Theories of Hyperarithmetic Analysis.

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*Some Systems of Second Order Arithmetic and Their Use.* 

Sections:
I. Axioms for arithmetic sets.
II. Axioms for hyperarithmetic sets.
III. Axioms for arithmetic recursion.
IV. Axioms for transfinite induction.
V. Axioms for the hyperjump.
Setting: Second order arithmetic.

Main Question: What axioms are necessary to prove the theorems of Mathematics?

Big Five Axiom systems:

RCA<sub>0</sub>. Recursive Comprehension + $\Sigma^0_1$-induction + Semiring ax.

WKL<sub>0</sub>.

ACA<sub>0</sub>. Arithmetic Comprehension + RCA<sub>0</sub>

Hyperarithmetic analysis (mostly here in between)

ATR<sub>0</sub>. Arithmetic Transfinite recursion + ACA<sub>0</sub>

$\Pi^1_1$-CA<sub>0</sub>.
A model of (the language of) 2nd order arithmetic is a tuple

\[ \langle X, \mathcal{M}, +_X, \times_X, 0_X, 1_X, \leq_X \rangle, \]

where \( \mathcal{M} \) is a set of subsets of \( X \).

Such a model is an \( \omega \)-model if

\[ \langle X, +_X, \times_X, 0_X, 1_X, \leq_X \rangle = \langle \omega, +, \times, 0, 1, \leq \rangle. \]

\( \omega \)-models are determined by their 2nd order parts \( \mathcal{M} \subseteq \mathcal{P}(\omega) \).
Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an $\omega$-models of RCA$_0$ $\iff$ 
$\mathcal{M}$ is closed under Turing reduction and $\oplus$

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an $\omega$-models of ACA$_0$ $\iff$ 
$\mathcal{M}$ is closed under Arithmetic reduction and $\oplus$

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an $\omega$-models of ATR$_0$ $\Rightarrow$ 
$\mathcal{M}$ is closed under Hyperarithmetic reduction and $\oplus$

The class of $HYP$, of hyperarithmetic sets, is not a model of ATR$_0$: 
There is a linear ordering $\mathcal{L}$ which isn’t an ordinal but looks like one in $HYP$ (the Harrison l.o.), so, 
$HYP \models \mathcal{L}$ is an ordinal but $0(\mathcal{L})$ does not exist.
Notation: Let $\omega_{1}^{CK}$ be the least non-computable ordinal.

Proposition [Suslin-Kleene, Ash] For a set $X \subseteq \omega$, T.F.A.E.:

- $X$ is $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$.
- $X$ is computable in $0^{(\alpha)}$ for some $\alpha < \omega_{1}^{CK}$.

$(0^{(\alpha)}$ is the $\alpha$th Turing jump of 0.)

- $X \in L(\omega_{1}^{CK})$.
- $X = \{x : \varphi(x)\}$, where $\varphi$ is a computable infinitary formula.

(Computable infinitary formulas are 1st order formulas which may contain infinite computable disjunctions or conjunctions.)

A set satisfying the conditions above is said to be hyperarithmetic.
Definition: $X$ is hyperarithmetic in $Y$ ($X \leq_H Y$) if $X \in \Delta^1_1(Y)$, or equivalently, if $X \leq_T Y^{(\alpha)}$ for some $\alpha < \omega_1^Y$.

Let $HYP$ be the class of hyperarithmetic sets. Let $HYP(Y)$ be the class of set hyperarithmetic in $Y$.

We say that $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is hyperarithmetically closed if it is closed downwards under $\leq_H$ and is closed under $\oplus$. 
The class of ω-models of a theory

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω-models of RCA$_0$ ⇔
$\mathcal{M}$ is closed under Turing reduction and ⊕

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω-models of ACA$_0$ ⇔
$\mathcal{M}$ is closed under Arithmetic reduction and ⊕

Observation: $\mathcal{M} \subseteq \mathcal{P}(\omega)$ is an ω-models of ATR$_0$ ⇒
$\mathcal{M}$ is hyperarithmetically closed.

Question: Are there theories whose ω-models are the hyperarithmetically closed ones?
Theories of Hyperarithmetic Analysis.

