# DECIDABILITY AND UNDECIDABILITY OF THE THEORIES OF CLASSES OF STRUCTURES

ASHER M. KACH AND ANTONIO MONTALBÁN

ABSTRACT. Many classes of structures have natural functions and relations on them: concatentation of linear orders, direct product of groups, disjoint union of equivalence structures, and so on. Here, we study the (un)decidability of the theory of several natural classes of structures with appropriate functions and relations. For some of these classes of structures, the resulting theory is decidable; for some of these classes of structures, the resulting theory is bi-interpretable with second-order arithmetic.

## 1. INTRODUCTION

Given a mathematical structure, as part of trying to understand it, a natural question to ask is whether its theory is decidable. On the one hand, the existence of an algorithm to decide the truth of any sentence about a structure can, of course, tell us a lot about the structure. On the other hand, knowing that such algorithms do not exist also gives us information. It tells us, for instance, that there are questions about the structure which are going to be hard to solve, and also that the structure itself is inherently very complex.

The authors started this project trying to answer a question from Ketonen [Ket]: Is the theory of the class of countable Boolean algebras, denoted by $\mathrm{BA}_{\aleph_0}$, with the direct sum operation, denoted by $\oplus$, decidable? When he posed the question, Ketonen had recently answered the following question:

Tarski's Cube Problem: Does there exist a countable Boolean algebra $\mathcal{B}$ such that $\mathcal{B} \cong \mathcal{B} \oplus \mathcal{B} \oplus \mathcal{B}$ but $\mathcal{B} \not\cong \mathcal{B} \oplus \mathcal{B}$?

Then, this was a well-known question which was open for a few decades before Ketonen [Ket78] resolved it by giving a decision procedure for all existential formulas about $(\mathrm{BA}_{\aleph_0}; \oplus)$: Ketonen proved that every countable commutative semi-group is embeddable in $(\mathrm{BA}_{\aleph_0}; \oplus)$, yielding a positive answer to the Tarski's cube problem. We show here that the full theory of $(\mathrm{BA}_{\aleph_0}; \oplus)$ is far from decidable; it is as complex as it can be.

*Date*: January 12, 2012.

The authors thank Steffen Lempp for early conversations on this subject and Julia Knight for bringing Feferman and Vaught [FV59] to their attention.

**Theorem.** *The first-order theory of the class of countable Boolean algebras under the direct sum operation, i.e., the first-order theory of the structure* $(BA_{\aleph_0};\oplus)$*, is 1-equivalent to true second-order arithmetic.*

We then look at the class of countable linear orderings, denoted by $\mathrm{LO}_{\aleph_0}$, with the concatentation operation, denoted by $+$. This time we do much more than just interpreting second order arithmetic.

**Theorem.** *The structure* $(LO_{\aleph_0};+)$ *of countable linear orderings under concatenation is bi-interpretable with second-order arithmetic.*

That two structures are bi-interpretable means that each is interpretable in the other, and also that the compositions of the interpretations are definable. Bi-interpretability with second-order arithmetic implies, in addition to the theory being 1-equivalent to true second-order arithmetic, that the structure is rigid and that every subset definable in second-order arithmetic is first-order definable in the structure.

We also look at the class of computable linear orderings and obtain the following result.

**Theorem.** *The theory of the structure* $(LO_{rec};+)$ *of computable linear orderings under concatenation is* 1*-equivalent to the* $\omega$*-jump of Kleene's* $\mathcal{O}$*.*

Next, we look at the class of groups. Here, we look at the class of countable groups, denoted by $\mathrm{GR}_{\aleph_0}$, under the direct product operation, denoted by $\times$, and the subgroup relation, denoted by $\leq$.

**Theorem.** *The first-order theory of countable groups under the direct product operation and the subgroup relation, i.e., the first-order theory of the structure* $(GR_{\aleph_0};\times,\leq)$*, is 1-equivalent to true second-order arithmetic.*

To break the pattern, and to contrast with these results, we give examples of theories which are decidable.

