Saved: June 23rd
Compiled: October 2, 2011

RESEARCH STATEMENT

ANTONIO MONTALBÁN

I am interested in studying the complexity of mathematical practice. In mathematics, as we all know, some structures are more complicated than others, some constructions more complicated than others, and some proofs more complicated than others. I am interested in understanding how to measure this complexity and in measuring it. The motivations for this come from different areas. Form a foundational viewpoint, we want to know what assumptions we really need to do mathematics (ZFC is way much more than we usually use), and we are also interested in knowing what assumptions are used in the different areas of mathematics. Form a computational viewpoint, it is important to know what part of mathematics can be done by mechanical algorithms, and, even for the part that can't be done mechanically, we want to know how constructive are the objects we deal with. Furthermore, it is sometimes the case that this computational analysis allows us to find connections between constructions in different areas of mathematics, and in many cases to obtain a deeper understanding of mathematical objects being analyzed.

My work is quite diverse in terms of the techniques I have used, the approaches I have taken, and the areas of mathematics that I have analyzed. However, my background area is Computability Theory, and most of my work can be considered as part of this branch of Mathematical Logic.

Inside computability theory, I have worked in various different areas. I have been particularly interested in the programs of Computable Mathematics, Reverse Mathematics and Turing Degree Theory. The former studies the computability aspects of mathematical theorems and structures. The second one analyzes the complexity of mathematical theorems in terms of the complexity of the constructions needed for their proofs. The latter studies the partial ordering induced by the relation "computable from" in an abstract way. I have also written papers in other areas like effective randomness, automata theory, the lattice of Π^0_1-classes, etc..

If I had to choose favorites, tools that are recurrent over my work are the iterations of the Turing jump and hyperarithmetic theory. Structures that are recurrent over my work are linear orderings, well-quasi-orderings and Boolean algebras.

In the next three sections I describe my work in each of the areas of Computable Mathematics, Reverse Mathematics, and Turing Degree Theory. The fourth section is dedicated to the part of my work that does not fit in this classification. Each of these sections starts by describing the general ideas of the subject and becomes more technical inside each subsection; for the most part, the reader can skip sections and subsections without loosing in understanding.

All my papers are available on my web page at www.math.uchicago.edu/∼antonio.

Computable Mathematics

Effective mathematics is concerned with the computable aspects of mathematical objects and constructions. I have been interested on general questions like the following: When can a algebraic structure be represented computably? How difficult is it to recognize a certain structure? Can information be encoded into the isomorphism type of a structure? We search for answers that connect computational properties with structural or algebraic properties.

The Jump of a structure. The jump of a structure $\mathcal{A}$ is another structure $\mathcal{A}'$ (only dependent on the isomorphism type) which contains all the Σ^0_1 information about $\mathcal{A}$. I defined this notion in [Mon09b], and then noticed that similar notions had already been independently defined in Bulgaria [Bal06, Sos07, SS09] and in Russia [Puz09, Stu09, Stu10]. We say that a relation R on a structure $\mathcal{A}$ is r.i.c.e. (relatively intrinsically c.e.) if in every copy $(\mathcal{B}, R^{\mathcal{B}})$ of $(\mathcal{A}, R)$, $R^{\mathcal{B}}$ is c.e. in $\mathcal{B}$. We define $\mathcal{A}'$

by adding to $\mathcal{A}$ a complete set of r.i.c.e. relations, as, for instance, the set of all computably infinitary Σ_1-definable relations. Even though this notion is new, it is behind many ideas and constructions in computable model theory from the last few decades. We have showed various results about the jump of a structure that show it is indeed a good analog of the Turing jump. One question that was open for a few years is whether there is a structure equivalent to its own jump. Puzarenko [Puz11] and me [Mona] have recently, independently show that such a fixed point for he jump exists. Puzarenko's proof is an elaborate construction that works in ZFC; my proof is quite simple, but uses the existence of $0^{\#}$. But, for me, even more interesting than the question itself, was the complexity of its answer:

Theorem 1 (Montalbán [Mona])**.** *The existence of a fixed point for the jump of a structure (that is, of structure $\mathcal{A}$ such that an oracle computes a copy of $\mathcal{A}$ if and only if it computes a copy of $\mathcal{A}'$) cannot be proved in full n-order arithmetic for any n.*

Recall that almost all of classical mathematics outside set theory or model theory can be developed in n-order arithmetic for some n (usually $n = 2$, or at most 3 or 4). So, this results can be read as saying that none of the structures that occur in mathematics outside logic can be the fixed point for the jump.

Structure simplicity versus coding. Let $\mathbb{K}$ be a nice (Borel for instance) class of countable structures and $n \in \omega$. Following the theme of the questions mentioned above, we want to know what can be said about the structures in $\mathbb{K}$ if we only have n Turing jumps available. (For simplicity, the reader may only consider the case $n = 1$, which is already interesting enough.) In [Monb] I prove the following dichotomy result.

Theorem 2 (Montalbán [Monb])**.** *Relative to some oracle, either*

(1) *with n jumps, we can distinguish no more that countably many different structures in $\mathbb{K}$; and*
(2) *no non-trivial information can be encoded in the $(n-1)$st jump of the structures in $\mathbb{K}$; but*
(3) *there is a nice characterization of all the relatively intrinsically Σ^0_{n+1} relations on the structures of $\mathbb{K}$;*

or

(1) *with n jumps, we can distinguish continuum many different structures in $\mathbb{K}$; and*
(2) *any set can be encoded in the $(n-1)$st jump of some structure in $\mathbb{K}$; but*
(3) *there is no nice characterization of all the relatively intrinsically Σ^0_{n+1} relation on the structures of $\mathbb{K}$.*

The statements labeled (1) are formalized by counting the number of n-back-and-forth relations on tuples in the structures of $\mathbb{K}$. For the statements labeled (2), we say that a set D can be coded in the mth jump of a structure $\mathcal{A}$ if D is left-c.e. in the mth jump of every presentation of $\mathcal{A}$. The notions needed to explain (3) are quite interesting and I will explain them below. Before going further, let me notice that, even if the theorem states the dichotomy relative to some oracle, when applied to natural classes of structures this oracle is always 0: I also address this issue in [Monb].

