Theories of Hyperarithmetic Analysis.

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CONFERENCE IN HONOR OF THE 60th BIRTHDAY OF HARVEY M. FRIEDMAN

Friedman's ICM address

Harvey Friedman.
Some Systems of Second Order Arithmetic and Their Use.
Proceedings of the International Congress of Mathematicians, Vancouver 1974.

Sections:
I. Axioms for arithmetic sets.
II. Axioms for hyperarithmetic sets.
II. Axioms for hyperarithmetic sets.
III. Axioms for arithmetic recursion.
IV. Axioms for transfinite induction.
V. Axioms for the hyperjump.

Reverse Mathematics

Setting: Second order arithmetic.

Main Question: What axioms are necessary to prove the theorems of Mathematics?

Big Five Axiom systems:

RCA_0. Recursive Comprehension + Σ^0_1-induction + Semiring ax.
WKL_0.
ACA_0. Arithmetic Comprehension + RCA_0
Hyperarithmetic analysis (mostly here in between)
ATR_0. Arithmetic Transfinite recursion + ACA_0.
Π^1_1-CA_0.

Models

A model of (the language of) 2nd order arithmetic is a tuple

$$\langle X, \mathcal{M}, +_X, \times_X, 0_X, 1_X, \leqslant_X \rangle,$$

where $\mathcal{M}$ is a set of subsets of X.

Such a model is an *ω-model* if

$$\langle X, +_X, \times_X, 0_X, 1_X, \leqslant_X \rangle = \langle \omega, +, \times, 0, 1, \leqslant \rangle.$$

ω-models are determined by their 2nd order parts $\mathcal{M} \subseteq \mathcal{P}(\omega)$.

The class of ω-models of a theory

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω-models of $\mathsf{RCA}_0 \Leftrightarrow$
$\mathcal{M}$ is closed under Turing reduction and $\oplus$

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω-models of $\mathsf{ACA}_0 \Leftrightarrow$
$\mathcal{M}$ is closed under Arithmetic reduction and $\oplus$

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω-models of $\mathsf{ATR}_0 \Rightarrow$
$\mathcal{M}$ is closed under Hyperarithmetic reduction and $\oplus$

The class of *HYP*, of hyperarithmetic sets, is not a model of ATR_0:
There is a linear ordering $\mathcal{L}$ which isn't an ordinal but looks like one in *HYP* (the Harrison l.o.), so,

$$HYP \models \mathcal{L} \text{ is an ordinal but } 0^{(\mathcal{L})} \text{ does not exist.}$$

Hyperarithmetic sets.

Notation: Let ω_1^{CK} be the least non-computable ordinal.

Proposition [Suslin-Kleene, Ash] For a set $X \subseteq \omega$, T.F.A.E.:

- X is $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$.
- X is computable in $0^{(\alpha)}$ for some $\alpha < \omega_1^{CK}$.
  ($0^{(\alpha)}$ is the αth Turing jump of 0.)
- $X \in L(\omega_1^{CK})$.
- $X = \{x : \varphi(x)\}$, where φ is a computable infinitary formula.
  (Computable infinitary formulas are 1st order formulas which may contain infinite computable disjunctions or conjunctions.)

A set satisfying the conditions above is said to be hyperarithmetic.

Hyperarithmetic reducibility

Definition: X is *hyperarithmetic in* Y ($X \leqslant_H Y$) if $X \in \Delta^1_1(Y)$,
or equivalently, if $X \leqslant_T Y^{(\alpha)}$ for some $\alpha < \omega_1^Y$.

Let *HYP* be the class of hyperarithmetic sets.
Let $HYP(Y)$ be the class of set hyperarithmetic in Y.

We say that $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is *hyperarithmetically closed* if it is closed downwards under $\leqslant_H$ and is closed under $\oplus$.

The class of ω-models of a theory

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω-models of RCA_0 $\Leftrightarrow$
$\mathcal{M}$ is closed under Turing reduction and $\oplus$

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω-models of ACA_0 $\Leftrightarrow$
$\mathcal{M}$ is closed under Arithmetic reduction and $\oplus$

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω-models of ATR_0 $\Rightarrow$
$\mathcal{M}$ is hyperarithmetically closed.

Question: Are there theories whose ω-models are the hyperarithmetically closed ones?

Theories of Hyperarithmetic analysis.

Definition

We say that a theory T is a theory of hyperarithmetic analysis if

- every ω-model of T is hyperarithmetically closed, and
- for every Y, $HYP(Y) \models$ T.