Definition

We say that a theory $T$ is a **theory of hyperarithmetic analysis** if

- every $\omega$-model of $T$ is hyperarithmetically closed, and
- for every $Y$, $\text{HYP}(Y) \models T$.

Note that $T$ is a theory of hyperarithmetic analysis $\iff$
for every set $Y$, $\text{HYP}(Y)$ is the **least $\omega$-model** of $T$ containing $Y$, and every $\omega$-model of $T$ is closed under $\oplus$.

Hence, $\text{HYP}$, and the relation $\leq_H$ can be **defined in terms of $T$**.
The following are theories of hyperarithmetic analysis

[Kreisel 62]
$\Sigma_1^1$-AC$_0$ ($\Sigma_1^1$-choice):
\[ \forall n \exists X (\varphi(n, X)) \Rightarrow \exists X \forall n (\varphi(n, X[n])), \]
where $\varphi$ is $\Sigma_1^1$.

[Kleene 59]
$\Delta_1^1$-CA$_0$ ($\Delta_1^1$-comprehension):
\[ \forall n (\varphi(n) \iff \neg \psi(n)) \Rightarrow \exists X = \{ n \in \mathbb{N} : \varphi(n) \}, \]
where $\varphi$ and $\psi$ are $\Sigma_1^1$.

**Theorem:** [Steel 78]
$\Delta_1^1$-CA$_0$ is strictly weaker than $\Sigma_1^1$-AC$_0$.

**Pf:** Steel’s forcing with Tagged trees.
The following is a theory of hyperarithmetic analysis

\[ \Sigma^1_1-\text{DC}_0 \ (\Sigma^1_1\text{-dependent choice}): \]
\[ \forall Y \exists Z (\varphi(Y, Z)) \implies \exists X \forall n (\varphi(X[n], X[n+1])), \]
where \( \varphi \) is \( \Sigma^1_1 \).

**Theorem:** [Friedman Ph.D. thesis 1967]
\( \Sigma^1_1\text{-DC}_0 \), is \( \Sigma^0_2 \)-conservative over \( \Sigma^1_1\text{-AC}_0 \).
\( \Sigma^1_1\text{-DC}_0 \) is strictly stronger than \( \Sigma^1_1\text{-AC}_0 \).

**Thm:** [Simpson 82]
\( \Sigma^1_1\text{-DC}_0 \) is equivalent to \( \Pi^1_1 \)-Transfinite induction.
The following is a theory of hyperarithmetic analysis

\[ \text{weak-}\Sigma^1_1\text{-AC}_0 \quad \text{(weak } \Sigma^1_1\text{-choice):} \]
\[ \forall n \exists ! X(\varphi(n, X)) \Rightarrow \exists X \forall n(\varphi(n, X[n])), \quad \text{where } \varphi \text{ is arithmetic.} \]

**Theorem:** [Van Wesep 77]

weak-\(\Sigma^1_1\text{-AC}_0\) is strictly weaker than \(\Delta^1_1\text{-CA}_0\).

**Pf:** Steel’s forcing with Tagged trees.
The bad news

There is NO theory $T$ whose $\omega$-models are exactly the hyperarithmetically closed ones.

**Theorem:** [Van Wesep 77]

For every theory $T$ whose $\omega$-models are all hyp. closed, there is a weaker one $T'$ whose $\omega$-models are all hyp. closed and which has more $\omega$-models than $T$.
**Obs:** ACA₀ is implied by all examples of theories of HA.

**Thm:** [Barwise, Schlipf 75]
Σ₁¹-AC₀ is Π₂¹-conservative over ACA₀.

**Corollary:** [Friedman, Barwise, Schlipf]
All examples of theories of HA are equi-consistent with PA.

**Thm:** [Friedman 67] ATR₀ ⊢ Σ₁¹-AC₀

**Thm:** [Friedman 67] ATR₀ ⊬ Σ₁¹-DC₀
**Definition**

S is a **sentence of hyperarithmetic analysis** if RCA$_0$ + S is a theory of hyperarithmetic analysis.
Hyperarithmetic analysis in the 70’s
Hyperarithmetic analysis in the 00’s

Background
Known theories

Statements of hyperarithmetic analysis

[Friedman ICM 74] **Arithmetic Bolzano-Weierstrass**

**ABW**: Every arithmetic bounded infinite class in $\mathbb{R}$ has an accumulation point.

$\Sigma^1_1$-$AC_0 \Rightarrow$ ABW

More on ABW later.