Let me explain now what I mean by a *nice characterization of the relatively intrinsically Σ^0_2 relations.* First of all, recall that a relation in a structure is *relatively intrinsically Σ^0_2* if it can be enumerated in the Turing jump of the structure independently of the presentation of the structure. It was showed by Ash, Knight, Manasse, Slaman and independently Chisholm (see [AK00, Theorem 10.1]) that the relatively intrinsically Σ^0_2 relations are exactly the ones defined by computably infinitary Σ_2 formulas, which we will denote by Σ^c_2. Hence, we will restrict our attention to the computably infinitary language. In [Mon09b] I introduced the notion of *complete set of r.i.c.e relations* for a class of structures $\mathbb{K}$, essentially, as a set of formulas that captures all the information that is available with one Turing jump. When we have that $\{P_0, P_1, ...\}$ is a *complete set of r.i.c.e formulas* for $\mathbb{K}$, we have that every Σ^c_2 formula is uniformly equivalent to a $0'$-disjunction of finitary existential formulas in the language that includes $\{P_0, P_1, ...\}$. Thus, if $\{P_0, P_1, ...\}$ is a natural class of formulas, then we have nice a characterization of all Σ^c_2 relations in $\mathbb{K}$. For example, for the class of linear orderings, it can be shown that the *adjacency* relation alone is a complete set of Π^c_1 formulas. This says that every relatively intrinsically Σ^0_2 relation on a linear ordering is a $0'$-union of relations defined by finitary existential formulas in the language $\{\leq, adj\}$. Other natural examples of complete set of Π^c_1 formulas are the linearly independence relations on $\mathbb{Q}$-vector spaces, and the atom relation on Boolean algebras. The idea of reducing the complexity of a formula by adding predicates is often used in model theory for the finitary first order language. But the finitary language behaves quite differently and doesn't reflect the computational properties we are interested in.

I have obtained the following application of complete sets of Π^c_n formulas.

Theorem 3 (Montalbán [Mon09b])**.** *Let $\{P_0, P_1, ...\}$ be a complete set of Π^c_n formulas for a class $\mathbb{K}$. If $Y \geq_T 0'$ can compute a structure $\mathcal{A}$ in $\mathbb{K}$ and all the interpretations in $\mathcal{A}$ of the relations P_i, then there exists an oracle X such that $X' \equiv_T Y$ and X computes a copy $\mathcal{A}$.*

A similar result, stated in a very different way, was obtained independently by I. Soskov and A. Soskova [SS09]. For the particular case of linear orderings and $n = 1$, Frolov [Fro06] had already independently observed the following corollary of the theorem above: If for a linear ordering $\mathcal{L}$ we have that $0'$ computes a copy of $(\mathcal{L}, \leq, adj)$, then $(\mathcal{L}, \leq)$ has a low copy.

The low$_n$ conjecture. A question that has captivated my attention is the well-known *low$_n$ Boolean algebra question*:

Question 1. [DJ94] Does every low$_n$ Boolean algebra have a computable copy?
(A set $X \subseteq \omega$ is low$_n$ if its nth Turing jump is as low as possible, namely $X^{(n)} = 0^{(n)}$.)

This is part of the more general problem of understanding which structures have computable presentations, and also of understanding the possible shapes of degree spectra. Boolean algebras seem to have an interesting behavior in this respect.

This question has been answered positively up to $n = 4$ by Downey and Jockusch; Thurber; and Knight and Stob [DJ94, Thu95, KS00], and is open for $n = 5$ and onwards. Kenneth Harris and I observed that the sets of relations considered in [DJ94, Thu95, KS00] for $n = 1, 2, 3, 4$ are equivalent by Boolean combinations to complete sets of Π^c_n formulas, and that their results can be restated as follows: For $n = 1, 2, 3, 4$, if a Boolean algebra, together with a complete set of Π^c_n relations, is computable in $0^{(n)}$, then the Boolean algebra has a computable copy.

Theorem 4 (Harris and Montalbán [HMa])**.** *For every $n \in \omega$, there exists a* finite *complete set of Π^c_n formulas for Boolean algebras.*

More interesting than the existence of such a set is the analysis that we do of these formulas, which we expect to be useful in the study of Boolean algebras. From our analysis, we could observe that the level five of the low$_n$ conjecture presents some extra essential difficulties not present at the previous levels. To show that these difficulties are real, we turned them around and proved the following theorem:

Theorem 5 (Harris and Montalbán [HMb])**.** *There is a low$_5$ Boolean algebra that is not $0^{(7)}$-isomorphic to any computable one.*

This contrasts with previous results, where for each $n = 1, 2, 3, 4$, every low$_n$ Boolean algebra is $0^{(n+2)}$-isomorphic to a computable one. The proof uses just a zero-double-jump priority argument, and it doesn't deal with $0^{(7)}$ or anything similar. This is due to a new machinery developed by Harris and I to build Boolean algebras.

A relation on Boolean algebras that has been particularly studied by computability theorist is the *atom* relation. (For a survey see [Rem89].) Stepping on known results, I [Mon08b] proved that for every computable Boolean algebra with infinitely many atoms, and for every high$_3$ c.e. degree X, there is a computable copy of the Boolean algebra where the atom relation has degree X.

As we already mentioned, one of the motivations for studying the low$_n$ conjecture is that it would give an example of a class of structures whose *degree spectra* have a very interesting and unusual property. Kach and I [KM] found another class where the low_n conjecture holds, namely the class of linear ordering with finitely many descending cuts. We have also shown that there is such a linear ordering of intermediate degree with no computable copy.

Hirschfeldt, Kach and I [KHM] studied a class of degrees that is slightly larger than the class of all low_n degrees. We call the degrees in this class *low for* Δ*-Feiner*. We say that a set S is $\Delta^0_{(n\mapsto n)}(X)$ if membership of n in S is a $\Delta^0_n(X)$ question, uniformly in n. So, X' can tell if 1 is in S, X'' can decide if 2 is in S, etc.. A set X is *low for* Δ*-Feiner* if every set S that is $\Delta^0_{(n\mapsto n)}(X)$ is also $\Delta^0_{(n\mapsto n)}(\emptyset)$. One of the motivations for this definition comes from results in [Kac07] that imply that X it is low for Δ-Feiner if and only if every *depth zero* Boolean algebra computable in X has a computable copy. It is not hard to see that every low_n set is low for Δ-Feiner. Hirschfeldt, Kach and I showed that the converse is not true by constructing a c.e. intermediate degree that is low for Δ-Feiner. We also studied variations of this notion, like the classes of degrees that are $\Delta^0_{(n\mapsto bn+a)}(X)$ or $\Sigma^0_{(n\mapsto bn+a)}(X)$, and the classes of sets that are low, intermediate, and high for them.

Equimorphism types of linear orderings. The study of the ordinals which have computable presentations has been fundamental for computability theory, higher recursion theory and hyperarithmetic theory. One of the most important results here is due to Spector [Spe55]: every hyperarithmetic well ordering is isomorphic to a computable one. (The hyperarithmetic sets form a complexity class that is quite large. One could say that hyperarithmetic sets are an effective version of Borel sets, as computably enumerable sets are an effective version of open sets.) I proved the following generalization to all countable linear orderings:

Theorem 6. [Mon05] *Every hyperarithmetic linear ordering is equimorphic with a computable one. (Two linear orderings are* equimorphic *if they can be embedded in each other.)*

The proof of this theorem requires a deep analysis of the structure of the countable linear orderings modulo equimorphisms. Using this analysis, in [Mon06b], I found equimorphism invariants for the class of scattered linear ordering of any size. These invariants are finite trees whose nodes are labeled by ordinals. They are equimorphism invariants in the sense that two linear orderings are equimorphic if and only if they are assigned the same invariant. One can then study the embeddability relation on linear orderings using these invariants.

