When low information is no information.

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**Definition:** The degree Spectrum of a structure $\mathcal{A}$ is

$$\text{Spec}(\mathcal{A}) = \{ \deg(B) : B \cong \mathcal{A} \}$$

and when $\mathcal{A}$ is non-trivial Knight showed that

$$\text{Spec}(\mathcal{A}) = \{ \deg(X) : X \text{ can compute a copy of } \mathcal{A} \}.$$
Theorem: [Downey, Jockusch 94]
Every low Boolean Algebra has a computable copy.

Relativized version: If $X' \equiv_T Y'$ and $B$ is a Boolean Alg., then $B$ has copy $\leq_T X \iff B$ has copy $\leq_T Y$.

Lemma: [Downey, Jockusch 94] For every Boolean Alg $B$ and set $X$, $B$ has copy $\leq_T X \iff (B, \text{atom}^B) \text{ has copy } \leq_T X'$
where $\text{atom}^B = \{x \in B : \forall y \in B \ (0 < y < x)\}$. 
Definition

A structure $\mathcal{A}$ admits Jump Inversion if there are relations $P_0, P_1, \ldots$ in $\mathcal{A}$ such that for every $X$, $(\mathcal{A}, P_0, P_1, \ldots)$ has copy $\leq_T X' \iff \mathcal{A}$ has copy $\leq_T X$

Observation If $\mathcal{A}$ admits Jump Inversion and $X' = Y'$, then $\mathcal{A}$ has copy $\leq_T X \iff \mathcal{A}$ has copy $\leq_T Y$. 
A \textbf{admits Jump Inversion} if there are \(P_0, P_1, \ldots\) in \(A\) s.t. \(\forall X (A, P_0, P_1, \ldots)\) has copy \(\leq_T X' \iff A\) has copy \(\leq_T X\).

\textbf{Theorem ([M 08])}

Let \(A\) be a structure. TFAE

- For every \(X, Y\) with \(X' \equiv_T Y'\), \(A\) has copy \(\leq_T X \iff A\) has copy \(\leq_T Y\).
- \(A\) admits Jump Inversion.

\textbf{Lemma ([M 08])}

If the computably infinitary \(\Sigma^0_1\) diagram of \(A\) is comp. in \(Z \geq_T 0'\). Then there is \(Y\) such that \(Y' = Z\) and \(A\) has copy \(\leq_T Y\).

\textbf{Pf:} Computably in \(Z\), we build a copy \(B\) of \(A\), and we use the \(\Sigma^0_1\) diagram of \(A\) to force the jump of \(B\).
Definition: The degree Spectrum of a relation $R$ on a structure computable $\mathcal{A}$ is

$$DgSp_{\mathcal{A}}(R) = \{\text{deg}(Q) : (\mathcal{B}, Q) \cong (\mathcal{A}, R), \mathcal{B} \text{ computable}\}$$
**Def:** \( \text{atom}^B = \{ x \in B : \forall y \in B \ (0 < y < x) \} = \text{DgSp}_B(\text{atom}) \)

Suppose \( B \) has infinitely many atoms

- \( \text{atom}^B \) is co-c.e., so \( \text{DgSp}_B(\text{atom}) \subseteq \text{c.e. degrees} \).
- There is \( B \) with \( 0 \not\in \text{DgSp}_B(\text{atom}) \). [Goncharov 75]
- \( \text{DgSp}_B(\text{atom}) \) is closed upwards in the c.e. degrees [Remmel 81]
- \( \text{DgSp}_B(\text{atom}) \) always contains some incomplete c.e. degree. [Downey 93]

**Theorem ([M07])**

*Every high\(_3\) c.e. degree is in \( \text{DgSp}_B(\text{atom}) \).*
Lemma [Thurber 95] \((B, \text{atom}^B)\) admits jump inversion. 
\((B, \text{atom}^B)\) has copy \(\leq_T X \iff (B, \text{atom}^B, \text{atmoless}^B, \text{infinite}^B)\) has copy \(\leq_T X'\)

Lemma [Knigh Stob 00] \((B, \text{atom}^B, \text{atomless}^B, \text{infinite}^B)\) admits double jump inversion.

Therefore, if \(X\) is high\(_3\) and \(B\) computable. Then \((B, \text{atom}^B)\) has copy \(\leq_T 0' \implies (B, \text{atom}^B)\) has copy \(\leq_T X\).

Lemma ([M], extending [Downey Jockusch 94])

*If \(X\) is c.e. and \((B, \text{atom}^B)\) has copy \(\leq_T X\), then \(B\) has computable copy \(A\) where \(\text{atom}^A \leq_T X\)*.
On the Triple jump of the Atom relation

Theorem ([M07])

Every high$_3$ c.e. degree is in $DgSp_B(\text{atom})$.

Questions: Is it true for every high$_n$ c.e. degree?
Do other relations, like atomless, have similar behavior?
Open Question:
Does every low\(_n\) Boolean Algebra have a computable copy?

Theorem: [Knight, Stob 00]
Every low\(_4\) Boolean Algebra has a computable copy.

Q: Do we know other structures with the low\(_n\) property?

Theorem [Spector 55]:
Every hyperarithmetic well ordering has a computable copy.

Theorem [M 05]: Every hypearithmetic linear ordering is equimorphic (bi-embeddable) to a computable one.
**Def:** A *descending cut* of a lin. ord. $\mathcal{A}$ is a partition $(L, R)$ of $\mathcal{A}$ where $R$ is closed upwards and has no least element.

**Theorem ([Kach, Miller, M 08])**

Every low$_n$ lin. ord. with finitely many descending cuts has a computable copy.

**Theorem**

There is a lin. ord. of intermediate with finitely many descending cuts and no computable copy.
Given a set $A \subseteq \omega$ let
\[ L_A = \omega^\omega + (\ldots \omega^2 \cdot A(2) + \omega \cdot A(1) + A(0)). \]

**Theorem [Kach, Miller 08]:** $L_A$ has copy $\leq_T X \iff \exists e$ such that $\forall n (n \in A \iff n \in W^X_e(2n+2))$.

**Definition:** [Hirschfeldt, Kach, M 08]. $X$ is **low for Feiner** if $\forall e \exists i$ such that $\forall n (n \in W^X_e(2n+2) \iff n \in W^0_i(2n+2))$.

**Obs:** $X$ is low$_n \implies X$ is low for Feiner.

**Theorem ([Hirschfeldt, Kach, M 08]):** There is an intermediate $X$ degree that is not low for Feiner.