The complexity within well-partial-orderings

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1. Background on WQOs

2. WQOs in Proof Theory
   - Kruskal's theorem and the graph-minor theorem
   - Linear orderings and Fraïssé's Conjecture

3. WPOs in Computability Theory

Well-quasi-orderings

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A *well-partial-ordering (WPO)*, is a WQO which is a partial ordering.

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The reverse mathematics and computability theory of these equivalences was been studied in [Cholak-Marcone-Solomon 04].

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- Finite strings over a WPO are a WPO (Higman, 1952)
- Finite trees with labels from a WPO are a WPO (Kruskal, 1960)
- Transfinite sequences with labels from a WPO which use only finitely many labels are a WPO (Nash-Williams, 1965)

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Theorem: [De Jongh, Parikh 77] $o(\mathcal{P}) + 1 = \mathrm{rk}(\mathbb{B}\mathrm{ad}(\mathcal{P}))$.

Kruskal's theorem

Theorem: [Kruskal 60] Let $\mathcal{T}$ be the set of finite trees ordered by $T \preccurlyeq S$ if there is an embedding : $T \to S$ preserving $\leqslant$ and $g.l.b.$
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Corollary: [Friedman] (RCA_0) Kruskal's theorem $\Rightarrow$ Γ_0 well-ordered.
Therefore,

$$\mathrm{ATR}_0 \nvdash \text{Kruskal's theorem.}$$

The "big five" subsystems of 2nd-order arithmetic

Axiom systems:

RCA_0:

WKL_0:

ACA_0:

ATR_0:

$\Pi^1_1\text{-}\mathsf{CA}_0$:

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Thm: [M.–Weiermann 2006] The following are equivalent over RCA_0

- ATR_0
- For every $\mathcal{P}$, if $\mathcal{P}$ is a WQO, then so is $\mathcal{T}(\mathcal{P})$, where $\mathcal{T}(\mathcal{P})$ is the set of finite trees with labels in $\mathcal{P}$, ordered by homomorphism.

The minor-graph theorem

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Corollary: [Friedman, Robertson, Seymour]
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Therefore,

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Fraïssé's Conjecture

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Theorem[Shore '93]
FRA implies ATR_0 over RCA_0.

Conjecture:[Clote '90][Simpson '99][Marcone]
FRA is equivalent to ATR_0 over RCA_0.

$$
\begin{tikzcd}
\Pi^1_2\text{-}CA_0 \arrow[d] \arrow[ddr] & \\
\Pi^1_1\text{-}CA_0 \arrow[dd] & \\
 & FRA \arrow[dl] \\
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Claim

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Theorem

The following are equivalent over RCA_0

- *FRA;*
- *Every scattered lin. ord. is a finite sum of indecomposables;*
- *Every indecomposable lin. ord. is either an ω-sum or an ω^*-sum of indecomposable l.o. of smaller rank.*
- *Jullien's characterization of extendible linear orderings*

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FRA is implied by Laver's Theorem above over RCA_0.

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Theorem ([Kach–Marcone–M.–Weiermann 2011])

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Question: Given α, what is the length of $\mathbb{L}_\alpha$?

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Theorem ([Marcone, M 08])

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Theorem ([Marcone, M 08])

The following are equivalent over ACA^+*:*

- $\epsilon_{\epsilon_{\epsilon_{\dots}}}$ *is well-ordered*
- $\mathbb{L}_\omega$ *is a WQO*

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Question [Schmidt 1979]:
Is the length of a computable WPO computable?

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Computable Length

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We mentioned that $o(\mathcal{P}) + 1 = \mathrm{rk}(\mathbb{B}\mathrm{ad}(\mathcal{P}))$, where

$$\mathbb{B}\mathrm{ad}(\mathcal{P}) = \{\langle x_0, ..., x_{n-1}\rangle \in W^{<\omega} : \forall i < j < n \ (x_i \not\leq_P x_j)\},$$

Since $\mathbb{B}\mathrm{ad}(\mathcal{P})$ is computable and well-founded, it has rank $< \omega_1^{CK}$.
So, $o(\mathcal{P})$ is a computable ordinal.

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Q:
Does every computable WPO have a computable maximal linearization?

A computable maximal linearization

Theorem ([M 2007])

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Theorem ([M 2007])

Let **a** *be a Turing degree. TFAE:*

1. **a** *uniformly computes maximal linearizations of comp. WPOs.*
2. **a** *uniformly computes* $0^{(\beta)}$ *for every* $\beta < \omega_1^{CK}$.

The height of a WPO

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If $\mathcal{P}$ is well founded, its *height* is

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Q: How difficult is it to compute maximal chains?

Computing maximal chains

Theorem ([Marcone-Shore 2010])

Every computable WPO $\mathcal{P}$ has a hyperarithmetic maximal chain.

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Maximal chains aren't easy to compute:

Theorem ([Marcone–M.–Shore 2012])

Let $\alpha < \omega_1^{\mathrm{CK}}$.
There exists a computable WPO $\mathcal{P}$ such that
$0^{(\alpha)}$ *does not compute any maximal chain of $\mathcal{P}$.*

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- The key observation is that all downward closed subsets of P are computable.
- Then, build an operator $\Phi_{\alpha}^{\mathcal{P},G}$, that returns a sequence of computable sub-partial orderings $P_0 \leqslant P_1 \leqslant \ldots$, such that, if $\mathcal{P}$ has cofinality $\omega^{\alpha+1}$, and G is generic, then infinitely many of the P_i will have cofinality ω^{α}.

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- Then use effective transfinite recursion.