Hausdorff showed that for every cardinality κ, every scattered linear ordering of size κ has a subset isomorphic to either κ or κ^*. I extended this result from cardinals to all ordinal by providing a construction, for each ordinal β, of the finitely many linear orderings that are minimal of Hausdorff rank β [Mon06b].

A third non-computability result that followed from my deep analysis of linear orderings is an answer to an open question from Rosenstein's book on Linear Orderings [Ros82, page 178]. I proved that, given countable linear orderings $\mathcal{L}$ and $\mathcal{C}_0 \preccurlyeq \mathcal{C}_1 \preccurlyeq \cdots$ such that for every $\mathcal{C}$ $\big(\forall n(\mathcal{C}_n \preccurlyeq \mathcal{C}) \implies \mathcal{L} \preccurlyeq \mathcal{C}\big)$ (where $\preccurlyeq$ is the embeddability relation), we have that there is some n such that $\mathcal{L} \preccurlyeq \mathcal{C}_n$ [Mon06a].

I wrote a paper surveying all these results for the BSL [Mon07b].

Well-quasi-orderings and linear orderings. Well-quasi-orderings have been widely studied by combinatorists, computer scientists, proof theorists, etc.. A quasi-ordering, P, is a *well-quasi-ordering* (abbreviated wqo) if, for every sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of P, there exist i and j such that $i < j$ and $x_i \leq_P x_j$, or equivalently, if P has no infinite descending sequences and no infinite anti-chains.

The length of a wqo is used to measure its well-quasi-orderedness: The *length* of a wqo is defined to be the supremum of the order types of its linearizations. Note that this definition makes sense because every linearization of a wqo is a well ordering. Moreover, this supremum is actually reachable: De Jongh and Parikh [dJP77] showed that every wqo has a linearization of maximal order type. The length can also be obtained as the well-founded rank of the tree of finite sequences $\langle x_0, ..., x_k\rangle$ of elements of P such that there no $i < j \leq k$ with $x_i \leq_P x_j$. Notice that P is wqo if and only if this tree has no infinite paths. So far, people have computed lengths of various wqos for different applications, but always using different methods and new ideas, and it was unknown whether the length of a wqo could be found computably. Diana Schmidt posed this question in [Sch79]. I showed [Mon07a] that computable wqos have computable maximal linearizations. However, I showed that the process of finding such linearizations is not computably uniform, not even hyperarithmetically.

Other work in computable mathematics.

Elementary equivalence of Boolean algebras. With Csima and Shore [CMS06] we studied the complexity of the question: Given two computable Boolean algebras, are they elementarily equivalent? This work included a complete analysis of the complexity of index sets of Boolean algebras with a certain Tarski invariant.

Fraïssé limits. Fraïssé studied countable structures $\mathcal{S}$ through analysis of the *age* of $\mathcal{S}$, i.e., the set of all finitely generated substructures of $\mathcal{S}$ ([Fra86], see also [Hod97]). B. Csima, V. Harizanov, R. Miller, and I [CMHM] investigated the effectiveness of his analysis. We provide sufficient and necessary conditions for the Fraïssé limit to be computable. (Let me notice that the interesting case is when the finitely generated structures aren't necessarily finite, as otherwise, under reasonable effectiveness hypothesis, it is easy to see that the Fraïssé limit is always computable.) One interesting result we get is that degree spectra of relations on a sufficiently nice Fraïssé limit are always upward closed unless the relation is definable by a quantifier-free formula.

Torsion-free Abelian groups and Vector Spaces. With Downey, Hirschfeldt, Kach, Lempp, and Mileti [DHK+07], we studied the computational complexity of finding proper subspaces of a given computable vector space. With Downey [DM08a], we studied the complexity of the isomorphism problem for Torsion free abelian groups, topic that had been studied by Descriptive Set theorists like Hjorth, Kechris and Thomas. Both papers were published in the Journal of Algebra.

Equivalence relations on computable structures. With Fokina, S. Friedman, Knight and McCoy [FFH+], we studied the equivalence relations given by isomorphism and by-embedabillity on a class of computable structures. The idea is to translate the question from the study of equivalence relations on the reals ordered by Borel reducibilities, to the study of equivalence relations on natural numbers (usually representing indices for computable structures) ordered by either computable or hyperarithmetic reducibilities. We show, for instance, that the isomorphism relation on computable torsion Abelian groups is complete among all Σ^1_1 equivalence relations on ω, while in the classic case it is know to be incomplete among isomorphism relations on classes of countable structures [FS89].

A Countably categorical theory with a complicated theory. With Khoussainov [KM10], we build a computable $\aleph_0$-categorical structure whose theory is 1-equivalent with true arithmetic. Before this result, all known computable $\aleph_0$-categorical structures had arithmetic theories, and most of them were actually decidable. Because of the homogeneity of these structures it was conceivable that the complexity of their theories couldn't get much higher; Our result shows that, unfortunately, they can.

Reverse Mathematics

Introduction. The questions of which axioms are necessary to do mathematics is of great importance in Foundations of Mathematics and is the main question behind Friedman and Simpson's program of Reverse Mathematics. To analyze this question formally it is necessary to fix a logical system. Reverse Mathematics deals with the system of second-order arithmetic. Second-order arithmetic, though much weaker than set theory, is rich enough to be able to express an important fragment of classical mathematics. This fragment includes number theory, calculus, countable algebra, real and complex analysis, differential equations, separable metric spaces and combinatorics among others. Almost all of mathematics that can be modeled with, or coded by, countable objects can be done in second-order arithmetic. Notice that, for instance, a real number or a continuous function on the reals can be coded by a countable object.

The idea of Reverse Mathematics goes as follows. We start by fixing a basic system of axioms as a base. The most commonly used base system is called RCA_0, that more or less says that the natural numbers are an ordered semi-group (which satisfy a bit of induction) and that computable sets exists. Now, given a theorem of "ordinary" mathematics, the question we ask is what axioms do we need to add to the base system to prove this theorem. It is often the case in Reverse Mathematics that we can show that a certain set of axioms is necessary to prove a theorem by showing, using the base system, that the axioms follow from the theorem. Because of this idea, this program is called Reverse Mathematics.