Note that
T is a theory of hyperarithmetic analysis $\Leftrightarrow$
for every set Y, $HYP(Y)$ is the least ω-model of T containing Y,
and every ω-model of T is closed under $\oplus$.

Hence, HYP, and the relation $\leqslant_H$ can be defined in terms of T.

Choice and Comprehension schemes

The following are theories of hyperarithmetic analysis

$$\Sigma^1_1\text{-AC}_0 \Downarrow \Delta^1_1\text{-CA}_0$$

[Kreisel 62]
Σ^1_1-AC_0 (Σ^1_1-choice):

$$\forall n \exists X (\varphi(n, X)) \Rightarrow \exists X \forall n (\varphi(n, X^{[n]})),$$

where φ is Σ^1_1.

[Kleene 59]
Δ^1_1-CA_0 (Δ^1_1-comprehension) :

$$\forall n (\varphi(n) \Leftrightarrow \neg\psi(n)) \Rightarrow \exists X = \{n \in \mathbb{N} : \varphi(n)\},$$

where φ and ψ are Σ^1_1.

Theorem:[Steel 78]
Δ^1_1-CA_0 is strictly weaker than Σ^1_1-AC_0.
Pf: Steel's forcing with Tagged trees.

Choice and Comprehension schemes

Σ^1_1-DC$_0$ $\Downarrow$ Σ^1_1-AC$_0$ $\Downarrow$ Δ^1_1-CA$_0$

The following is a theory of hyperarithmetic analysis

[Harrison 68]
Σ^1_1-DC$_0$ (Σ^1_1-dependent choice):
$$\forall Y \exists Z(\varphi(Y,Z)) \Rightarrow \exists X \forall n(\varphi(X^{[n]}, X^{[n+1]})),$$
where φ is Σ^1_1.

Theorem:[Friedman Ph.D. thesis 1967]
Σ^1_1-DC$_0$, is Π^0_2-conservative over Σ^1_1-AC$_0$.
Σ^1_1-DC$_0$ is strictly stronger than Σ^1_1-AC$_0$.

Thm:[Simpson 82]
Σ^1_1-DC$_0$ is equivalent to Π^1_1-Transfinite induction.

Choice and Comprehension schemes

Σ^1_1-DC$_0$ $\Downarrow$ Σ^1_1-AC$_0$ $\Downarrow$ Δ^1_1-CA$_0$ $\Downarrow$ weak-Σ^1_1-AC$_0$

The following is a theory of hyperarithmetic analysis

weak-Σ^1_1-AC$_0$ (weak Σ^1_1-choice):

$$\forall n \exists! X(\varphi(n,X)) \Rightarrow \exists X \forall n(\varphi(n, X^{[n]})),$$

where φ is arithmetic.

Theorem:[Van Wesep 77]
weak-Σ^1_1-AC$_0$ is strictly weaker than Δ^1_1-CA$_0$.
Pf: Steel's forcing with Tagged trees.

The bad news

There is NO theory T whose ω-models are exactly the hyperarithmetically closed ones.

Theorem: [Van Wesep 77]

For every theory T whose ω-models are all hyp. closed, there is a weaker one T' whose ω-models are all hyp. closed and which has more ω-models than T.

Σ^1_1-DC_0

$\Downarrow$

Σ^1_1-AC_0

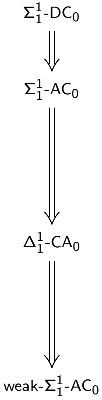
$\Downarrow$

Δ^1_1-CA_0

$\Downarrow$

weak-Σ^1_1-AC_0

Between ACA_0 and ATR_0



Obs: ACA_0 is implied by all examples of theories of HA.

Thm:[Barwise, Schlipf 75]
Σ^1_1-AC_0 is Π^1_2-conservative over ACA_0.

Corollary: [Friedman, Barwise, Schlipf]
All examples of theories of HA are equi-consistent with PA.

Thm: [Friedman 67] $ATR_0 \vdash \Sigma^1_1$-AC_0

Thm: [Friedman 67] $ATR_0 \nvdash \Sigma^1_1$-DC_0

Statements of hyperarithmetic analysis

Σ^1_1-DC_0

$\Downarrow$

Σ^1_1-AC_0

$\Downarrow$

Δ^1_1-CA_0

$\Downarrow$

weak-Σ^1_1-AC_0

Definition

S is a sentence of hyperarithmetic analysis if RCA_0+S is a theory of hyperarithmetic analysis.