**Theorem.** *The theories of the following structures are decidable.*

- *The class of countable* $F$*-vector spaces under direct sum, for any fixed countable field* $F$*.*
- *The class of countable equivalence structures under disjoint union.*
- *The class of finitely generated abelian groups under direct sum.*

The main tools to prove the decidability results of this theorem are due to Tarski [Tar49] and Feferman and Vaught [FV59]. Using completely different techniques, we show that the existential theory of the class of countable linear orderings, under the relation "being a convex suborder of," is decidable.

The restriction to countable structures is non-essential for some of these results. For example, if $\kappa$ is an infinite cardinal, then the first-order theory of the class of linear orders of size at most $\kappa$, denoted $\mathrm{LO}_\kappa$, under concatenation is 1-equivalent to true second-order arithmetic.[1] We also observe that the

[1] Note that bi-interpretability is not possible for cardinality reasons if $\kappa > \aleph_0$.

theory is dependent on the infinite cardinal $\kappa$. For example, the first-order theory of $(\mathrm{LO}_{\aleph_0};+)$ and the first-order theory of $(\mathrm{LO}_\kappa;+)$ are distinct if $\kappa > \aleph_0$. Finally, in the case of linear orderings we note that if $\kappa = \beth_n$, then $(\mathrm{LO}_\kappa;+)$ interprets $(n+2)$nd-order arithmetic. Thus, for linear orderings, the theories get more and more complex as $\kappa$ grows. On the other hand, for equivalence structures, the theory cycles as $\kappa$ grows, though it always remains decidable.

Surprisingly, this type of investigation of the theories of classes of algebraic structures seems to be in its infancy. Indeed, the only example in the literature the authors are knowledgeable about is the Ketonen [Ket78] result already mentioned. However, a vast amount of the literature by computability theorists has focused on understanding the structure of the Turing degrees $\mathcal{D}$ and other related structures. For instance, Simpson [Sim77] showed that the full theory of $\mathcal{D}$ in the language $\{\leq\}$ is recursively isomorphic to true second-order arithmetic, and whether this structure is bi-interpretable with second-order arithmetic is among the main open questions in the field [Sla08].

Throughout, we denote the standard first-order model of arithmetic by $\mathcal{N}_1 = (\mathbb{N};+,\times,\leq)$, where $+\subset\mathbb{N}^3$, $\times\subset\mathbb{N}^3$, and $\leq\subset\mathbb{N}^2$ are interpreted as the usual addition, multiplication, and less than relations. We denote the standard second-order model of arithmetic by $\mathcal{N}_2 = (\mathbb{N},\mathcal{P}(\mathbb{N});+,\times,\leq,\in)$, where $+\subset\mathbb{N}^3$, $\times\subset\mathbb{N}^3$, $\leq\subset\mathbb{N}^2$, and $\in\subset\mathbb{N}\times\mathcal{P}(\mathbb{N})$ are interpreted as the usual addition, multiplication, less than, and membership relations.

## 2. Linear Orders Under Addition

It has long been known that the class of linear orders is deceptively rich. Amongst countable order types, the *scattered* / *nonscattered* dichotomy, together with Hausdorff's analysis of scattered linear orders, yield a relatively straightforward means of understanding the countable order types. This dichotomy and analysis applies to uncountable order types as well, though it fails to characterize the uncountable order types as succinctly. Consequently, it might seem the class of countable order types fails to be as rich as the class of uncountable order types.

Here, we show that the class of countable order types under concatenation is already rather rich in that its theory is as complicated as possible. We also show the first-order theory of the countable linear orders under concatenation differs from the first-order theory of the uncountable linear orders under concatenation. Finally, we show that the class of computable order types under concatenation is also rather rich in that its theory is also as complicated as possible.

**Definition 2.1.** Fix an infinite cardinal $\kappa$. Define $\mathrm{LO}_\kappa$ to be the set of all isomorphism types of linear orders of size at most $\kappa$ and $\mathbb{LO}^+_\kappa = (\mathrm{LO}_\kappa;+)$ to be the monoid of linear orders of size at most $\kappa$ under concatenation.

Throughout, we operate under the convention that the set $\mathrm{LO}_\kappa$ includes the *empty linear order* (with empty universe). Being the identity element of the monoid $\mathbb{LO}_\kappa^+$, we denote the empty linear order by $\mathbf{0}$.