Many different systems of axioms have been defined, and many theorems from all over mathematics have been analyzed. A very interesting fact is that most of the theorems that have been analyzed have been proved equivalent to one of five systems, which we will call the *main five*. The reason for this is not known. It could be that for some reason mathematicians find it easier to prove theorems that are equivalent to one of the main five systems. Or it could be that is easier for logicians to show something is equivalent to one of these five systems than proving it's not, and therefore our sample of theorems successfully analyzed is biased. For sure, there is something special about these five main systems, and I am interested knowing what it is. I believe that studying the notion of *robust system*, as a system that is equivalent to small perturbations of itself, could help to find an answer.

I'm currently writing a paper on open questions on Reverse Mathematics.

0.1. **Determinacy.** Statements about determinacy of games have attracted logicians for many decades because of the high complexity of the winning strategies, and also because they have been a useful combinatorial tool in a wide range of areas. Consider a set A of sequences of natural numbers. We define a game $G(A)$ played as follows. Players I and II play natural numbers alternatively for infinitely many turns, forming an infinite sequence of natural numbers; player I wins if the sequence is in A, and, otherwise, player II wins. We say that $G(A)$, or just that A, is *determined* if one of the players has a winning strategy for $G(A)$. Not every game $G(A)$ is determined, but most of the games we might encounter are. It requires a simple proof to show that if A is open, then $G(A)$ is determined. Martin's celebrated theorem says whenever A is a Borel set, $G(A)$ is determined. (We're referring to the natural topology on the space of sequences of natural numbers given by the product topology of the discrete topology on $\mathbb{N}$.) H. Friedman showed that to prove that all Borel sets are determined one needs and unusual large amount of ZFC, and that even the fifth level in the Borel hierarchy (Σ^0_5) is not provable in second-order arithmetic. This is quite remarkable, as almost all the theorems about countable objects can be proved in second-order arithmetic. With Shore, we obtained the precise limit of how much determinacy can be proved in second-order arithmetic. To state our result, we need a couple definitions: A set of sequences is $\mathbf{G}_{\delta_\sigma}$ if it is the countable union of countable intersections of open sets (i.e. it is in the third level of the Borel hierarchy); a set is a *Boolean combination* of n sets $A_1, ..., A_n$ if it can be defined form them using (finite) intersections, unions and complements.

Theorem 7 (Montalbán, Shore [MS])**.** *Given a fixed number n, second-order arithmetic can prove that every Boolean combination of n $\mathbf{G}_{\delta_\sigma}$-sets is determined. However, for each n a different proof is needed and no single proof works for all n: second-order arithmetic cannot show that all Boolean combinations of any number of $\mathbf{G}_{\delta_\sigma}$-sets are determined.*

Linear orderings. My work in Reverse Mathematics started when trying to find the proof-theoretic strength of Jullien's Theorem, which is a classification of the countable extendible linear orderings [Jul69]. It is a theorem that seems to require more complex axioms than most of the theorems in classical mathematics. Without finding its exact proof-theoretic strength in terms of logical axioms, I ended up finding that it is equivalent to many other statements about embeddability of linear orderings. The proof-theoretic strength of one of these statements, Fraïssé's conjecture, have been studied before. Fraïssé's conjecture (also known as Laver's Theorem [Lav71]) is the statement that says that the countable linear orderings form a wqo with respect to embeddablity. (Recall that a *well-quasi-ordering*, or *wqo*, is a quasi-ordering without infinite descending sequences or infinite antichains.) Fraïssé's conjecture has interested logicians for many years also because of the difficulty of its proof in terms of reverse mathematics. From my work, it follows that this statement has a *robustness property* in the sense that it is equivalent to many other statements talking about the same type of objects. So far, the only systems with this robustness property were the main five (and WWKL_0), but we do not know that Fraïssé's conjecture is equivalent to one of these five. Furthermore, from my work, one could conclude the following claim.

Claim 1. [Mon06c] Fraïssé's conjecture is a sufficient and necessary assumption to develop a reasonable theory of linear orderings and the embeddability relation.

Question 2. Is Fraïssé's conjecture equivalent to Arithmetic Transfinite Recursion?

It was conjectured by Clote [Clo90], Simpson [Sim99, Remark X.3.31] and Marcone [Mar05] that that the answer is positive, but the problem is still open. One possible approach is to study the length of the wqo's involved, since this usually gives proof-theoretic information. Together with Marcone and Weiermann, we have recently made some progress on this approach. Marcone and I proved that the length of the wqo of linear orders of finite Hausdorff rank under embeddability is $\epsilon_{\epsilon_{\epsilon_{..}}}$, the first fixed point of the *epsilon-ordinal-function*, $\alpha \mapsto \epsilon_\alpha$ (where ϵ_α is the αth fixed point of the ordinal function $\beta \mapsto \omega^\beta$). We then showed that Fraïssé's conjecture restricted to linear orders of finite Hausdorff rank is equivalent to "$\epsilon_{\epsilon_{\epsilon_{..}}}$ is well-ordered" over ACA_0^+, where ACA_0^+ is $\mathsf{RCA}_0 +\ \forall X\ (X^{(\omega)}$ exists). We note that the statement "$\epsilon_{\epsilon_{\epsilon_{..}}}$ is well-ordered" implies the consistency of ACA_0^+ and hence is not provable in ACA_0^+.

Ordinal Notations. The *proof theoretic ordinal* of a theory is an extremely useful notion when trying to measure its consistency strength. Ordinal notations are the main tool to deal with these ordinals. The proof theoretic ordinal of a theory is the least ordinal that the theory cannot show is well-ordered. Usually, it is also the least ordinal such that the consistency of the theory can be proved using transfinite induction along this ordinal. For instance, Gentzen's proof of the consistency of Peano Arithmetic used transfinite induction along ϵ_0, and it can be shown that Peano Arithmetic proves transfinite induction along any ordinal less than ϵ_0. This makes ϵ_0 the proof theoretic ordinal of Peano Arithmetic. Ordinals are naturally defined in set theory, but to deal with ϵ_0 inside first or second order arithmetic we use ordinal notations: We use a a string of symbols to represent each ordinal below ϵ_0. In this representation we should be able to easily compare two strings and decide which one corresponds to a larger ordinal. This way we have a linear ordering that is representing ϵ_0. When we refer to the statement that says that ϵ_0 is well-ordered we actually mean the statement that this particular linear ordering formed of notations is well-ordered.

Statements about the well-orderedness of a certain ordinal are Π^1_1 and have no set-existence implications. However, when we use *ordinal notation operations* we can get set-existence implications. For instance, Hirst [Hir94] proved that the statement that says "If X is well-ordered, then ω^X is also well-ordered" is equivalent to ACA_0, where ω^X is a linear ordering whose elements are of the form $\omega^{x_0} \cdot n_0 + ... + \omega^{x_k} \cdot n_k$, with $x_0, ..., x_k \in X$ and $n_0, ..., n_k \in \mathbb{N}$, and the ordering on these elements is defined in the obvious way.

