On the jump of a structure.

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Idea

In computable mathematics we want to understand the interaction between computational notions and structural notions.

We want to consider the Turing jump.

Def: The *degree spectrum* of a structure $\mathcal{L}$ is

$$DegSp(\mathcal{L}) = \{deg(\mathcal{A}) : \mathcal{A} \cong \mathcal{L}\} = \{\mathbf{x} : \mathbf{x} \text{ computes copy of } \mathcal{L}\}.$$

We want the *jump of $\mathcal{L}$* to be a structure $\mathcal{L}'$ such that

$$DegSp(\mathcal{L}') \cap \mathcal{D}_{(\geq 0')} = \{\mathbf{x}' : \mathbf{x} \in DegSp(\mathcal{L})\}.$$

Succesivities on Linear orderings

Let $\mathcal{A}$ be a linear ordering.
Let $Succ = \{(a, b) \in \mathcal{A}^2 : a < b \;\&\; \nexists c \; (a < c < b)\}$.

Obs: If $\mathcal{A} \leq_T \mathbf{x}$, then $(\mathcal{A}, Succ) \leq_T \mathbf{x}'$.

Thm: If $0' \geq_T \mathbf{y}'$ and $\mathbf{y} \geq_T (\mathcal{A}, Succ)$, then
$\exists \mathbf{x}$ s.t. $\mathbf{x}' \equiv_T \mathbf{y}$ and $\mathbf{x}$ computes copy of $\mathcal{A}$.

So,

$$DegSp(\mathcal{A}, Succ) \cap \mathcal{D}_{(\geq 0')} = \{\mathbf{x}' : \; \mathbf{x} \in DegSp(\mathcal{A})\}$$

We would like to set $\mathcal{A}' = (\mathcal{A}, Succ)$.

Atoms on Boolean algebras

Let $\mathcal{B}$ be a Boolean algebra.
Let $At = \{a \in \mathcal{B} : 0 < a \,\&\, \nexists c\ (0 < c < a)\}$.

Obs: If $\mathcal{B} \leq_T \mathbf{x}$, then $(\mathcal{B}, At) \leq_T \mathbf{x}'$.

Thm[Downey, Jockusch 94]:
If $(\mathcal{B}, At) \leq_T \mathbf{x}'$ then $\mathbf{x}$ computes copy of $\mathcal{B}$.

So,

$$DegSp(\mathcal{B}, At) \cap \mathcal{D}_{(\geq 0')} = \{\mathbf{x}' : \ \mathbf{x} \in DegSp(\mathcal{B})\}$$

We would like to set $\mathcal{B}' = (\mathcal{B}, At)$.

Atomless and inftinite on Boolean algebras

Let $\mathcal{B}$ be a Boolean algebra.
Let $Atless = \{a \in \mathcal{B} : 0 < a \,\&\, \nexists c\ (c < a \,\&\, c \in At)\}$.
Let $Inf = \{a \in \mathcal{B} : \exists^\infty c\ (c < a)\} = \{a : a \text{ is not a finite sum of atoms}\}$.

Obs: If $(\mathcal{B}, At) \leq_T \mathbf{x}$, then $(\mathcal{B}, At, Atless, Inf) \leq_T \mathbf{x}'$.

Thm[Thurber 95]:
If $(\mathcal{B}, At, Atless, Inf) \leq_T \mathbf{x}'$ then $\mathbf{x}$ computes a copy of $(\mathcal{B}, At)$.

So,

$$DegSp(\mathcal{B}, At, Atless, Inf) \cap \mathcal{D}_{(\geq 0')} = \{\mathbf{x}' : \ \mathbf{x} \in DegSp(\mathcal{B}, At)\}$$

We would like to set $(\mathcal{B}, At, Atless, Inf) = (\mathcal{B}, At)' = \mathcal{B}''$.

Complete set of Π^c_1 relations

A Π^c_1 *$\mathcal{L}$-formula* is of the form $\bigwedge_{j\in\omega} \forall \bar{y}\ \psi_j(\bar{z},\bar{y})$
where $\{\psi_j : j \in \omega\}$ is a comp. list of finitary quantifier-free $\mathcal{L}$-formulas.

Let P_0, P_1,... be relations Π^c_1 on $\mathcal{A}$.

Definition ([M])

$\{P_0, P_1, ...\}$ is a *complete set of Π^c_1 relations on* $\mathcal{A}$ if
every Π^c_1 $\mathcal{L}$-formula is equivalent to a $\Sigma^{c,0'}_1$ $(\mathcal{L} \cup \{P_0, ...\})$-formula.