**Definition 2.2.** For $u, v \in \mathrm{LO}_\kappa$, we write $u \trianglelefteq v$ if $(\exists w_1)(\exists w_2)\,[v = w_1 + u + w_2]$, $u \trianglelefteq_I v$ if $(\exists w_2)\,[v = u + w_2]$, and $u \trianglelefteq_E v$ if $(\exists w_1)\,[v = w_1 + u]$.

We write $u \triangleleft v$ if $u \trianglelefteq v$ and $v \not\trianglelefteq u$.

We emphasize the relation $\trianglelefteq$ is not a partial order as there exist distinct $a, b \in \mathrm{LO}_\kappa$ with $a \trianglelefteq b$ and $b \trianglelefteq a$, for example $a := \eta$ and $b := 1 + \eta + 1$. On the other hand, it is immediate the relation $\trianglelefteq$ is reflexive and transitive, so a preorder.

2.1. **Interpreting Second-Order Arithmetic.** As preparation to interpreting second-order arithmetic in $\mathbb{LO}_\kappa^+$, we develop a small repertoire of definable subsets of $\mathrm{LO}_\kappa$.

**Lemma 2.3.** *Fix an infinite cardinal $\kappa$. Each of the following subsets of $LO_\kappa$ is first-order definable in $\mathbb{LO}_\kappa^+$:*

(1) $\{\mathbf{n}\}$ *for* $n \in \mathbb{N}$
(2) $\{\omega\}$, $\{\omega^*\}$, $\{\zeta\}$
(3) $FIN$ *(the set of finite order types)*
(4) $ORD_\kappa$ *(the set of ordinals of cardinality at most $\kappa$)*
(5) $RAI_\kappa$ *(the set of right additively indecomposable linear orders of cardinality at most $\kappa$)*
(6) $\{\zeta^n\}$, $\{\zeta^n \cdot \omega\}$, $\{\zeta^n \cdot \omega^*\}$ *for* $n \in \mathbb{N}$

*Proof.* We exhibit a first-order formula witnessing the definability of each subset.

(1) The formula

$$\psi_0(x) := (\forall y)[y = x + y]$$

is easily seen to define the set $\{\mathbf{0}\}$.

The formula

$$\psi_1(x) := y \neq \mathbf{0} \wedge (\forall y \trianglelefteq x)\,[y = \mathbf{0} \vee y = x]$$

defines the set $\{\mathbf{1}\}$. The reason is the second conjunct implies $x$ has size at most one as both $\mathbf{0}$ and $\mathbf{1}$ are $\triangleleft$-below all order types of size two or greater.

The formula

$$\psi_n(x) \;:=\; x = \mathbf{1} + \cdots + \mathbf{1}$$

is easily seen to define the set $\{\mathbf{n}\}$.

(2) The formulas

$$\begin{aligned}
\psi_\omega(x) &:= x = \mathbf{1} + x \wedge (\forall z)\,[z = \mathbf{1} + z \implies x \trianglelefteq z]\,,\\
\psi_{\omega^*}(x) &:= x = x + \mathbf{1} \wedge (\forall z)\,[z = z + \mathbf{1} \implies x \trianglelefteq z]\,,\\
\psi_\zeta(x) &:= x = \omega^* + \omega
\end{aligned}$$

define the sets $\{\omega\}$, $\{\omega^*\}$, and $\{\zeta\}$, respectively. For $\psi_\omega(x)$, the first conjunct implies $\omega \trianglelefteq x$ by induction: As $\mathbf{1}+x$ has a least element, the order type $x$ has a least element. Because $x$ has a least element, the order type $\mathbf{1}+x$ has a second smallest element. Hence $x$ has a second smallest element. Continuing, this implies $\omega \trianglelefteq x$. The second conjunct implies $x \trianglelefteq \omega$ by choice of $\omega$ for $z$.

(3) The formula

$$\psi_{FIN}(x) := x \lhd \omega$$

is easily seen to define the set of finite natural numbers.