Theorem 8 (Marcone, Montalbán [MM]). *The statement "If X is well-ordered, then ϵ_X is also well-ordered" is equivalent to ACA_0^+, over RCA_0, where ϵ_X is, in a sense, the X'th fixed point of the function $x \mapsto \omega^x$, and ACA_0^+ is $\mathsf{RCA}_0 + \forall X\ (X^{(\omega)}$ exists).*

Even though proofs with ordinal notations usually involve techniques from Proof Theory, the proof of this theorem is purely computability theoretical. We build a computable linear ordering X such that ϵ_X has a computable descending sequence, but any descending sequence in X computes $0^{(\omega)}$. Afshari and Rathjen have recently shown the same result using only proof theoretic methods [AR]. It is remarkable that the same result can be proved using such different methods, and we haven't yet fully understand the connection between the two proofs.

Marcone and I then extended this result to all computable ordinals and showed that the statement "If X is well-ordered, then $\varphi(\alpha, X)$ is also well-ordered" equivalent to the Comprehension Axiom scheme for infinitary $\Pi^0_{\omega^\alpha}$ formulas. As a corollary of the uniformity in our proof, we obtained a new, purely computability-theoretic proof of Friedman's result that the statement "If X is well-ordered, then $\varphi(X,0)$ is also well-ordered" is equivalent to ATR_0, where $\varphi(\cdot,\cdot)$ is the Veblen ordinal function (Veblen 1908 [Veb08], see also [Sch77]). We see our results with Marcone as a way of exhibiting the already known interesting properties of Veblen functions from a computability viewpoint.

Rathjen has formulated some conjectures on the strength of statements about operators that map operators that preserve well-orderness into ordinals. Based on these, I have then formulated conjectures of what statements should be equivalent to Π^1_1-CA_0, one using the Howard-Bachmann notation system, and one using Carlson's patterns of resemblance [Monc].

Arithmetic Transfinite Recursion. With Greenberg, in [GM08] we show how for some type of structures, ATR_0 is the natural system to work with them. These structures are superatomic Boolean algebras, reduced p-groups, compact countable topological spaces and well-founded trees. This paper provides a good understanding of what ATR_0 is able to prove. It also reinforces the idea that ATR_0 is a robust system.

Statements of hyperarithmetical analysis. These are statements whose minimal ω-model consists of the hyperarithmetic sets. They are mostly in between ACA_0 and ATR_0. These theories have been studied in the seventies (see [Fri75], [Van77] and [Ste78]), but there was no natural example of a statement of hyperarithmetic analysis. In [Mon06d] I give the first such natural example, called the *Indecomposability Theorem* which is about linear orderings and due to Jullien [Jul69]. Other statements of hyperarithmetic analysis concerning determinacy of games are also discussed in [Mon06d] and many questions are left open.

In [Mon08a], answering a question of Tanaka, I showed that the Π^1_1-*separation axiom scheme* lies strictly in between the Δ^1_1-comprehension and Σ^1_1-choice axiom schemes. Recently Neeman has proved that Jullien's Indecomposability Theorem lies strictly in between weak-Σ^1_1-choice and Δ^1_1-comprehension [Nee]. Working with me as a student, Chris Conidis has recently obtain interesting results comparing a version of the Bolzano-Wierstrass's theorem with other theories of hyperarithmetical analysis [Con].

Turing Degree Theory

Introduction. The Turing degree structure is a very natural object introduced by Kleene and Post in [KP54]. The objective is to study the relation "computable from" abstracting out the interpretation of the object being computed. It is defined as follows: We say that a set A is *computable in* a set B, and write $A \leq_T B$ if there is a program that can decide membership in A using information about membership in B. The relation $\leq_T$ is a quasi-ordering on $\mathcal{P}(\mathbb{N})$, the set of subsets of $\mathbb{N}$. It induces an equivalence relation ($A \equiv_T B \iff A \leq_T B$ & $B \leq_T A$) and a partial ordering on the equivalence classes. The equivalence classes are called *Turing degrees*. We use $(\mathbf{D}, \leq_T)$ to denote this partial ordering. One of the goals of computability theory is to understand the structure of $(\mathbf{D}, \leq_T)$.

The Turing degrees form an *upper semilattice*; that is, every pair of elements $\mathbf{a}$, $\mathbf{b}$ has a least upper bound $\mathbf{a}\vee\mathbf{b}$. Intuitively, $\mathbf{a}\vee\mathbf{b}$ contains all the information that $\mathbf{a}$ and $\mathbf{b}$ have. There is another naturally defined operation called the *Turing jump* (or just *jump*). The jump of a degree $\mathbf{a}$, denoted $\mathbf{a}'$, is given by the degree of the *Halting Problem* relativized to some set in $\mathbf{a}$. It can be shown that the jump operation is strictly increasing (i.e., $\forall \mathbf{a}(\mathbf{a} <_T \mathbf{a}')$) and monotonic (i.e., $\forall \mathbf{a},\mathbf{b}(\mathbf{a} \leq_T \mathbf{b} \implies \mathbf{a}' \leq_T \mathbf{b}')$). A *jump upper semilattice* is an upper semilattice together with a strictly increasing, monotonic function.

Embeddings. One approach to understanding the shape of the Turing Degree Structure has been by studying the structures that can be embedded into it. I've written a survey paper on this approach for the Logic Colloquium 2006 [Mon09a] Kleene and Post, in the same paper where they introduced the Turing degree structure [KP54], proved that every finite upper semilattice can be embedded into $(\mathbf{D}, \leq_T)$. Since then, various other embeddablity results have been proved. Abraham and Shore [AS86] proved in that every upper semilattice of size at most $\aleph_1$ with the countable predecessor property can be embedded into $(\mathbf{D}, \leq_T, \vee)$, extending a previous result of Sacks [Sac61]. Hinman and Slaman [HS91] proved that every countable jump partial ordering is embeddable in $(\mathbf{D}, \leq_T, ')$. For countable structures, the most general result proved so far is the following:

Theorem 9 (Montalbán [Mon06c]). *Every countable jump upper semilattice can be embedded into the Turing Degrees $(\mathbf{D}, \leq_T, \vee, ')$ (of course, preserving join and jump).*

It follows from my result that the quantifier-free formulas that are always true in $(\mathbf{D}, \leq_T, \vee, ')$ are the exactly the ones that follow from the definition of jump upper semilattice. Therefore, it follows that the existential theory of $(\mathbf{D}, \leq_T, \vee, ')$ is decidable. More results about embeddings into the Turing degrees of both jump upper semilattices with $\mathbf{0}$ and uncountable jump upper semilattices can also be found in [Mon06c]. For example, I proved that the question of whether it is possible to embed every jump upper semilattice of size $\aleph_1$ satisfying the countable predecessor property into $(\mathbf{D}, \leq_T, \vee, ')$ is independent of ZFC.

