Well-quasi-orderings and computability theory.

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Well-quasi-orderings

Definition: A *well-quasi-ordering (wqo)*, is quasi-ordering which has no infinite descending sequences and no infinite antichains.

Example: The following sets are WQO under an embeddability relation:

- finite strings over a finite alphabet [Higman 52];
- finite trees [Kruskal 60],
- labeled transfinite sequences with finite labels [Nash-Williams 65];
- scattered linear orderings [Laver 71];
- finite graphs [Robertson, Seymour 04].

An Equivalent Definition

Theorem: Let $\mathcal{Q} = (Q, \leqslant_Q)$ be a quasi-ordering. TFAE:

1. for every sequence $\{x_n\}_{n\in\omega} \subseteq Q$, $\exists i < j \ (x_i \leqslant_Q x_j)$.
2. $\mathcal{Q}$ has no infinite descending seq. and no infinite antichains.
3. Every linearization of $\mathcal{Q}$ is well-ordered.
4. For every $X \subseteq Q$ there exists a finite $F \subseteq X$ such that
$$\forall x \in X \exists y \in F \ \ (y \leqslant_Q x).$$

Graph minor Theorem

Theorem: [Robertson, Seymour 04] The set of finite graphs, ordered by $G \preccurlyeq H$ if G is a minor or H, is a Well-Quasi-Ordering.

Corollary: If P is a set of graphs which is closed upwards, then there exists graphs $G_1, ..., G_k$ such that a graph G is in P iff it has some of the $G_1, ..., G_k$ as a minor.

Example: A graph is planar
(i.e. it can be drawn without auto-intersections),
$\Leftrightarrow$ it doesn't have neither $K_{3,3}$ nor K_5 as a minor.

Jump Upper semilattices

Definition: A *Jump Upper semilatice* is a structure $\mathcal{J} = (J, \leqslant, \vee, j(\cdot))$, where

- $(J, \leqslant)$ is a partial ordering,
- $\vee$ is the least upper bound
- and $j(\cdot)$ is a monotonic and increasing operator.

Example: The Turing degree structure $\mathcal{D} = (D, \leqslant_T, \oplus, ')$

Lemma (M. 03)

Every finitely generated Jump Upper semilattice is WQO.

This was used in [M. 03] to prove that
Every ctble. Jump Upper semilattice embeds into $\mathcal{D}$.

Kruskal's Theorem

Theorem: [Kruskal 60] Let $\mathcal{T}$ be the set of finite trees ordered by $T \preccurlyeq S$ if there is an embedding $: T \to S$ preserving $\leqslant$ and $g.l.b.$ Then $\mathcal{T}$ is a WQO.

Theorem: [Friedman]
Kruskal's theorem is can't be proved in ATR_0.

Reverse Mathematics

Setting: Second order arithmetic.

Main Question: What axioms are necessary to prove the theorems of Mathematics?

Axiom systems:

RCA_0: Recursive Comprehension + Σ^0_1-induction + Semiring ax.

WKL_0: Weak König's Lemma + RCA_0

(Every infinite subtree of $2^{<\omega}$ has a path.)

ACA_0: Arithmetic Comprehension + RCA_0

$\Leftrightarrow$ "for every set X, X' exists".

ATR_0: Arithmetic Transfinite Recursion + ACA_0.

$\Leftrightarrow$ " $\forall X$, $\forall$ ordinal α, $X^{(\alpha)}$ exists".

Π^1_1-CA_0: Π^1_1-Comprehension + ACA_0.

$\Leftrightarrow$ "$\forall X$, the hyper-jump of X exists".

Which definition do we use?

Cholak, Marcone, Simpson and Solomon studied how hard is it to prove the equivalence between the definitions of WQO.