A $\Sigma^{c,0'}_1$ *$(\mathcal{L} \cup \{P_0, ...\})$-formula* is of the form $\bigvee_{j\in\omega} \exists \bar{y}_j\ \psi_j(\bar{z},\bar{y})$
where $\{\psi_j : j \in \omega\}$ is a $0'$-comp. list of finitary quantifier-free
$(\mathcal{L} \cup \{P_0, ...\})$-formulas.

Examples:

On a Boolean algebra, the atom relation is a complete Π^c_1 relation.
On a linear order, the successor relation is a complete Π^c_1 relation.

Complete set of Π^c_1 relations

Let P_0, P_n,... be relations uniformly Π^c_n on $\mathcal{A}$.

Definition ([M])

$\{P_0, P_1, ...\}$ is a *complete set of Π^c_n relations on* $\mathcal{A}$ if every $\Pi^{c,Z}_n$ $\mathcal{L}$-form. is unif- equivalent to a $\Sigma^{c,Z'}_1$ $(\mathcal{L} \cup \{P_0, ...\})$form.

Theorem (Harris, M.)

On Boolean algebras,

$\forall n$, there is a finite complete set of Π^c_n relations.

More examples haven been cooked up for applications.
Q: What are other natural examples?

The jump of a structure

Lemma ([M])

*Let P_0, P_1,... be a complete set of Π^c_1 relations on $\mathcal{A}$.
If $Y \geq_T 0'$ computes a copy of $(\mathcal{A}, P_0, P_1, ...)$, then
$\exists$ X that computes a copy of $\mathcal{A}$ and $X' \equiv_T Y$.*

So, $$DegSp(\mathcal{A}, P_0, P_1, ...) \cap \mathcal{D}_{(\geq 0')} = \{\mathbf{x}' : \mathbf{x} \in DegSp(\mathcal{A})\}$$

Definition ([M])

Let $\mathcal{A}$ be an $\mathcal{L}$-structure.
The *jump of $\mathcal{A}$* is an $\mathcal{L}_1$-structure $\mathcal{A}'$ where:
$\mathcal{L}_1$ is $\mathcal{L} \cup \{P_0, P_1, ...\}$,
and $\mathcal{A}' = (\mathcal{A}, P_0, P_1, ...)$.

Obs: The jump of a structure is not unique,
but it is essentially unique in a sense.

restating previous lemma:

Lemma ([M])

$\mathcal{A}'$ has a Y-comp. copy $\implies$

$\exists X\ (X' \equiv_T Y)$ *and $\mathcal{A}$ has X-comp copy.*

Proof.

Use Ash,Knight, Mennasse,Slaman; Chisholm ideas to build a 1-generic copy of $\mathcal{A}$ computable in $\mathcal{A}'$.
The point is that $\mathcal{A}'$ has enough information to find conditions deciding Σ_1-facts of the generic. $\square$

Second jump inversions

Definition ([M])

A structure $\mathcal{A}$ *admits Jump Inversion* if
for every X,

$$\mathcal{A}' \text{ has copy } \leq_T X' \iff \mathcal{A} \text{ has copy } \leq_T X$$

Observation If $\mathcal{A}$ admits Jump Inversion and $X' = Y'$, then

$$\mathcal{A} \text{ has copy } \leq_T X \iff \mathcal{A} \text{ has copy } \leq_T Y.$$

Example: Boolean algebras

Let $\mathcal{B}$ be a Boolean algebra.

Lemma ([Harris, M. 09])

$\mathcal{B}' = (\mathcal{B}, At^{\mathcal{B}})$
$\mathcal{B}'' = (\mathcal{B}, At^{\mathcal{B}}, Inf^{\mathcal{B}}, Atless^{\mathcal{B}})$.
$\mathcal{B}''' = (\mathcal{B}, At^{\mathcal{B}}, Inf^{\mathcal{B}}, Atless^{\mathcal{B}}, atomic^{\mathcal{B}}, 1\text{-}atom^{\mathcal{B}}, atominf^{\mathcal{B}})$.
$\mathcal{B}^{(4)} = (\mathcal{B}, At^{\mathcal{B}}, Inf^{\mathcal{B}}, Atless^{\mathcal{B}}, atomic^{\mathcal{B}}, 1\text{-}atom^{\mathcal{B}}, atominf^{\mathcal{B}}, \sim\text{-}inf^{\mathcal{B}}, Int(\omega+\eta)^{\mathcal{B}}, infatomicless^{\mathcal{B}}, 1\text{-}atomless^{\mathcal{B}}, nomaxatomless^{\mathcal{B}})$.
Furthermore, $\forall n$ there is a finite complete set of Π^c_n relations

These relations for $\mathcal{B}^{(4)}$ where used by Downey, Jockusch, Thurber, Knight and Stob

Low_4 Boolean algebras

Lemma: $\mathcal{B}$ admits double-triple-fouth-jump inversion.