Jockusch and Posner [JP78] defined the *generalized high/low hierarchy* with the intention of classifying the Turing degrees depending on how close a degree is to being computable, and on how close it is to computing the Halting Problem. This notion has been useful in many results. However, I've showed that, surprisingly, there is no order at all on these classes.

Theorem 10 (Montalbán [Mon06e]). *Every finite partial ordering, whose elements are labeled in any way with classes from the set $\{GL_1, GL_2, \dots, GI, \dots, GH_2, GH_1\}$, can be embedded into the Turing degrees preserving labels.*

Extensions of embeddings. Let $\mathbf{D}_{(\leq 0')}$ be the set of degrees below $0'$. We do know this is a complicated structure; its theory is 1-equivalent to true first order arithmetic [Sho81]. On the other hand, if we look only at existential sentences, we can decide which sentences are true (as follows from results in [KP54]). In order to understand where the complexity lies, we ask what fragments of its theory are decidable. It is known that the $\exists\forall$-theory of $(\mathbf{D}_{(\leq 0')}, \leq_T)$ is decidable [LS88], but the one of $(\mathbf{D}_{(\leq 0')}, \leq_T, \vee, \wedge)$ is not [MNS04]. The one quantifier theories of these structures are all decidable [KP54, LL76], and the three quantifier ones aren't [Ler83]. The only question left open is whether the $\exists\forall$-theory of $(\mathbf{D}_{(\leq 0')}, \leq_T, \vee)$ is decidable.

Downey, Greenberg, Lewis and I [DGLM] have found a good number of necessary and sufficient conditions that we expect will eventually lead to a solution of the problem. Many of the theorems we proved for this purpose are interesting on their own right, and provide a better understanding of the structure $\mathcal{D}_{(\leq 0')}$:

1. Simultaneous 1-genericity below c.e. sets: For every c.e. set C and every sequence $\{A_i : i \in \omega\}$ of sets uniformly computable in C, there exists a set $G \leq_T C$ that is simultaneously 1-generic relative to each A_i such that $A_i <_T C$.
2. No-least-join theorem: Consider degrees $\mathbf{a}, \mathbf{b} \leq_T \mathbf{c}$ with $\mathbf{c}$ c.e. such that $\mathbf{a} \not\leq_T \mathbf{b}$, $\mathbf{b} \not\leq_T \mathbf{0}$. Then, $\mathbf{b}$ is not the least degree below $\mathbf{c}$ that joins $\mathbf{a}$ up to $\mathbf{a} \vee \mathbf{b}$.
3. Join property for non-GL_2 degrees: Let $\mathbf{c}$ be a non-GL_2 degree. Then, for every degree $\mathbf{a} < \mathbf{c}$, there exists $\mathbf{x} < \mathbf{c}$ such that $\mathbf{a} \vee \mathbf{x} = \mathbf{c}$.
4. There exist c.e. sets A, B, C, D and E such that A, B, D and E are all Turing reducible to C and pairwise incomparable, and such that any Δ^0_2 set X which is computable in C and joins A above B also joins D above E.

Complexity vs structure. Another way of analyzing $(\mathbf{D}, \leq_T)$ has been by finding relations between the computational complexity of a degree $\mathbf{a}$ and the structure $\mathbf{D}_{(\leq_T \mathbf{a})} = \{\mathbf{x} \in \mathbf{D} : \mathbf{x} \leq_T \mathbf{a}\}$.

Posner [Pos81] asked if for every generalized high degree $\mathbf{a}$, the upper semilattice $\mathbf{D}_{(\leq_T \mathbf{a})}$ has the complementation property. Greenberg, Shore and I answered this question affirmatively in [GMS04].

Another way of analyzing the structures of $(\mathbf{D}_{(\leq_T \mathbf{a})}, \leq_T)$ is by studying the complexity of their theory. Shore [Sho81] proved that $Theory(\langle \mathbf{D}_{(\leq_T \mathbf{a}), \leq_T \rangle})$, is 1-equivalent to true first order arithmetic whenever $\mathbf{a}$ is arithmetic and $\geq_T \mathbf{0}'$, c.e., or high. With Greenberg [GM03], we extended this result to $\mathbf{a}$ being n-CEA, 1-generic and below $\mathbf{0}'$, 2-generic and arithmetic, or arithmetically generic.

Misselaneous

Effective randomness. I only obtained a few little results in this popular area of computability theory. With Csima [CM06] we have showed that there is a minimal pair of Kolmogorov degrees. With Kjos-Hanssen we observed that ranked sets are Never-Continuously-Random. With Slaman, we showed that every K-trivial is Never-Continuously-Random, result that was later extended by Barmpalias and Greenberg to all sets computed from incomplete c.e. sets [BGMS]. With Lewis and Nies, we showed that there is a weakly-2-random set that is not generalized-low [LMN07].

Orbits of the lattice of Π^0_1*-classes.* A Π^0_1 class P is called *thin* if, for every Π^0_1 subclass P' of P, there is a clopen C with $P' = P \cap C$. This property is preserved under automorphisms of the lattice of Π^0_1 classes under inclusion, as it is definable in this structure. Cholak, Coles, Downey and Herrmann [CCDH01] found sufficient conditions that make two thin Π^0_1 classes automorphic: They proved that if P and Q are thin Π^0_1 classes and their lattices of subclasses are isomorphic, and these lattices are a Boolean algebra with finitely many atoms, then P and Q are automorphic. Downey and I [DM08b] proved that this is the only case when the lattices of subclasses determine the automorphism orbit of thin Π^0_1 classes: We showed that if P is a thin Π^0_1 class and its lattice of subclasses is not a Boolean algebra with finitely many atoms, then there is another thin Π^0_1 class Q whose lattice of subclasses is isomorphic to the one of P, but such that P and Q are not automorphic. This was conjectured by Cholak and Downey in [CD04].

Büchi and Borel structures. With Hjorth, Khoussainov and Nies we started to work on the effective model theory of Büchi and Borel structures [HKMN08],. We analyzed continuum size structures that can be presented using a Büchi automata. This interaction between automata theorists, computability theorists and a descriptive set theorist led to interesting results. For example, among our results, we solved an open question in the area of whether every Büchi structure, where equality is represented as a Büchi equivalence relation, has a presentation where equality is just the the identity equivalence relation. To solve this question we use Borel presentable structures, and the fact that the Borel equivalence relation E_0 is not smooth.

Nies and I have written a survey paper on Borel structures [MN].

References

[AK00] C.J. Ash and J. Knight. *Computable Structures and the Hyperarithmetical Hierarchy.* Elsevier Science, 2000.

[AR] Bahareh Afshari and Michael Rathjen. Reverse mathematics of well ordering proofs. In preparation.

[AS86] Uri Abraham and Richard A. Shore. Initial segments of the degrees of size $\aleph_1$. *Israel Journal of Mathematics*, 53:1–51, 1986.

