\gamma$.
$\epsilon_\beta =$ the βth fixed point of the function $\gamma \mapsto \omega^\gamma$

Thm [Gentzen 36]:
Finitiary methods + Transfinite-Ind. up to $\epsilon_0 \vdash$ PA is consistent.

Obs: For $\alpha < \epsilon_0$, PA $\vdash$ Trans.-Ind. up to α.

This makes ϵ_0 the *proof theoretic ordinal* of PA.
ϵ_0 is also the *proof theoretic ordinal* of ACA_0.

Obs: ϵ_0 can be represented by a prim. rec. relation $\leqslant_{\epsilon_0}$ on ω.

Proof-theoretic ordinals of various theories have been calculated.

Example:
$\mathsf{ACA}_0^+ \quad \equiv \quad \mathsf{ACA}_0 + \forall X\ (X^{(\omega)}$ exists).

The proof-theoretic ordinal of ACA_0^+ is $\epsilon_{\epsilon_{\epsilon_{\dots}}} = \sup\{\epsilon_0, \epsilon_{\epsilon_0}, \epsilon_{\epsilon_{\epsilon_0}}, \dots\}$,

That means that,

- Finitiary methods + Trans-Ind. up to $\epsilon_{\epsilon_{\epsilon_{\dots}}} \vdash$ Cons(ACA_0^+)
- For $\alpha < \epsilon_{\epsilon_{\epsilon_{\dots}}}$, $\mathsf{ACA}_0^+ \vdash \alpha$ is an ordinal.

Definition [Veblen 1908]:
For each α, we define a function $\varphi_\alpha \colon \mathit{Ord} \to \mathit{Ord}$.

- $\varphi_0(\beta) = \omega^\beta$.
- $\varphi_1(\beta) = \epsilon_\beta$.
- $\varphi_{\alpha+1}(\beta)$ is the βth fixed point of φ_α.
- $\varphi_\lambda(\beta)$ is the βth simultaneous fixed point of φ_α for all $\alpha < \lambda$.

Def: $\Gamma_0 = \sup\{\varphi_0(0), \varphi_{\varphi_0(0)}(0), \varphi_{\varphi_{\varphi_0(0)}(0)}(0), ...\}$
is the first ordinal s.t. $\forall \alpha, \beta < \Gamma_0 \;\; \varphi_\alpha(\beta) < \Gamma_0$.

Obs: Γ_0 can be represented by a prim. rec. relation $\leqslant_{\Gamma_0}$ on ω.

Γ_0 is the proof theoretic ordinal of ATR_0.

Obs: There is a natural way of defining an operation that given a linear ordering $\mathcal{X}$, returns a linear ordering $\omega^{\mathcal{X}}$.

Thm [Girard 87]: TFAE over RCA_0.

- For every well ordering $\mathcal{X}$, $\omega^{\mathcal{X}}$ is also a well-ordering.
- ACA_0 (Arithmetic Comprehension).

Obs: There is a natural way of defining an operation that given a linear orderings $\mathcal{X}, \mathcal{Y}$, returns a linear ordering $\varphi_{\mathcal{X}}(\mathcal{Y})$.

Thm [Friedman]: TFAE over RCA_0.

- For every well ordering $\mathcal{X}$, $\varphi_{\mathcal{X}}(0)$ is also a well-ordering.
- ATR_0 (Arithmetic Transfinite Recursion).

Let $\mathcal{F}$ be an operator : LO $\rightarrow$ LO.
(LO $\equiv$ Linear Orderings)

WOP($\mathcal{F}$): If $\mathcal{X}$ is well-ordering, so is $\mathcal{F}(\mathcal{X})$.

Q: What is the proof theoretic complexity of WOP($\mathcal{F}$)?

Q: What is the computability theoretic complexity of WOP($\mathcal{F}$)?

Given a LO $\mathcal{X}$ and a descending sequence in $\mathcal{F}(\mathcal{X})$,
how difficult is it to find a descending sequence in $\mathcal{X}$?

Thm: Let $\mathcal{X}$ be a comp. LO with a comp. desc. sequence in $\omega^{\mathcal{X}}$. Then, there is a desc. seq. in $\mathcal{X}$ computable in $0'$.

Pf. We have $a_0 > a_1 > .... > a_n > ... \in \omega^{\mathcal{X}}$ where

$$
\begin{aligned}
a_0 &= \omega^{x_{0,0}} + \omega^{x_{0,1}} + ... + \omega^{x_{0,k_0}} \\
&\vdots \\
a_n &= \omega^{x_{n,0}} + \omega^{x_{n,1}} + ... + \omega^{x_{n,k_0}} \\
a_{n+1} &= \omega^{x_{n+1,0}} + \omega^{x_{n+1,1}} + ... + \omega^{x_{n+1,k_1}} \\
&\vdots
\end{aligned}
$$

and $x_{0,0} \geqslant x_{0,1} \geqslant ... \geqslant x_{0,k_0} \in \mathcal{X}$

- Therefore, $x_{0,0} \geqslant x_{1,0} \geqslant x_{2,0} \geqslant ... \geqslant x_{n,0} \geqslant ...$

If this doesn't stabilize, it has a comp. desc. subsequence in $\mathcal{X}$.
If it stabilizes, $0'$ can find a point n_0 after which

$$x_{n_0,0} = x_{n_0+1,0} = x_{n_0+2,0} = ...$$

- Then $x_{n_0,1} \geqslant x_{n_0+1,1} \geqslant x_{n_0+2,1} \geqslant ...$

If it doesn't stabilize, it has a comp. desc. subsequence in $\mathcal{X}$.
If it stabilizes, $0'$ can find a point n_1 when it does.