Theorem: ACA_0 can prove:
Let $\mathcal{Q} = (Q, \leqslant_Q)$ be a quasi-ordering. TFAE:

1. for every sequence $\{x_n\}_{n\in\omega} \subseteq Q$, $\exists i < j\ (x_i \leqslant_Q x_j)$.
2. $\mathcal{Q}$ has no infinite descending seq. and no infinite antichains.
3. Every linearization of $\mathcal{Q}$ is well-ordered.
4. For every $X \subseteq Q$ there exists a finite $F \subseteq X$ such that $\forall x \in X \exists y \in F (y \leqslant_Q x)$.

Theorem: WKL_0 can only prove $1 \Rightarrow 2$, $1 \Leftrightarrow 3$ and $1 \Leftrightarrow 4$.

Theorem: RCA_0 can only prove $1 \Rightarrow 2$, $1 \Rightarrow 3$ and $1 \Leftrightarrow 4$.

Fraïssé's conjecture

Theorem [Fraïsé's Conjecture '48; Laver '71]
FRA: The countable linear orderings form a WQO with respect to embeddablity.

Laver's proof of FRA can be carried out in Π^1_2-CA_0.

Obs: Since FRA is a Π^1_2 statement, it cannot imply Π^1_1-CA_0.

Theorem (Shore '93)
The fact that the well-orderings form a WQO implies ATR_0 over RCA_0.

Corollary: FRA implies ATR_0 over RCA_0.

Fraïssé's conjecture

$$
\begin{tikzcd}
\Pi^1_2\text{-}\mathsf{CA}_0 \arrow[d] \arrow[ddr] & \\
\Pi^1_1\text{-}\mathsf{CA}_0 \arrow[dd] & \\
& \mathsf{FRA} \arrow[dl] \\
\mathsf{ATR}_0 \arrow[d] & \\
\mathsf{ACA}_0 \arrow[d] & \\
\mathsf{WKL}_0 \arrow[d] & \\
\mathsf{RCA}_0 &
\end{tikzcd}
$$

Conjecture:[Clote '90]
[Simpson '99][Marcone]
FRA is equivalent to ATR_0 over RCA_0.

Theory for linear orderings.

Claim (Simposon 99)

"ATR_0 is the least system where it is possible to develop a reasonable theory of ordinals."

Claim (M. 05)

" RCA_0+FRA is the least system where it is possible to develop a reasonable theory of linear orderings and the relation of embeddability."

Theorem: [Laver 71, 73]

1. FRA;
2. Every scattered linear ordering can be written as a finite sum of indecomposable ones;
3. Every indecomposable linear ordering can be written either as an ω-sum or as an ω^* sum of indecomposable l.o. of smaller rank.

Theorem (M. 05)

The three statements above are equivalent over RCA_0

Other statements about L.O.s have been proved to be equivalent to FRA like Jullien's classifcation of the extendible linear orderings, and the fact that there are only countably many equimorphism types below a countable L.O. [M. 05]

Signed Trees

Definition:

- A *signed tree* is a well founded tree $T \subset \omega^{<\omega}$ together with a map $s_T\colon T \to \{+,-\}$.
- $h\colon T \to T'$ is a *homomorphism* if $\forall \sigma, \tau \in T$

$$\sigma \subsetneq \tau \Rightarrow h(\sigma) \subsetneq h(\tau) \text{ and } s_{T'}(h(\sigma)) = s_T(\sigma).$$

- Let $T \preccurlyeq T'$ if such an h exists.

Obs: That the (unsigned) well-founded trees are WQO under $\preccurlyeq$ is equivalent to ATR_0 over RCA_0.

Theorem (M. 05)

TFAE over RCA_0.

- *FRA;*
- *The signed trees are WQO under $\preccurlyeq$.*

Well-partial-orderings

Let $\mathcal{Q} = (Q, \leqslant_Q)$ be a quasi-ordering.
Consider the partial-ordering associated to it in the usual way:

Let $x \equiv_Q y \Leftrightarrow x \leqslant_Q y \;\&\; y \leqslant_Q x$, and
let $\mathcal{W} = \mathcal{Q}/\equiv_Q$, where $[x] \leqslant_W [y] \Leftrightarrow x \leqslant_Q y$

Definition: A *well-partial-ordering (wpo)*, is a partial-ordering with no infinite descending sequences and no infinite antichains.