[Friedman ICM 74] **Sequential Limit system**

**SL**: Every accumulation point of an arithmetic class in $\mathbb{R}$ is a limit of some sequence of points in that class.

$\Sigma^1_1$-$AC_0 \Leftrightarrow$ SL

[Van Wesep 1977] **Determined-Open-Game Axiom-of-Choice.**

**DOG-AC**: If we have a sequence of open games such that player II has a winning strategy in each of them, then there exists a sequence of strategies for all of them.

$\Sigma^1_1$-$AC_0 \Leftrightarrow$ DOG-AC

\[ \Sigma^1_1$-$DC_0 \downarrow \Sigma^1_1$-$AC_0 \]

\[ \Sigma^1_1$-$AC_0 \downarrow \Delta^1_1$-$CA_0 \]

\[ \Delta^1_1$-$CA_0 \downarrow \text{weak-} \Sigma^1_1$-$AC_0 \]}
Hyperarithmetic analysis in the 70's
Hyperarithmetic analysis in the 00's

A weaker statement of HA

[M 04] The Jump Iteration statement:
\[ \forall X \forall \alpha (\alpha \text{ an ordinal } \& \forall \beta < \alpha (X^{(\beta)} \text{ exists}) \Rightarrow X^{(\alpha)} \text{ exists}) \]

JI is a statement of hyperarithmetic analysis.

Theorem ([M 04])

\[ \text{JI is strictly weaker than weak-} \Sigma^1_1-\text{AC}_0. \]

\textbf{Pf:} Steel’s forcing w Tagged trees.

\[ \Sigma^1_1-\text{DC}_0 \quad \downarrow \quad \Sigma^1_1-\text{AC}_0 \quad \downarrow \quad \Delta^1_1-\text{CA}_0 \quad \downarrow \quad \text{weak-} \Sigma^1_1-\text{AC}_0 \quad \downarrow \quad \text{JI} \]
A natural mathematical statement

$A$, $B$ denote linear orderings. If $A$ embeds into $B$, we write $A \preceq B$.

**Theorem [Jullien ’69] INDEC:** If $\mathcal{L}$ is a l.o. such that $\mathbb{Q} \not\preceq \mathcal{L}$ and whenever $\mathcal{L} = A + B$, either $\mathcal{L} \preceq A$ or $\mathcal{L} \preceq B$, then either whenever $\mathcal{L} = A + B$, $\mathcal{L} \preceq A$, or whenever $\mathcal{L} = A + B$, $\mathcal{L} \preceq B$.

**Theorem ([M 04])**

**On $\omega$-models, $\Delta^1_1$-CA$_0 \Rightarrow$ INDEC $\Rightarrow$ JI.**

**Pf:** Uses results of Ash-Knight which use the Ash’s method of $\alpha$-systems.

INDEC is, so far, the most natural statement of HA as it doesn’t use notions from logic.
Theorem: [Neeman 08]  
$\text{RCA}_0 + \Sigma^1_1$-induction $\vdash \Delta^1_1$-$\text{CA}_0 \Rightarrow \text{INDEC} \Rightarrow \text{weak-} \Sigma^1_1$-$\text{AC}_0$.  

Moreover, the implications can't be reversed.  

Pf: Steel's forcing with Tagged trees.
$L_{\omega_1,\omega}$-Comprehension

$L_{\omega_1,\omega}$ is the set of infinitary formulas or arithmetic where one is allowed to have infinitary disjunctions or conjunctions.

A formula $\varphi$ is *determined* if there is a map $\nu : \text{Subformulas}(\varphi) \to \{T, F\}$ such that...(the obvious logic rules hold.)

$\varphi$ is *true* if it is determined by $\nu$ and $\nu(\varphi) = T$.

$L_{\omega_1,\omega}$-$\text{CA}_0$: Let $\{\varphi_i : i \in \mathbb{N}\}$ be determined $L_{\omega_1,\omega}$-sentences. Then, there exists a set $X = \{i \in \mathbb{N} : \varphi_i \text{ is true}\}$.