(4) The formula

$$\psi_{ORD}(x) := (\forall y)(\forall z)\,[x = y + z \ \wedge\ z \neq 0 \implies (\exists w)\,[z = \mathbf{1} + w]]$$

is easily seen to define the set of well-orders.

(5) The formula

$$\psi_{RAI}(x) := (\forall y)(\forall z)\,[x = y + z \ \wedge\ z \neq 0 \implies x = z]\,.$$

defines the set of right additively indecomposable linear orders as the right additively indecomposable linear orders are defined by this property.

(6) The formulas

$$\begin{aligned}
\psi_{\zeta^n\cdot\omega}(x) &:= x = \zeta^n + x \ \wedge\ (\forall z)\,[z = \zeta^n + z \implies x \trianglelefteq z]\,,\\
\psi_{\zeta^n\cdot\omega^*}(x) &:= x = x + \zeta^n \ \wedge\ (\forall z)\,[z = z + \zeta^n \implies x \trianglelefteq z]\,,\\
\psi_{\zeta^n\cdot\zeta}(x) &:= x = \zeta^n\cdot\omega^* + \zeta^n\cdot\omega
\end{aligned}$$

define the sets $\{\zeta^n\cdot\omega\}$, $\{\zeta^n\cdot\omega^*\}$, and $\{\zeta^n\cdot\zeta\}$, respectively, by analysis similar to Part (2). Indeed, the base case of the induction is Part (2).

Hence, the enumerated subsets are first-order definable in $\mathbb{LO}_\kappa^+$. □

These definable subsets will be exploited in our encoding of the standard model of arithmetic into $\mathbb{LO}_\kappa^+$. Indeed, we will encode the natural number $n \in \mathbb{N}$ by the order type $\mathbf{n}$. Thus, the set of natural numbers $FIN$ is definable by Lemma 2.3(3). Further, the order on the natural numbers is definable as $m \leq n$ if and only if $\mathbf{m} \trianglelefteq \mathbf{n}$, and addition is definable as $m+n=p$ if and only if $\mathbf{m}+\mathbf{n}=\mathbf{p}$.

**Definition 2.4.** If $(n_1,\dots,n_k) \in \mathbb{N}^k$ is an ordered $k$-tuple, let $t_k(n_1,\dots,n_k)$ be the order type

$$t_k(n_1,\dots,n_k) := \zeta^2 + \mathbf{n_1} + \zeta + \mathbf{n_2} + \zeta + \cdots + \mathbf{n_{k-1}} + \zeta + \mathbf{n_k} + \zeta + \zeta^2.$$

If $z \in \mathrm{LO}_\kappa$ and $k \in \mathbb{N}$, let $S_k(z)$ be the subset

$$S_k(z) := \{(n_1,\dots,n_k) \in \mathbb{N}^k : t_k(n_1,\dots,n_k) \trianglelefteq z\}$$

and say that $z$ *codes* the set $S_k(z)$.

An element $m \in \mathrm{LO}_\kappa$ is a *multiplicative code for* $\mathcal{N}_1$ if, with

- $y_1 \cdot y_2 = y_3$ if and only if $(y_1,y_2,y_3) \in S_3(m)$, the structure $\mathbb{LO}_\kappa^+$ satisfies the sentence that says $S_3(m)$ defines a function $\cdot\colon \mathbb{N}^2 \to \mathbb{N}$ with $a \cdot 0 = 0$ and $a \cdot (b+1) = a \cdot b + b$ for all $a, b \in \mathbb{N}$.

Careful inspection of Definition 2.4 shows that the property of being a multiplicative code for $\mathcal{N}_1$ is first-order definable in $\mathbb{LO}_\kappa^+$.

**Definition 2.5.** Fix a set $X = \{x_i\}_{i \in I} \subseteq \mathbb{N}$. The *code for* $X$ is the order type $t_I(X)$ given by

$$t_I(X) := \sum_{i \in I} t_1(x_i)$$

We note that, as $t_I(X)$ is countable for any $X \subseteq \mathbb{N}$, every subset $X \subseteq \mathbb{N}$ has a code in $\mathrm{LO}_\kappa$. Moreover, we have that $X = S_1(t_I(X))$ for all $X \subseteq \mathbb{N}$.