Statements of hyperarithmetic analysis

[Friedman ICM 74] *Arithmetic Bolzano-Weierstrass*

ABW: Every arithmetic bounded infinite class in $\mathbb{R}$ has an accumulation point.

$\Sigma^1_1\text{-AC}_0 \Rightarrow \text{ABW}$

More on ABW later.

[Friedman ICM 74] *Sequential Limit system*

SL: Every accumulation point of an arithmetic class in $\mathbb{R}$ is a limit of some sequence of points in that class.

$\Sigma^1_1\text{-AC}_0 \Leftrightarrow \text{SL}$

[Van Wesep 1977] *Determined-Open-Game Axiom-of-Choice.*

DOG-AC: If we have a sequence of open games such that player *II* has a winning strategy in each of them, then there exists a sequence of strategies for all of them.

$\Sigma^1_1\text{-AC}_0 \Leftrightarrow \text{DOG-AC}$

$\Sigma^1_1\text{-DC}_0$
$\Downarrow$
$\Sigma^1_1\text{-AC}_0$
$\Downarrow$
$\Delta^1_1\text{-CA}_0$
$\Downarrow$
weak-$\Sigma^1_1\text{-AC}_0$

A weaker statement of HA

[M 04] *The Jump Iteration statement*:
JI: $\forall X \forall \alpha (\alpha$ an ordinal & $\forall \beta < \alpha (X^{(\beta)}$ exists$) \Rightarrow X^{(\alpha)}$ exists$)$

JI is a statement of hyperarithmetic analysis.

Theorem ([M 04])

JI is strictly weaker than weak-Σ^1_1-AC_0.

Pf: Steel's forcing w Tagged trees.

Σ^1_1-DC_0
$\Downarrow$
Σ^1_1-AC_0
$\Downarrow$
Δ^1_1-CA_0
$\Downarrow$
weak-Σ^1_1-AC_0
$\Downarrow$
JI

A natural mathematical statement

$\mathcal{A}$, $\mathcal{B}$ denote linear orderings. If $\mathcal{A}$ embeds into $\mathcal{B}$, we write $\mathcal{A} \preccurlyeq \mathcal{B}$.

Theorem[Jullien '69] INDEC: If $\mathcal{L}$ is a l.o. such that $\mathbb{Q} \not\preccurlyeq \mathcal{L}$ and whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$, either $\mathcal{L} \preccurlyeq \mathcal{A}$ or $\mathcal{L} \preccurlyeq B$, then
either whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$, $\mathcal{L} \preccurlyeq \mathcal{A}$,
or whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$, $\mathcal{L} \preccurlyeq \mathcal{B}$.

Theorem ([M 04])

On ω-models, Δ^1_1-CA_0 $\Rightarrow$ INDEC $\Rightarrow$ JI.

Pf: Uses results of Ash-Knight which use the Ash's method of α-systems.

INDEC is, so far, the most natural statement of HA as it doesn't use notions from logic.

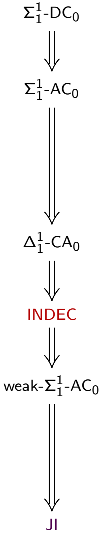
Σ^1_1-DC_0
$\Downarrow$
Σ^1_1-AC_0
$\Downarrow$
Δ^1_1-CA_0
$\Downarrow$
weak-Σ^1_1-AC_0
$\Downarrow$
JI

Neeman's work

Theorem: [Neeman 08]
$RCA_0 + \Sigma^1_1$-induction$\vdash$ Δ^1_1-$CA_0 \Rightarrow$ INDEC $\Rightarrow$ weak-Σ^1_1-AC_0.

Moreorver, the implications can't be reversed.

Pf: Steel's forcing w Tagged trees.



$L_{\omega_1,\omega}$-Comprehension

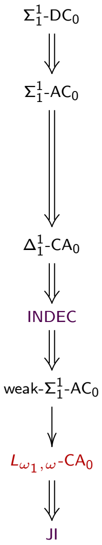
$L_{\omega_1,\omega}$ is the set of infinitary formulas or arithmetic where one is allowed to have infinitary disjunctions or conjunctions.

A formula φ is *determined* if there is a map $v\colon \text{Subformulas}(\varphi) \to \{T, F\}$ such that...(the obvious logic rules hold.)
φ *is true* if it is determined by v and $v(\varphi) = T$.