**Thm:** [M 04] $\text{RCA}_0 \vdash \text{weak-}\Sigma_1^1$-$\text{AC}_0 \Rightarrow L_{\omega_1,\omega}$-$\text{CA}_0 \Rightarrow \text{JI}$. The second implication cannot be reversed.

[M 04] used games instead of $L_{\omega_1,\omega}$, and considered various versions...
\[ \Pi^1_1\text{-sep: } \forall n (\neg \varphi(n) \lor \neg \psi(n)) \Rightarrow \exists X \forall n (\varphi(n) \Rightarrow n \in X \land \psi(n) \Rightarrow n \notin X) \]

where \( \varphi \) and \( \psi \) are \( \Pi^1_1 \).

**Observation** [Tanaka] \( \Sigma^1_1\text{-AC}_0 \Rightarrow \Pi^1_1\text{-sep} \Rightarrow \Delta^1_1\text{-CA}_0 \)

**Theorem ([M07])**

\( \Pi^1_1\text{-sep is strictly in between } \Sigma^1_1\text{-AC}_0 \text{ and } \Delta^1_1\text{-CA}_0 \)
Hyperarithmetic analysis in the 70’s
Hyperarithmetic analysis in the 00’s

A curiosity

\[ \Sigma^1_1 \text{-collection} \]
\[ \Sigma^1_1 \text{-COL}: \forall n \exists X(\varphi(n, X)) \Rightarrow \exists X \forall n \exists m(\varphi(n, X[m])) \]

where \( \varphi \) is \( \Sigma^1_1 \).

**Obs [Tanaka]:** ACA\(_0^+\Sigma^1_1\text{-COL} \iff \Sigma^1_1\text{-AC}_0.\)

**Thm [M, Tanaka, Yamazaki]:**
\( \Sigma^1_1\text{-COL} \) is \( \Pi^1_1 \)-conservative over RCA\(_0^+\).

\[ \Sigma^1_1\text{-DC}_0 \]
\[ \Sigma^1_1\text{-AC}_0 \]
\[ \Pi^1_1\text{-sep} \]
\[ \Delta^1_1\text{-CA}_0 \]
\[ \text{INDEC} \]
\[ \text{weak-}\Sigma^1_1\text{-AC}_0 \]
\[ L_{\omega_1, \omega^-}\text{-CA}_0 \]
\[ \text{JI} \]
[Friedman ICM 74] *Arithmetic Bolzano-Weierstrass*

**ABW**: Every arithmetic bounded infinite class in \( \mathbb{R} \) has an accumulation point.

**Obs**[Conidis 09]: \( G_\delta \)-BW \( \Rightarrow \) weak-\( \Sigma^1_1 \)-AC\(_0\).

**Work in progress** [Conidis 09]: ABW \( \not\Rightarrow \) INDEC.

**Obs**[Friedman ICM 74]: \( \Sigma^1_1 \)-AC\(_0\) \( \Rightarrow \) ABW.

**Work in progress**[Conidis 09]: \( \Delta^1_1 \)-CA\(_0\) \( \not\Rightarrow \) ABW.

**Pf**: Steel’s forcing w Tagged trees
Necessity of $\Sigma^1_1$-induction

**Work in progress** [Neeman]:
Without $\Sigma^1_1$-induction, $\text{RCA}_0 + \text{INDEC}$ $\not\vdash$ weak-$\Sigma^1_1$-$\text{AC}_0$.

**Pf:** Build a non-standard model, using elementary extensions of Steel’s forcing with Tagged trees.

\[
\begin{align*}
\Sigma^1_1\text{-DC}_0 & \downarrow \downarrow \\
\Sigma^1_1\text{-AC}_0 & \downarrow \\
\Pi^1_1 \text{-sep} & \downarrow \\
\Delta^1_1\text{-CA}_0 \times \text{ABW} & \downarrow \\
\text{INDEC} & \downarrow \\
\text{weak-}\Sigma^1_1\text{-AC}_0 & \downarrow \\
L_{\omega_1, \omega}\text{-CA}_0 & \downarrow \\
\text{JI} &
\end{align*}
\]
Questions

Q1: What are other natural statements of HA?

Q2: Does $L_{\omega_1,\omega}$-$CA_0 \Rightarrow$ weak-$\Sigma^1_1$-$AC_0$?