Theories of Classes of Structures

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Ketonen's question

Let BA be the class of countable Boolean algebras,
and $\oplus$ the product operation

Question ([Ketonen 78])

Is the theory of (BA; $\oplus$) decidable?

Tarski's Cube Problem (1950's):
Is there $\mathcal{A} \in$ BA with $\mathcal{A} \cong \mathcal{A} \oplus \mathcal{A} \oplus \mathcal{A} \not\cong \mathcal{A} \oplus \mathcal{A}$

Thm:[Ketonen 78] Any commutative semigroup embeds into (BA; $\oplus$).

Corollary: The $\exists$-theory of (BA; $\oplus$) is decidable.

Th(BA; $\oplus$) is undecidable

Theorem ([Kach, M])

The theory of $(BA; \oplus)$ is 1-equivalent to true 2nd-order arithmetic.

Proof:

- We encode $(\mathbb{N}, \mathcal{P}(\mathbb{N}^3); \leqslant)$ instead of $(\mathbb{N}, \mathcal{P}(\mathbb{N}); \leqslant, +, \times)$.
- Encode an integer $n \in \mathbb{N}$ by the interval algebra of $\omega^n \cdot (1 + \eta)$.
- Given $\mathcal{B} \in$ BA, we define $S_3(\mathcal{B}) \subseteq \mathbb{N}^3$ as follows:

$$S_3(\mathcal{B}) = \{(n_1, n_2, n_3) \in \mathbb{N}^3 : \text{IntAlg}\Big(\sum_{i \in 1+\eta} (\omega^{n_1} \cdot (1+\eta) + \omega^{n_2} \cdot (1+\eta) + \omega^{n_3} \cdot (1+\eta))\Big) \text{ is a direct summand of } \mathcal{B}\}.$$

More Questions About $\mathbb{BA}_{\aleph_0}^{\oplus}$...

Conjecture

The theory of $(\mathsf{BA}_\kappa; \oplus)$, for $\kappa > \aleph_0$, computes true 2nd-order arithmetic.

Remark The theories of $(\mathsf{BA}; \oplus)$ and $(\mathsf{BA}_\kappa; \oplus)$ differ for $\kappa > \aleph_0$:
The former has exactly two [nontrivial] minimal elements, namely the atom and the atomless algebra; the latter has more.

Our proof is not known to work for $\kappa > \aleph_0$.

Question

Is the structure $(\mathsf{BA}; \oplus)$ bi-interpretable with 2nd-order arithmetic?

(LO; $+$) is undecidable.

Let LO be the class of countable linear orderings,
and $+$ the concatenation operation

Theorem ([Kach, M])

The theory of (LO; $+$) is 1-equivalent to true 2nd-order arithmetic.

Proof:

- We encode $(\mathbb{N}, \mathcal{P}(\mathbb{N}^3); \leqslant)$.
- Encode $n \in \mathcal{N}$ by the linear ordering $\mathbf{n}$ with n elements.
- Every lin. ord. $\mathcal{A}$ encodes a set
$$S_3(\mathcal{A}) = \{(n_1, n_2, n_3) \in \mathbb{N}^3 : \zeta^2 + \mathbf{n_1} + \zeta + \mathbf{n_2} + \zeta + \mathbf{n_3} + \zeta^2 \text{ is a segment of } \mathcal{A}\}$$

Bi-interpretability

Theorem

The structure (LO; $+$) is bi-interpretable with 2nd-order arithmetic.

That is, the set
$\{(\mathcal{A},\mathcal{L}) : S_2(\mathcal{A}) \subseteq \mathbb{N}^2$ codes a lin.ord. isomorphic to $\mathcal{L}\} \subseteq \mathrm{LO}^2$
is definable in (LO; $+$).

Corollary: The structure (LO; $+$) is rigid.

Corollary: Every $K \subseteq \mathrm{LO}^n$ definable in 2nd-order arithmetic
is definable in (LO; $+$).

Examples The following are definable in 2nd-order arithmetic:

- The set of scattered LO.
- The set of triples (x, y, z) of order types such that $x \cdot y = z$.
- The set pairs (x, y) such that x has Hausdorff rank y.

Answers for $\mathbb{LO}_c^+$...

Let LO_c be the set of all computable order types.

Theorem ([Kach, M])

The theory of $(\mathrm{LO}_c; +)$ is 1-equivalent to the ωth jump of Kleene's $\mathcal{O}$.

Proof:

- For $\leqslant_1$, note that $\mathcal{O}$ suffices to determine if two computable order types are isomorphic. So, $\mathcal{O}$ computes a presentation of $(\mathrm{LO}_c; +)$.
- For $\geqslant_1$, we code $(\mathbb{N}; \leqslant, +, \times, \mathcal{O})$ in $(\mathrm{LO}_c; +)$.

Answers for $\mathbb{GR}_{\kappa}^{\times,\leqslant}$...

Let GR be the class of countable groups,
$\times$ the product operation, and $\leqslant$ the sub-group relation.

Theorem ([Kach, M])

The theory of $(GR; \times, \leqslant)$ is 1-equivalent to true 2nd-order arithmetic.

Proof

- Let z be a minimal group. I.e. $z \cong \mathbb{Z}$ or $z \cong \mathbb{Z}_p$.
- Encode the integer $n \in \mathbb{N}$ by the group z^n.
- Coding sets of triples becomes tricky.
- Decoding using Kurosch's Theorem about the sub-groups of a free produce.

Further results

Theorem ([Tamvana Makuluni])

The theory of $(GR; \leqslant)$ is 1-equivalent to true 2nd-order arithmetic.

Theorem ([Tamvana Makuluni])

The first-order theory of countable fields with the subfield relation is 1-equivalent to true 2nd-order arithmetic.