[Downey Jockusch 94]	$\mathcal{B}$ has copy $\leq_T X \iff \mathcal{B}'$ has copy $\leq_T X'$
[Thuruber 95]	$\mathcal{B}'$ has copy $\leq_T X \iff \mathcal{B}''$ has copy $\leq_T X'$
[Knight Stob 00]	$\mathcal{B}''$ has copy $\leq_T X \iff \mathcal{B}'''$ has copy $\leq_T X'$
[Knight Stob 00]	$\mathcal{B}'''$ has copy $\leq_T X \iff \mathcal{B}^{(4)}$ has copy $\leq_T X'$

Corollary: [KS00] If $\mathcal{B}$ has a low_4 copy, it has a computable copy.

Proof: $\mathcal{B}$ has low_4 copy $\implies$ $\mathcal{B}^{(4)}$ has copy $\leq_T 0^{(4)}$ $\implies$ $\mathcal{B}'''$ has copy $\leq_T 0'''$ $\implies$ $\mathcal{B}''$ has copy $\leq_T 0''$ $\implies$ $\mathcal{B}'$ has copy $\leq_T 0'$ $\implies$ $\mathcal{B}$ has copy $\leq_T 0$

Q: Does every low_n-BA have a computable copy? [DJ 94]
Q: Do BAs admit nth jump inversion? [Harris, M]

Example: Linear ordering with few descending cuts

Def: A *descening cut* of a lin. ord. $\mathcal{A}$ is a partition (L, R) of $\mathcal{A}$ where R is closed upwards and has no least element.

Thm: Ordinals admit αth-jump inversion $\forall \alpha < \omega_1^{CK}$. [Spector 55]

Theorem ([Kach, M])

Lin. ord. with finitely many desc. cuts admit nth-jump inversion.

Every low_n lin. ord. with finitely many descending cuts has a computable copy.

There is a lin. ord. of intermediate degree with finitely many descending cuts and no computable copy.

Work in progress [Kach, M.]

Scattered linear orderings admit double-jump inversion.

Every low_2 scattered linear ord. has a computable copy.

Jump inversions of structures

[Goncharov, Harizanov, Knight, MaCoy, Miller, Solomon '05]
used the following result

For every $\mathcal{A}$ and every succ. ordinal α, there exists $\mathcal{B}$ such that

$$\mathcal{B}^{(\alpha)} = \mathcal{A} \quad \text{essentially.}$$

plus other properties to show the following
For successor ord. α,

- Δ^0_α-categorical $\neq$ relatively Δ^0_α-categorical
- intrinsically Σ^0_α relations $\neq$ explicitly Σ^0_α relations

Jump Inversion vs Low property

$\mathcal{A}$ *admits Jump Inversion* $\forall X$
$\mathcal{A}'$ has copy $\leq_T X' \iff \mathcal{A}$ has copy $\leq_T X$

Theorem ([M])

Let $\mathcal{A}$ be a structure. TFAE

- *For every X, Y with $X' \equiv_T Y'$,*
  $$\mathcal{A} \text{ has copy } \leq_T X \iff \mathcal{A} \text{ has copy } \leq_T Y.$$
- *$\mathcal{A}$ admits Jump Inversion.*

Corollary: The following questions are equivalent:
Does every X-low$_n$-Boolean algebra have a X-computable copy?
[Downey Jockusch 94]
Do Boolean algebras admit nth jump inversion? [Harris, M.]

Questions

Q: Do Boolean algebras admit *n*th jump inversion?

Q: What are other structures that admit jump inversion?

Q: What are natural structures that have finite complete set of Π^c_n-relations?
What are the jumps of other natural structures?

Q: How does $DegSp(\mathcal{A}')$ look outside $\mathcal{D}_{(\geq 0')}$ for the different choices for $\mathcal{A}'$?