Length

Obs: Every linearization of a wpo is well-ordered.
(A *linearization* of $(P, \leqslant_P)$ is a linear ordering $\leqslant_L$ of P
such that $x \leqslant_P y \Rightarrow x \leqslant_L y$.)
So, if $\{x_n\}$ is $\leqslant_L$ decreasing, $\forall i < j \; (x_i \nleqslant_P x_j)$

Definition: The *length* of $\mathcal{W} = (W, \leqslant_W)$ is
$o(\mathcal{W}) = \sup\{\mathrm{ordTy}(W, \leqslant_L) : \text{ where } \leqslant_L \text{ is a linearization of } \mathcal{W}\}$.

Def: $\mathbb{B}\mathrm{ad}(\mathcal{W}) = \{\langle x_0, \ldots, x_{n-1}\rangle \in W^{<\omega} : \forall i < j < n \; (x_i \nleqslant_W x_j)\}$,
Note: $\mathcal{W}$ is a wpo $\Leftrightarrow$ $\mathbb{B}\mathrm{ad}(\mathcal{W})$ is well-founded.

Theorem: [De Jongh, Parikh 77] $o(\mathcal{W}) = \mathrm{rk}(\mathbb{B}\mathrm{ad}(\mathcal{W}))$

Friedman's result

Theorem: [Kruskal 60] Let $\mathcal{T}$ be the set of finite trees ordered by $T \preccurlyeq S$ if there is an embedding : $T \to S$ preserving $\leqslant$ and $g.l.b.$ Then $\mathcal{T}$ is a WQO.

Theorem: [Friedman] The length of $\mathcal{T}$ is $\geqslant \Gamma_0$.
(where Γ_0 the the proof-theoretic ordinal of ATR_0.
it's the "least ordinal" that ATR_0 can't prove it's an ordinal.)
Moreover, there is a computable map $\mathcal{T} \to \Gamma_0$ which preserves $\leqslant$.

Corollary: [Friedman] (RCA_0) Kruskal's theorem implies that Γ_0 is well-ordered. Therefore, ATR_0 cannot imply Kruskal's theorem.

Theorem: [Marcone] Kruskal's theorem can be proved in Π^1_1-CA_0.

Extensions of Friedman's result

Theorem: [Rathjen, Weiermann 93]
Over ACA_0, Kruskal's theorem is equivalent to the uniform Π^1_1 reflection principle for Π^1_1-Transfinite induction.

Theorem: [Friedman, Robertson, Seymour 87]
The Graph minor theorem is not provable in Π^1_1-CA_0.

Maximal order types

Theorem: [De Jongh, Parikh 77]
Every wpo $\mathcal{W}$ has a linearization of order type $o(\mathcal{W})$.

We call such a linearization, a *maximal linearization* of $\mathcal{W}$.

This is why $o(\mathcal{W})$ if often called the *maximal order type* of $\mathcal{W}$.

Such linearizations have been found in many of the examples, always by different methods.

Schmidt's questions

Schmidt, in her Habilitationsschrift [1979], computed the maximal order type of the wpo investigated by Higman, and gave upper bounds for the maximal order types of the wpo's investigated by Kruskal and Nash-Williams.

Schmidt posed two questions:

Question:

Is there a non-trivial relation between rank and length of a wpo?

Question:

Is the length of a computable wpo, a computable ordinal?

Computable Length

Q: Is the maximal order type of a computable wpo, computable?

