When low information is no information.

Antonio Montalbán.
U. of Chicago

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Degree Spectrum

Definition: The degree Spectrum of a structure $\mathcal{A}$ is

$$Spec(\mathcal{A}) = \{deg(\mathcal{B}) : \mathcal{B} \cong \mathcal{A}\}$$

and when $\mathcal{A}$ is non-trivial Knight showed that

$$Spec(\mathcal{A}) = \{deg(X) : X \text{ can compute a copy of } \mathcal{A}\}.$$

Theorem: [Downey, Jockusch 94]
Every low Boolean Algebra has a computable copy.

Relativized version: If $X' \equiv_T Y'$ and $\mathcal{B}$ is a Boolean Alg., then
$$\mathcal{B} \text{ has copy } \leq_T X \iff \mathcal{B} \text{ has copy } \leq_T Y.$$

Lemma: [Downey, Jockusch 94] For every Boolean Alg $\mathcal{B}$ and set X,
$$\mathcal{B} \text{ has copy } \leq_T X \iff (\mathcal{B}, atom^{\mathcal{B}}) \text{ has copy } \leq_T X'$$
where $atom^{\mathcal{B}} = \{x \in \mathcal{B} : \nexists y \in \mathcal{B}\ (0 < y < x)\}$.

Jump Inversion

Definition

A structure $\mathcal{A}$ *admits Jump Inversion* if
there are relations $P_0, P_1, ...$ in $\mathcal{A}$ such that for every X,

$$(\mathcal{A}, P_0, P_1, ...) \text{ has copy } \leq_T X' \iff \mathcal{A} \text{ has copy } \leq_T X$$

Observation If $\mathcal{A}$ admits Jump Inversion and $X' = Y'$, then

$$\mathcal{A} \text{ has copy } \leq_T X \iff \mathcal{A} \text{ has copy } \leq_T Y.$$

Jump Inversion vs Low property

$\mathcal{A}$ *admits Jump Inversion* if there are $P_0, P_1, \ldots$ in $\mathcal{A}$ s.t. $\forall X$
$(\mathcal{A}, P_0, P_1, \ldots)$ has copy $\leq_T X' \iff \mathcal{A}$ has copy $\leq_T X$

Theorem ([M 08])

Let $\mathcal{A}$ be a structure. TFAE

- *For every X, Y with $X' \equiv_T Y'$,*
  $\mathcal{A}$ *has copy* $\leq_T X \iff \mathcal{A}$ *has copy* $\leq_T Y$.
- $\mathcal{A}$ *admits Jump Inversion.*

Lemma ([M 08])

If the computably infinitiary Σ^0_1 diagram of $\mathcal{A}$ is comp. in $Z \geq_T 0'$. Then there is Y such that $Y' = Z$ and $\mathcal{A}$ has copy $\leq_T Y$.

Pf: Computably in Z, we build a copy $\mathcal{B}$ of $\mathcal{A}$, and
we use the Σ^0_1 diagram of $\mathcal{A}$ to force the jump of $\mathcal{B}$.

Spectrum of a Relation

Definition: The *degree Spectrum of a relation* R on a structure computable $\mathcal{A}$ is

$$DgSp_{\mathcal{A}}(R) = \{deg(Q) : (\mathcal{B}, Q) \cong (\mathcal{A}, R), \mathcal{B} \text{ computable}\}$$

Atom Relation

Def: $atom^{\mathcal{B}} = \{x \in \mathcal{B} : \nexists y \in \mathcal{B}\ (0 < y < x)\} = DgSp_{\mathcal{B}}(atom)$

Suppose $\mathcal{B}$ has infinitely many atoms

- $atom^{\mathcal{B}}$ is co-c.e., so $DgSp_{\mathcal{B}}(atom) \subseteq$ c.e. degrees.
- There is $\mathcal{B}$ with $\mathbf{0} \notin DgSp_{\mathcal{B}}(atom)$. [Goncharov 75]
- $DgSp_{\mathcal{B}}(atom)$ is closed upwards in the c.e. degrees [Remmel 81]
- $DgSp_{\mathcal{B}}(atom)$ always contains some incomplete c.e. degree. [Downey 93]

Theorem ([M07])

Every high$_3$ c.e. degree is in $DgSp_{\mathcal{B}}(atom)$.

Lemma [Thurber 95] $(\mathcal{B}, atom^{\mathcal{B}})$ admits jump inversion.
$(\mathcal{B}, atom^{\mathcal{B}})$ has copy $\leq_T X \iff$
$(\mathcal{B}, atom^{\mathcal{B}}, atmoless^{\mathcal{B}}, infinite^{\mathcal{B}})$ has copy $\leq_T X'$

Lemma [Knigh Stob 00] $(\mathcal{B}, atom^{\mathcal{B}}, atomless^{\mathcal{B}}, infinite^{\mathcal{B}})$ admits double jump inversion.

Therefore, if X is high$_3$ and $\mathcal{B}$ computable. Then $(\mathcal{B}, atom^{\mathcal{B}})$ has copy $\leq_T 0' \implies (\mathcal{B}, atom^{\mathcal{B}})$ has copy $\leq_T X$.

Lemma ([M], extending [Downey Jockusch 94])

If X is c.e. and $(\mathcal{B}, atom^{\mathcal{B}})$ has copy $\leq_T X$,
then $\mathcal{B}$ has computable copy $\mathcal{A}$ where $atom^{\mathcal{A}} \leq_T X$.

On the Triple jump of the Atom relation

Theorem ([M07])

Every $high_3$ c.e. degree is in $DgSp_{\mathcal{B}}(atom)$.

Questions: Is it true for every high_n c.e. degree?
Do other relations, like *atomless*, have similar behavior?

Open Question:

Does every low_n Boolean Algebra have a computable copy?

Theorem: [Knight, Stob 00]

Every low_4 Boolean Algebra has a computable copy.

Q: Do we know other structures with the low_n property?

Theorem [Spector 55]:
Every hyperarithmetic well ordering has a computable copy.

Theorem [M 05]: Every hypearithmetic linear ordering is equimorphic (bi-embeddable) to a computable one.

Finite descending cuts

Def: A *descening cut* of a lin. ord. $\mathcal{A}$ is a partition (L, R) of $\mathcal{A}$ where R is closed upwards and has no least element.

Theorem ([Kach, Miller, M 08])

Every low_n *lin. ord. with finitely many descending cuts has a computable copy.*

Theorem

There is a lin. ord. of intermediate with finitely many descending cuts and no computable copy.

Low for Feiner

Given a set $A \subseteq \omega$ let

$$L_A = \omega^\omega + (...\omega^2 \cdot A(2) + \omega \cdot A(1) + \cdot A(0)).$$

Theorem [Kach, Miller 08]: L_A has copy $\leq_T X \iff$

$$\exists e \text{ such that } \forall n\ (n \in A \leftrightarrow n \in W_e^{X^{(2n+2)}}).$$

Definition: [Hirschfeldt, Kach, M 08]. X is *low for Feiner* if

$$\forall e \exists i \text{ such that } \forall n\ (n \in W_e^{X^{(2n+2)}} \leftrightarrow n \in W_i^{0^{(2n+2)}}).$$

Obs: X is $\text{low}_n \implies X$ is low for Feiner.

Theorem ([Hirschfeldt, Kach, M 08])

There is an intermediate X degree that is not low for Feiner.