$L_{\omega_1,\omega}$-CA_0: Let $\{\varphi_i : i \in \mathbb{N}\}$ be detemined $L_{\omega_1,\omega}$-sentences. Then, there exists a set $X = \{i \in \mathbb{N} : \varphi_i \text{ is true}\}$.

Thm:[M 04] $RCA_0 \vdash$ weak-Σ^1_1-$AC_0 \Rightarrow L_{\omega_1,\omega}$-$CA_0 \Rightarrow$ JI.
The second implication cannot be reversed.

[M 04] used games instead of $L_{\omega_1,\omega}$, and considered various versions



Pi-one-one separation

Π^1_1-sep: $\forall n(\neg\varphi(n) \vee \neg\psi(n)) \Rightarrow$

$$\exists X \forall n\ (\varphi(n) \Rightarrow n \in X\ \&\ \psi(n) \Rightarrow n \notin X)$$

where φ and ψ are Π^1_1.

Observation[Tanaka] Σ^1_1-$AC_0 \Rightarrow \Pi^1_1$-sep $\Rightarrow \Delta^1_1$-CA_0

Theorem ([M07])

Π^1_1-sep is strictly in between Σ^1_1-AC_0 and Δ^1_1-CA_0

Σ^1_1-DC_0
$\Downarrow$
Σ^1_1-AC_0
$\Downarrow$
Π^1_1-sep
$\Downarrow$
Δ^1_1-CA_0
$\Downarrow$
INDEC
$\Downarrow$
weak-Σ^1_1-AC_0
$\downarrow$
$L_{\omega_1,\omega}$-CA_0
$\Downarrow$
JI

A curiosity

Σ^1_1-collection
Σ^1_1-COL: $\forall n \exists X(\varphi(n, X)) \Rightarrow \exists X \forall n \exists m(\varphi(n, X^{[m]}))$ where φ is Σ^1_1.

Obs [Tanaka]: $\mathsf{ACA}_0 + \Sigma^1_1$-COL $\Leftrightarrow$ Σ^1_1-AC_0.

Thm [M, Tanaka, Yamazaki]:
Σ^1_1-COL is Π^1_1-conservative over RCA_0.

Σ^1_1-DC_0
$\Downarrow$
Σ^1_1-AC_0
$\Downarrow$
$\Pi^1_1 - sep$
$\Downarrow$
Δ^1_1-CA_0
$\Downarrow$
INDEC
$\Downarrow$
weak-Σ^1_1-AC_0
$\downarrow$
$L_{\omega_1,\omega}$-CA_0
$\Downarrow$
JI

Arithmetic Bolsano-Wierstrass

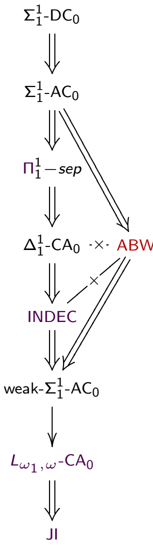
[Friedman ICM 74] *Arithmetic Bolzano-Weierstrass*
ABW: Every arithmetic bounded infinite class in $\mathbb{R}$ has an accumulation point.

Obs[Conidis 09]: G_δ-BW$\Rightarrow$ weak-Σ^1_1-AC_0.

Work in progress [Conidis 09]: ABW$\not\Rightarrow$ INDEC.

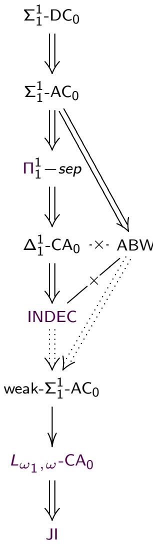
Obs[Friedman ICM 74]: Σ^1_1-$AC_0$$\Rightarrow$ABW.

Work in progress[Conidis 09]: Δ^1_1-$CA_0$$\not\Rightarrow$ ABW.
Pf: Steel's forcing w Tagged trees



Necessity of Σ^1_1-induction

Work in progress[Neeman]:
Without Σ^1_1-induction, RCA_0+INDEC$\not\vdash$ weak-Σ^1_1-AC_0.
Pf: Build a non-standard model, using elementary extensions of Steel's forcing w Tagged trees.



Questions

Q1: What are other natural statements of HA?

Q2: Does $L_{\omega_1,\omega}$-$CA_0 \Rightarrow$ weak-Σ^1_1-AC_0?