[Bal06] V. Baleva. The jump operation for structure degrees. *Arch. Math. Logic*, 45(3):249–265, 2006.

[BGMS] G. Barmpalias, N. Greenberg, A. Montalbán, and T. A. Slaman. K-trivials are never continuously random. submitted for publication.

[CCDH01] Peter Cholak, Richard Coles, Rod Downey, and Eberhard Herrmann. Automorphisms of the lattice of Π^0_1 classes: perfect thin classes and anc degrees. *Trans. Amer. Math. Soc.*, 353(12):4899–4924 (electronic), 2001.

[CD04] Peter A. Cholak and Rod Downey. Invariance and noninvariance in the lattice of Π^0_1 classes. *J. London Math. Soc. (2)*, 70(3):735–749, 2004.

[Clo90] Peter Clote. The metamathematics of Fraïssé's order type conjecture. In *Recursion theory week (Oberwolfach, 1989)*, volume 1432 of *Lecture Notes in Math.*, pages 41–56. Springer, Berlin, 1990.

[CM06] B. F. Csima and A. Montalbán. A minimal pair of k-degrees. *Proceedings of the American Mathematical Society*, 134(5):1499–1502, 2006.

[CMHM] B. Csima, R. Miller, V. Harizanov, and A. Montalbán. Computability of fraisse limits. To appear in the.

[CMS06] B. F. Csima, A. Montalbán, and R. A. Shore. Boolean algebras, tarski invariants, and index sets. *Notre Dame Journal of Formal Logic*, 47(1):1–23, 2006.

[Con] C. J. Conidis. The strength of the bolzano-weierstrass theorem the strength of the bolzano-weierstrass theorem the strength of the bolzano-weierstrass theorem the strength of the bolzano-weierstrass theorem. submitted for publication.

[DGLM] R. Downey, N. Greenberg, A. Lewis, and A. Montalbán. Extensions of embeddings below computably enumerable degrees. submitted for publication.

[DHK$^+$07] Downey, Hirschfeldt, Kach, Lempp, A. Montalbán, and Mileti. Subspaces of computable vector spaces. *Journal of Algebra*, 314(2):888–894, August 2007.

[DJ94] Rod Downey and Carl G. Jockusch. Every low Boolean algebra is isomorphic to a recursive one. *Proc. Amer. Math. Soc.*, 122(3):871–880, 1994.

[dJP77] D. H. J. de Jongh and Rohit Parikh. Well-partial orderings and hierarchies. *Nederl. Akad. Wetensch. Proc. Ser. A* **80**=*Indag. Math.*, 39(3):195–207, 1977.

[DM08a] R. Downey and A. Montalbán. The isomorphism problem for torsion-free abelian groups is analytic complete. *Journal of Algebra*, 320:2291–2300, 2008.

[DM08b] R. Downey and A. Montalbán. Slender classes. *Journal of the London Math. Society*, 78(1):36–50, 2008.

[FFH$^+$] E. B. Fokina, S. Friedman, V. Harizanov, J. F. Knight, C. McCoy, and A. Montalbán. Isomorphism and bi-embeddability relations on computable structures. submitted for publication.

[Fra86] R. Fraïssé. *Theory of Relations*. North-Holland, 1986.

[Fri75] Harvey M. Friedman. Some systems of second order arithmetic and their use. In *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1*, pages 235–242. Canad. Math. Congress, Montreal, Que., 1975.

[Fro06] A. N. Frolov. Δ^0_2-copies of linear orderings. *Algebra Logika*, 45(3):354–370, 376, 2006.

[FS89] Harvey Friedman and Lee Stanley. A Borel reducibility theory for classes of countable structures. *J. Symbolic Logic*, 54(3):894–914, 1989.

[GM03] N. Greenberg and A. Montalbán. Embedding and coding below a 1-generic degree. *Notre Dame Journal of Formal Logic*, 44(4):200–216, 2003.

[GM08] N. Greenberg and A. Montalbán. Ranked structures and arithmetic transfinite recursion. *Transactions of the AMS*, 360:1265–1307, 2008.

[GMS04] N. Greenberg, A. Montalbán, and R. A. Shore. Generalized high degrees have the complementation property. *Journal of Symbolic Logic*, 69(4):1200–1220, 2004.

[Hir94] Jeffry L. Hirst. Reverse mathematics and ordinal exponentiation. *Ann. Pure Appl. Logic*, 66(1):1–18, 1994.

[HKMN08] G. Hjorth, B. Khoussainov, A. Montalbán, and A. Nies. From automatic structures to borel structures. *Proceedings of the Twenty-Third Annual IEEE Symposium on Logic in Computer Science (LICS 2008)*, pages 431–441, June 2008.

[HMa] K. Harris and A. Montalbán. On the n-back-and-forth types of boolean algebras. ¡a href=”http://kaharris.org/Boolean/index.html”¿ Related material here¡/a¿. To appear in the.

[HMb] Kenneth Harris and A. Montalbán. Boolean algebra approximations. submitted for publication.

[Hod97] Wilfrid Hodges. *A shorter model theory*. Cambridge University Press, Cambridge, 1997.

[HS91] Peter G. Hinman and Theodore A. Slaman. Jump embeddings in the Turing degrees. *Journal of Symbolic Logic*, 56:563–591, 1991.

[JP78] Carl G. Jockusch, Jr. and David B. Posner. Double jumps of minimal degrees. *Journal of Symbolic Logic*, 43(4):715–724, 1978.

[Jul69] Pierre Jullien. *Contribution à l'étude des types d'ordre dispersés*. PhD thesis, Marseille, 1969.

[Kac07] Asher M. Kach. *Characterizing the computable structures: Boolean algebras and linear orders*. PhD in Mathematics, University of Wisconsin-Madison, 2007.

[KHM] A. Kach, D. Hirschfeldt, and A. Montalbán. A feiner look at the intermediate degrees. in preparation.

[KM] A. Kach and A. Montalbán. Cuts of linear orders. submitted for publication.

[KM10] B. Khoussainov and A. Montalbán. A computable $\aleph_0$-categorical structure whose theory computes true arithmetic. *Journal of Symbolic Logic*, 75(2):728–740, 2010.

[KP54] S.C. Kleene and E.L. Post. The upper semi-lattice of the degrees of recursive unsolvability. *Annals of Mathematics*, 59:379–407, 1954.

[KS00] Julia F. Knight and Michael Stob. Computable Boolean algebras. *J. Symbolic Logic*, 65(4):1605–1623, 2000.

[Lav71] Richard Laver. On Fraïssé's order type conjecture. *Annals of Mathematics (2)*, 93:89–111, 1971.

[Ler83] M. Lerman. *Degrees of Unsolvability*. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.

[LL76] A. H. Lachlan and R. Lebeuf. Countable initial segments of the degrees of unsolvability. *J. Symbolic Logic*, 41(2):289–300, 1976.