- We continue like this. If we never succeed this way, we get

Thm [Hirst]:
There exists a comp. LO $\mathcal{X}$ s.t. $\omega^{\mathcal{X}}$ has a comp. desc. sequence but all descending sequences in $\mathcal{X}$ compute $0'$.

Thm: Let $\mathcal{X}$ be a comp. LO with a comp. desc. sequence in $\epsilon_{\mathcal{X}}$. Then, there is a desc. seq. in $\mathcal{X}$ computable in $0^{(\omega)}$.

Pf. We have $a_0 > a_1 > .... > a_n > ...$ in $\epsilon_{\mathcal{X}}$. Let $x_0 \in \mathcal{X}$ be such that

$$\epsilon_{x_0} \leqslant a_0 < \epsilon_{x_0+1}.$$

Find n_0 such that

$$\epsilon_{x_0} \leqslant a_0 < \overbrace{\omega^{\omega^{\cdots^{\omega^{\epsilon_{x_0+1}}}}}}^{n \text{ tower}} < \epsilon_{x_0+1}.$$

Using previous proof n_0 times to $(a_n)_{n\in\mathbb{N}} \subseteq \omega^{\omega^{\cdots^{\omega^{\epsilon_{x_0+1}}}}}$,
get a sequence $b_0 > b_1 > .... > b_n > ... \subseteq \epsilon_{x_0}$ computable in $0^{(n_0)}$.
Let $x_1 < x_0$ be such that

$$\epsilon_{x_1} \leqslant b_0 < \epsilon_{x_1+1}.$$

... Continue like this and build $x_0 > x_1 > ... \in \mathcal{X}$ computable in $0^{(\omega)}$.

Theorem (MM)

There exists a comp. LO $\mathcal{X}$ s.t. $\epsilon_{\mathcal{X}}$ has a comp. desc. sequence, but all descending sequences in $\mathcal{X}$ compute $0^{(\omega)}$.

Corollary

TFAE over RCA_0.

- *If $\mathcal{X}$ is well ordered, then so is $\epsilon_{\mathcal{X}}$.*
- $\mathsf{ACA}_0^+ \equiv RCA_0 + \forall X, X^{(\omega)}$ *exists.*

Ashfari and Rathjen [2009] found a purely proof-theoretic proof of this corollary, using different logic systems, cut elimination, etc..

Let α be a computable ordinal.

Thm: Let $\mathcal{X}$ be a comp.LO with a comp.desc. sequence in $\varphi_\alpha(\mathcal{X})$. Then, there is a desc. seq. in $\mathcal{X}$ computable in $0^{(\omega^\alpha)}$.

Theorem (MM)

$\exists$ a comp. lin. $\mathcal{X}$ s.t. $\varphi_\alpha(\mathcal{X})$ has a comp. desc. sequence, but all descending sequences in $\mathcal{X}$ compute $0^{(\omega^\alpha)}$.

Corollary (MM)

TFAE over RCA_0.

- *If $\mathcal{X}$ is well ordered, then so is $\varphi_\alpha(\mathcal{X})$.*
- Π^0_α-$CA_0 \equiv RCA_0 + \forall X, X^{(\omega^\alpha)}$ *exists.*

Corollary 2 [Friedman]: TFAE over RCA_0.

- If $\mathcal{X}$ is well ordered, then so is $\varphi_{\mathcal{X}}(0)$.
- ATR_0 (Arithmetic Transfinite Recursion).

Rathjen and Weiermann [2009] found a purely proof theoretic proof of this corollary, using different logic systems, cut elimination, etc..

System	p.t.o.	$\mathbf{F}(\mathcal{X})$	references
ACA_0	ϵ_0	$\boldsymbol{\omega}^{\mathcal{X}}$	Girard; Hirst.
ACA_0^+	$\varphi_2(0)$	$\boldsymbol{\epsilon}_{\mathcal{X}}$	[MM]; Afshari-Rathjen
$\Pi^0_{\omega^\alpha}$-CA_0	$\varphi_{\alpha+1}(0)$	$\boldsymbol{\varphi}(\alpha, \mathcal{X})$	[MM].
ATR_0	Γ_0	$\boldsymbol{\varphi}(\mathcal{X}, 0)$	Friedman; Rathjen-Weiermann; [MM].

where:
p.t.o. is the proof theoretic ordinal of the system;
$\mathcal{F}$ is such that $\mathsf{RCA}_0 \vdash$ system $\Leftrightarrow$ $WOP(\mathcal{F})$;
references are in historical order.