A few results

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DETERMINACY

With Richard A. Shore, we obtained the precise limit of how much determinacy can be proved without the use of uncountable objects.

Theorem. *(Montalbán, Shore 2009) Given a fixed number n, second-order arithmetic can prove that every Boolean combination of n $\mathbf{G}_{\delta_\sigma}$-sets is determined. However, for each n a different proof is needed and no single proof works for all n: second-order arithmetic cannot show that all Boolean combinations of any number of $\mathbf{G}_{\delta_\sigma}$-sets are determined.*

Consider a set A of binary sequences, or equivalently, a subset of Cantor set. Players I and II play binary bits alternatively for infinitely many turns, forming an infinite binary sequence; player I wins if the sequence is in A, and, otherwise, player II wins. We say that A is *determined* if one of the players has a winning strategy for this game. Statements about determinacy of games have attracted logicians for many decades because of the high complexity of the winning strategies, and also because they have been a useful combinatorial tool in a wide range of areas. Martin's celebrated theorem says whenever A is a Borel subset of Cantor set, A is determined. But, as noticed by Friedman, Martin's proof requires unusually large cardinals almost never used in mathematics.

Second-order arithmetic, a widely studied logical system, captures all mathematics that can be done in the realm of countable objects. Countable objects include real numbers, continuous functions, real analysis, countable algebra, etc.. Almost all of mathematics that can be modeled with, or coded by, countable objects can be carried out in second-order arithmetic.

For the statement above we also need the following definitions: A subset of Cantor space is $\mathbf{G}_{\delta_\sigma}$ if it is the countable union of countable intersections of open sets (i.e. it is in the third level of the Borel hierarchy); a set is a *Boolean combination* of n sets $A_1, ..., A_n$ if it can be defined form them using (finite) intersections, unions and complements.

The study of the determinacy of $\mathbf{G}_{\delta_\sigma}$ sets seemed intractable for a long time, and so did the study of results at the boundary of second-order arithmetic. Our result broke those two barriers.

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LINEAR ORDERINGS

In 1955, Clifford Spector proved a central result in hyperarithmetic theory: that every hyperarithmetic well ordering is isomorphic to a computable one. In less technical terms this says that if an ordinal has a representation of a certain complexity (hyperarithmetic, which is quite high) then it also has a very simple (computable) representation. (The hyperarithmetic sets are an effective analog of the Borel sets: they are the smallest non-trivial class of sets of natural numbers which is closed under countable effective intersections and unions.) I proved the following unexpected generalization to all countable linear orderings:

Theorem. *(Montalbán 2005) Every hyperarithmetic linear ordering is bi-embeddable with a computable one.*

The proof of this Theorem requires a deep analysis of the structure of the countable linear orderings and the embeddability relation. This analysis led me to define equimorphism invariants, given by finite trees labeled by ordinals, for the class of scattered linear orderings of any size. The invariants provide a new description for the partial orderings induced by the embeddability relation on the class of scattered (and of all countable) linear orderings.

Fraïssé's conjecture (proved by Laver in 1971) is the statement that says that the countable linear orderings form a well-quasi-ordering with respect to embeddablity. It has interested logicians for many years because of the difficulty of its proof in terms of reverse mathematics; it uses constructions which are computationally more complicated than most of the theorems of mathematics. From my work, it follows that Fraïssé's conjecture is a sufficient and necessary assumption to develop a reasonable theory linear orderings and the embeddability relation. This means that statement has a *robustness property* in the sense that it is equivalent to most statements talking about embeddability of linear orderings.

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Boolean Algebras

With K. Harris we obtained a complete characterization of the relations on a Boolean algebra that are defined within a certain number of Turing jumps. Studying the definable relations on a structure is an important theme all throughout logic, and for the computability view point, it is useful to understand how many Turing jumps it takes to compute a certain relation. Researchers have been looking for a result of this sort for some time, as it is the first step to solve the well-known open question of whether every low$_n$ Boolean algebra has a computable isomorphic copy. We achieved our characterization by getting a very good understanding of the back-and-forth relations on Boolean algebras, and we believe this is a key step towards the solution of this open problem, and will also be a useful tool for other work on Boolean Algebras.

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Turing Degrees

Theorem. *(Montalbán 2003) Every countable jump upper semilattice can be embedded into the Turing Degrees* $(\mathbf{D}, \leq_T, \vee, ')$ *(of course, preserving join and jump).*

The Turing degrees form an *upper semilattice*; that is, every pair of elements $\mathbf{a}$, $\mathbf{b}$ has a least upper bound $\mathbf{a} \vee \mathbf{b}$. Intuitively, $\mathbf{a} \vee \mathbf{b}$ contains all the information that $\mathbf{a}$ and $\mathbf{b}$ have toghether. The other naturally defined operation is called the *Turing jump*. The jump of a degree $\mathbf{a}$, denoted $\mathbf{a}'$, is given by the degree of the *Halting Problem* relativized to some set in $\mathbf{a}$. It can be shown that the jump operation is strictly increasing (i.e., $\forall \mathbf{a}(\mathbf{a} <_T \mathbf{a}'))$ and monotonic (i.e., $\mathbf{a} <_T \mathbf{b} \implies \mathbf{a}' <_T \mathbf{a}'$). A *jump upper semilattice* is an upper semilattice together with a strictly increasing, monotonic function. The result above shows that the Truing degree structure is universal for countable jump upper semilattices, a result that gives us valuable information about the the shape of the Turing Degree Structure.

Another interesting result I proved along these lines is that the question of whether it is possible to embed every jump upper semilattice of size $\aleph_1$ satisfying the countable predecessor property into $(\mathbf{D}, \leq_T, \vee, ')$ is independent of ZFC.

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