[LMN07] A. Lewis, A. Montalbán, and A. Nies. A weakly 2-random set that is not generalized low. In *Computation and Logic in the Real World*, volume 4497, pages 474–477. Springer Berlin / Heidelberg, 2007.

[LS88] Manuel Lerman and Richard A. Shore. Decidability and invariant classes for degree structures. *Trans. Amer. Math. Soc.*, 310(2):669–692, 1988.

[Mar05] Alberto Marcone. Wqo and bqo theory in subsystems of second order arithmetic. In *Reverse mathematics 2001*, volume 21 of *Lect. Notes Log.*, pages 303–330. Assoc. Symbol. Logic, La Jolla, CA, 2005.

[MM] A. Marcone and A. Montalbán. The veblen functions for computability theorists. submitted for publication.

[MN] A. Montalbán and A. Nies. Borel structures: a brief survey. submitted for publication.

[MNS04] Russell G. Miller, Andre O. Nies, and Richard A. Shore. The $\forall\exists$-theory of $\mathcal{R}(\leq, \vee, \wedge)$ is undecidable. *Trans. Amer. Math. Soc.*, 356(8):3025–3067 (electronic), 2004.

[Mona] A fixed point for the jump operator on structures. submitted for publication.

[Monb] A. Montalbán. Counting the back-and-forth types. submitted for publication.

[Monc] A. Montalbán. Ordinal functors and pi-1-1-ca-knot. Unpublished notes dated December 7, 2009.

[Mon05] A. Montalbán. Up to equimorphism, hyperarithmetic is recursive. *Journal of Symbolic Logic*, 70(2):360–378, 2005.

[Mon06a] A. Montalbán. Countably complementable linear orderings. *Order*, 23(4):321–331 (2007), 2006.

[Mon06b] A. Montalbán. Equimorphism invariants for scattered linear orderings. *Fundamenta Mathematicae*, 191(2):151–173, 2006.

[Mon06c] A. Montalbán. Equivalence between fraïssé's conjecture and jullien's theorem. *Annals of Pure and Applied Logic*, 139(1-3):1–42, 2006.

[Mon06d] A. Montalbán. Indecomposable linear orderings and hyperarithmetic analysis. *Journal of Mathematical Logic*, 6(1):89–120, 2006.

[Mon06e] A. Montalbán. There is no ordering on the classes in the generalized high/low hierarchies. *Archive for Mathematical Logic*, 45(2):215–231, 2006.

[Mon07a] A. Montalbán. Computable linearizations of well-partial-orderings. *Order*, 24:39–48, 2007.

[Mon07b] A. Montalbán. On the equimorphism types of linear orderings. *Bulletin of Symbolic Logic*, 13(1):71–99, 2007.

[Mon08a] A. Montalbán. On the pi-1-1 separation principle. *Mathematical Logic Quarterly*, 54(6):563–578, 2008.

[Mon08b] A. Montalbán. On the triple jump of the set of atoms of a boolean algebra. *Proceedings of the American Mathematical Society*, 136:2589–2595, 2008.

[Mon09a] A. Montalbán. Embeddings into the turing degrees. In *Logic colloquium 2006*, Lecture Notes in Logic, pages 229–246. ASL, 2009. To appear in the.

[Mon09b] A. Montalbán. Notes on the jump of a structure. *Mathematical Theory and Computational Practice*, pages 372–378, 2009.

[MS] A. Montalbán and R. A. Shore. The limits of determinacy in second order arithmetic. submitted for publication.

[Nee] I. Neeman. Necessary use of σ^1_1 induction in a reversal. To appear.

[Pos81] David B. Posner. The upper semilattice of degrees $\leq \mathbf{0}'$ is complemented. *J. Symbolic Logic*, 46(4):705–713, 1981.

[Puz09] V. G. Puzarenko. On a certain reducibility on admissible sets. *Sibirsk. Mat. Zh.*, 50(2):415–429, 2009.

[Puz11] Vadim Puzarenko. Fixed points of the jump operator. *Algebra i Logika*, 50, 2011.

[Rem89] J. B. Remmel. Recursive Boolean algebras. In *Handbook of Boolean algebras, Vol. 3*, pages 1097–1165. North-Holland, Amsterdam, 1989.

[Ros82] Joseph Rosenstein. *Linear orderings*. Academic Press, New York - London, 1982.

[Sac61] Gerald E. Sacks. On suborderings of degrees of recursive unsolvability. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 7:46–56, 1961.

[Sch77] Kurt Schütte. *Proof theory*. Springer-Verlag, Berlin, 1977. Translated from the revised German edition by J. N. Crossley, Grundlehren der Mathematischen Wissenschaften, Band 225.

[Sch79] Diana Schmidt. Well-partial orderings and their maximal order types. Habilitationsschrift, 1979. University of Heidelberg.

[Sho81] Richard A. Shore. The theory of the degrees below $\mathbf{0}'$. *J. London Math. Soc.*, 24:1–14, 1981.

[Sim99] Stephen G. Simpson. *Subsystems of second order arithmetic*. Springer, 1999.

[Sos07] Alexandra Soskova. A jump inversion theorem for the degree spectra. In *Proceeding of CiE 2007*, volume 4497 of *Lecture Notes in Comp. Sci.*, pages 716–726. Springer-Verlag, 2007.

[Spe55] Clifford Spector. Recursive well-orderings. *Journal Symbolic Logic*, 20:151–163, 1955.

[SS09] Alexandra A. Soskova and Ivan N. Soskov. A jump inversion theorem for the degree spectra. *J. Logic Comput.*, 19(1):199–215, 2009.

[Ste78] John R. Steel. Forcing with tagged trees. *Annals of Mathematical Logic*, 15(1):55–74, 1978.

[Stu09] A. I. Stukachev. A jump inversion theorem for semilattices of Σ-degrees. *Sib. Èlektron. Mat. Izv.*, 6:182–190, 2009.

[Stu10] A. I. Stukachev. A jump inversion theorem for the semilattices of Sigma-degrees [translation of mr2586684]. *Siberian Adv. Math.*, 20(1):68–74, 2010.

[Thu95] John J. Thurber. Every low_2 Boolean algebra has a recursive copy. *Proc. Amer. Math. Soc.*, 123(12):3859–3866, 1995.

[Van77] Robert Alan Van Wasep. *Subsystems of Second-order Arithmetic, and Descriptive Set Theory under the Axiom of Determinateness*. PhD thesis, University of California, Berkeley, 1977.

[Veb08] Oswald Veblen. Continuous increasing functions of finite and transfinite ordinals. *Trans. Amer. Math. Soc.*, 9(3):280–292, 1908.

Department of Mathematics, University of Chicago, Chicago, IL, USA
E-mail address: antonio@math.uchicago.edu
URL: www.math.uchicago.edu/∼antonio