The boundary of determinacy within second order arithmetic.

Antonio Montalbán.
U. of Chicago

(with Richard A. Shore)
Berkeley, CA, March 2011

Special session in honor of Leo Harrington.

The Question

How much determinacy can be proved without using uncountable objects?

Fix a set $A \subseteq \omega^\omega$.

Player I	a_0		a_2		$\cdots$
Player II		a_1		a_3	$\cdots$

let $\bar{a} = (a_0, a_1, a_2, a_3, \ldots)$

Player I *wins* is $\bar{a} \in A$, and Player II *wins* if $\bar{a} \in \omega^\omega \setminus A$.
A *strategy* is a function $s \colon \omega^{<\omega} \to \omega$.
It's a *winning strategy for I* if $\forall a_1, a_3, a_5, \ldots . (f(\emptyset), a_1, f(a_1), a_3, \ldots) \in A$

$A \subseteq \omega^\omega$ is *determined* if there is a strategy for either player I or II.

For a class of sets of reals $\Gamma \subseteq \mathcal{P}(\omega^\omega)$, let

Γ-DET: Every $A \in \Gamma$ is determined.

History

Γ	Γ-DET	remark
Open (Σ^0_1)	[Gale Stwart 53]	
G_δ (Π^0_2)	[Wolfe 55]	
$F_{\sigma\delta}$ (Π^0_3)	[Davis 64]	
$G_{\delta\sigma\delta}$ (Π^0_4)	[Paris 72]	
$F_{\sigma\delta\sigma\delta}$ (Π^0_5)		needs Power-set axiom [Friedman 71]
Borel (Δ^1_1)	[Martin 75]	needs $\aleph_1$ iterations of Power-set axiom [Friedman 71]
Analitic (Σ^1_1)	$\forall x(x^\sharp \text{exists}) \vdash ..$ [Martin 70]	Martin's bound is sharp [Harrington 1978]
Full (ω^ω)	False in ZFC [Gale Stwart 53]	

Harrington's result

Sharps: We define the statement:
"$x^\sharp$ exists" as "In $\mathbb{L}(x)$, there is an ω_1-list of indiscernibles."
$x^\sharp$ is the the ω-type of this list.

Thm:[Kunen] [Jensen](ZFC) The following are equivalent:

1. $0^\sharp$ exists.
2. There is a proper embedding of $\mathbb{L}$ into $\mathbb{L}$.
3. There is an uncountable $X \subseteq ON$ such that $\forall Y$
$$Y \supseteq X \;\; \& \; |Y| = |X| \implies Y \notin \mathbb{L}.$$

Theorem ([Harrington 78])

$\mathbf{\Sigma}^1_1$-DET is equivalent to "$\forall x$ ($x^\sharp$ exists)".

Countable mathematics

Second order arithmetic Z_2 (a.k.a. analysis) consist of

- ordered semi-ring axioms for $\mathbb{N}$
- induction for all 2^{nd}-order formulas
- comprehension for all 2^{nd}-order formulas

Most of classical mathematics can be expressed and proved in Z_2.

Thm: ZFC^- is Σ^1_4-conservative over Z_2,
where ZFC^- is ZFC without the Power-set axiom.

(Obs: Borel-DET and $\mathbf{\Pi}^0_k$-DET are Π^1_3-statements.)

Determinacy without countable objects

Thm: [Friedman 71, Martin] $Z_2 \nvdash \Pi^0_4$-DET.

Theorem (essentially due to Martin)

Given $n \in \mathbb{N}$, Z_2 (and also ZFC$^-$) can prove that
every Boolean combination of n $\mathbf{\Pi}^0_3$ sets is determined

where $F_{\sigma\delta} = \mathbf{\Pi}^0_3 =$ intersection of unions of closed sets

But....
The larger the n, the more axioms are needed.

Theorem (MS)

Z_2 (and also ZFC$^-$) cannot prove that
every Boolean combination of $\mathbf{\Pi}^0_3$ sets is determined

Reverse Mathematics in a nutshell

The main question of Reverse Mathematics is:
What axioms of Z_2 are necessary for classical mathematics?

Using a base theory as RCA_0, one can often prove that

- theorems are equivalent to axioms.
- Most theorems are equivalent to one of 5 subsystems.

Most theorems of classical mathematics can be proved in Π^1_1-CA_0.
where in
Π^1_1-CA_0, induction and comprehension are restricted to Π^1_1-formulas.

No example of a classical theorem of Z_2 needed more than Π^1_3-CA_0.
We provide a hierarchy of natural statements
that need axioms all the way up in Z_2.