We mentioned that $o(\mathcal{W}) + 1 = \text{rk}(\mathbb{B}\text{ad}(\mathcal{W}))$, where

$$\mathbb{B}\text{ad}(\mathcal{W}) = \{\langle x_0, \ldots, x_{n-1}\rangle \in W^{<\omega} : \forall i < j < n \ (x_i \not\leq_W x_j)\},$$

Since $\mathbb{B}\text{ad}(\mathcal{W})$ is computable and well-founded, it has rank $< \omega_1^{CK}$. So, $o(\mathcal{W})$ is a computable ordinal.

Q: Does every computable wpo have a computable maximal linearization?

Uniformly computable linearization

Given $\mathcal{W}$, we computably uniformly define a linearization $\preceq^{\mathcal{W}}$ of it.

Theorem

If $\omega^\delta \leqslant o(\mathcal{W}) < \omega^{\delta+1}$, *then*
$\omega^\delta \leqslant \mathrm{ordTy}(W, \preceq^{\mathcal{W}}) \leqslant o(\mathcal{W}) < \omega^{\delta+1}$.
In particular, if $o(\mathcal{W}) = \omega^\delta$, *then* $\mathrm{ordTy}(W, \preceq^{\mathcal{W}}) = \omega^\delta$

Construction:

- We define a map $y \mapsto \sigma_y \colon W \to 2^W$:

$$\text{Let } \sigma_y(x) = \begin{cases} 1 & \text{if } x \leqslant_W y \\ 0 & \text{otherwise} \end{cases}, \qquad \text{for } y, x \in W.$$

- Order 2^W lexicographically, w.r.t. the enumeration of W.
- let $y \preceq^{\mathcal{W}} z \Leftrightarrow \sigma_y \leqslant_{lex} \sigma_z$.

A computable maximal linearization

Theorem

Every computable wpo has a computable maximal linearization.

Scketch of the Proof:

- Write $o(\mathcal{W})$ as $\omega^{\alpha_n} + ... + \omega^{\alpha_0}$.
- Find a partition $J_0, ..., J_k$ of W such that $o(J_i) = \omega^{\alpha_i}$ and if $x \in J_i, y \in J_j$ and $i \leqslant j$ then $y \nleqslant_W x$.
- Then, we note that any such partition is computable.
- Linearize each of the J_i's and put one after the other.

Q: Can we uniformly find computable maximal linearizations?

non-uniformity

Theorem

Let **a** *be a Turing degree. TFAE:*

1. **a** *uniformly computes maximal linearizations of comp. wpos.*
2. **a** *can decide whether two computable ordinals are isomorphic.*
3. **a** *uniformly computes* $0^{(\beta)}$ *for every* $\beta < \omega_1^{CK}$.

That (2) $\Leftrightarrow$ (3) follows from results of Ash and Knight.
For (3) $\Rightarrow$ (1) we note that **a** can do our construction of computable maximal linearizations.

non-uniformity

Proof of (1) $\Rightarrow$ (2):

- **a** unif. computes maximal linearizations of comp. wpos.
- Let α and β be computable ordinals.
- Consider $\mathcal{W} = (\omega^\alpha + \{\mathtt{a}\} + \omega^\alpha) \oplus (\omega^\beta + \{\mathtt{b}\} + \omega^\beta)$.
- Use **a** to get a maximal linearization $\trianglelefteq$ of $\mathcal{W}$.
- If $\alpha < \beta$, we have that $\mathrm{ordTy}(W, \trianglelefteq) = \omega^\beta + \omega^\beta + \omega^\alpha + \omega^\alpha$.
- Let $h(\alpha, \beta) = \begin{cases} 0 & \text{if } \mathtt{a} \trianglelefteq \mathtt{b} \\ 1 & \text{if } \mathtt{b} \trianglelefteq \mathtt{a}. \end{cases}$.
- If $\alpha < \beta$, then $h(\alpha, \beta) = 1$, and if $\alpha > \beta$, then $h(\alpha, \beta) = 0$.
- To tell whether $\alpha = \beta$ check if $2\alpha < 2\beta + 1$ and $2\beta < 2\alpha + 1$.