Strength of Determinacy in Second order arithmetic

Γ	strength of Γ-DET		base
Δ^0_1	ATR_0	[Steel 78]	RCA_0
Σ^0_1	ATR_0	[Steel 78]	RCA_0
$\Sigma^0_1 \wedge \Pi^0_1$	Π^1_1-CA_0	[Tanaka 90]	RCA_0
Δ^0_2	Π^1_1-TR_0	[Tanaka 91]	RCA_0
Π^0_2	Σ^1_1-ID_0	[Tanaka 91]	ATR_0
Δ^0_3	$[\Sigma^1_1]^{TR}$-ID_0	[MedSalem, Tanaka 08]	Π^1_1-TI_0
Π^0_3	Π^1_3-$\mathsf{CA}_0 \vdash$ .. Δ^1_3-$\mathsf{CA}_0 \nvdash$ ..	[Welch 09]	
Π^0_4	$Z_2 \nvdash$ ..	[Martin] [Friedman 71]	

Difference hierarchy

Def: $A \subseteq \omega^\omega$ is m-$\mathbf{\Pi}^0_3$ if there are $\mathbf{\Pi}^0_3$ sets $A_0 \supseteq A_1 \supseteq ... \supseteq A_m = \emptyset$
s.t.: $A = (...(((A_0 \setminus A_1) \cup A_2) \setminus A_3) \cup ...)$
i.e. $x \in A \iff$ (least i $(x \notin A_i)$) is odd.

Obs: (Boolean combinations of $\mathbf{\Pi}^0_3$) $= \bigcup_{m \in \omega} m$-$\mathbf{\Pi}^0_3$.

The difference hierarchy extends through the transfinite.

Thm: [Kuratowski 58] $\mathbf{\Delta}^0_4 = \bigcup_{\alpha \in \omega_1} \alpha$-$\mathbf{\Pi}^0_3$.

A closer look at our main theorem

Recall:
Π^1_n-CA$_0$ is Z_2 with induction and comprehension restricted to Π^1_n formulas.
Δ^1_n-CA$_0$ is Z_2 with induction and comprehension restricted to Δ^1_n sets.

Theorem (MS, following Martin's proof)

Π^1_{n+2}-$CA_0 \ \vdash \ n$-$\mathbf{\Pi}^0_3$$-DET.$

Theorem (MS)

Δ^1_{n+2}-$CA_0 \ \nvdash \ n$-$\mathbf{\Pi}^0_3$$-DET.$

[Welch 09] had already proved the cases $n = 1$.

Since $Z_2 = \bigcup_n \Pi^1_n\text{-CA}_0 = \bigcup_n \Delta^1_n\text{-CA}_0$:

Corollary: For each n, $Z_2 \vdash n\text{-}\mathbf{\Pi}^0_3 - DET$, but
$Z_2 \nvdash \forall n \ (n\text{-}\mathbf{\Pi}^0_3 - DET)$.

Reversals

Theorem (MS)

Reversals aren't possible: for each n

$\Delta^1_{n+2}\text{-}CA_0 \quad \subsetneq \quad \Delta^1_{n+2}\text{-}CA_0 + n\text{-}\boldsymbol{\Pi}^0_3\text{-}DET \quad \subsetneq \quad \Pi^1_{n+2}\text{-}CA_0$

Thm: [MedSalem, Tanaka 07] $\Pi^1_1\text{-CA}_0$ + Borel-DET $\not\Rightarrow \Delta^1_2\text{-CA}_0$.

Theorem (MS)

Let T be a true Σ^1_4 sentence. Then, for $n \geq 2$,

- $\Delta^1_n\text{-}CA_0 + T \nvdash \Pi^1_n\text{-}CA_0$
- $\Pi^1_n\text{-}CA_0 + T \nvdash \Delta^1_{n+1}\text{-}CA_0$ *(even for β-models)*

This also holds if T is a Σ^1_{n+2} theorem of ZFC.

Obs: Borel-DET and m-$\boldsymbol{\Pi}^0_3$−DET are Π^1_3 theorems of ZFC.

The techniques

Def: *α is n-admissible* if there is no unbounded, Σ_n-over-L_α-definable function $f : \delta \to \alpha$, with $\delta < \alpha$.

- α is n-admissible $\implies$ $2^\omega \cap L_\alpha \models \Delta^1_{n+1}$-$CA_0$ (for $n \geq 2$).
- Let α_n be the least n-admissible ordinal.
- Let Th_n =Theory of L_{α_n}.
- $Th_n \notin L_{\alpha_n}$ using Gödel-Tarski undefinability of truth.

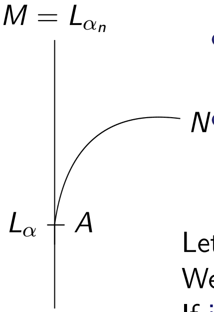
Lemma (MS)

For $n \geq 2$, there is a $(n\text{-}1)$-Π^0_3 game where each player plays a set of sentences, and

1. *if* I *plays* Th_n*, he wins.*
2. *if* I *does not play* Th_n *but* II *does, then* II *wins.*

A winning strategy for this game must compute Th_n*.*
Hence $2^\omega \cap L_{\alpha_n} \models \Delta^1_{n+1}\text{-}CA_0$ & $\neg(n-1)\text{-}\mathbf{\Pi}^0_3\text{-}DET$

Ideas in the proof.



- Each player has to play a complete, consistent set of formulas including ZF$+V = L_{\alpha_n}$.
- We consider the term models of these theories: M and N.
- L_{α_n} is the only
  well-founded model of ZF$+V = L_{\alpha_n}$.
- Using differences of Π^0_3 formulas we need to identify the player playing a well-founded model.

Let $L_\alpha = N \cap M$.
We find a Π^0_3 condition C_k and a property P_k s.t.:
If is α is k-admissible and P_k holds, then

- If C_k, we find a descending sequence in N.
- If $\neg C_k$, then α is $k+1$-admissible and P_{k+1}