**Theorem 2.6.** *Fix $\kappa \geq \aleph_0$. Then $Th(\mathcal{N}_2) \leq_1 Th(\mathbb{LO}_\kappa^+)$.*

*Proof.* Let $\varphi$ be a sentence in the language of $\mathcal{N}_2$. Let $\psi(m)$ be the formula with one free variable in the language of $\mathbb{LO}_\kappa^+$ obtained from $\varphi$ by replacing instances of

- $x \leq y$ with $x \trianglelefteq y$,
- $x + y = z$ with $x + y = z$,
- $x \cdot y = z$ with $\zeta^2 + x + \zeta + y + \zeta + z + \zeta + \zeta^2 \trianglelefteq m$,
- $x \in X$ with $\zeta^2 + x + \zeta + \zeta^2 \trianglelefteq v_X$,
- $\exists x$ with $\exists x \in FIN$, and
- $\exists X$ with $\exists v_X$.

Let $\chi$ be the sentence stating there is a multiplicative code $m$ for $\mathcal{N}_1$ and $\psi(m)$.

Then $\mathcal{N}_2 \models \varphi$ if and only if $\mathbb{LO}_\kappa^+ \models \chi$ as a multiplicative code for $\mathcal{N}_1$ codes a structure isomorphic to $\mathcal{N}_1$. □

*Remark* 2.7. The method of interpreting second-order arithmetic in $\mathbb{LO}_\kappa$ can be generalized to interpret higher-order arithmetic in $\mathbb{LO}_\kappa^+$ for sufficiently large $\kappa$.

For example, an ordered set of ordered sets of natural numbers

$$\mathcal{S} = \left\{S^j\right\}_{j \in J} = \left\{\left\{n_i^j\right\}_{i \in I}\right\}_{j \in J} \subseteq \mathcal{P}(\mathcal{P}(\mathbb{N}))$$

can be coded by the order type

$$\zeta^3 + \sum_{j \in J} \left( \sum_{i \in I} \left( n_i^j + \zeta \right) + \zeta^2 \right) + \zeta^3.$$

This order potentially has cardinality $2^{\aleph_0}$ as the set $J$ could be of size continuum. Thus, we need $\kappa \geq \beth_1 := 2^{\aleph_0}$ to interpret third-order arithmetic.

In a similar fashion, it is possible to interpret $(n+2)$nd-order arithmetic in $\mathbb{LO}_\kappa^+$ for $\kappa \geq \beth_n$.

2.2. **Bi-Interpretability of Second-Order Arithmetic.** In Section 2.1, we saw how to interpret the standard model of second-order arithmetic in $\mathbb{LO}^+_\kappa$. We also know how to interpret $\mathbb{LO}^+_{\aleph_0}$ in second-order arithmetic by using linear orders whose domain is a subset of $\mathbb{N}$. We now show these two interpretations are sufficiently compatible, enough to yield the bi-interpretability of second-order arithmetic.

We review the encoding of a countable linear order in arithmetic. For a set of pairs $A \subseteq \mathbb{N}^2$, we let

$$dom(A) := \{x \in \mathbb{N} : (x, x) \in A\}.$$

Provided $A$ specifies an antisymmetric, transitive, and total order on $dom(A)$, we view $A$ as encoding the linear order $\mathcal{L}_A := (dom(A); A)$.

The set $A$ will be the set $S_2(\mathcal{A})$ coded by a linear ordering $\mathcal{A}$. Consequently, the properties of antisymmetry, transitivity, and totality are first-order definable in $\mathbb{LO}^+_\kappa$. For example, totality of $\mathcal{L}_{S_2(\mathcal{A})}$ can be given by $(\forall x, y \in FIN)\,[t_2(x,x) \trianglelefteq \mathcal{A} \wedge t_2(y,y) \trianglelefteq \mathcal{A} \implies t_2(x,y) \trianglelefteq \mathcal{A} \vee t_2(y,x) \trianglelefteq \mathcal{A}]$.

**Definition 2.8.** Let $B \subseteq \mathrm{LO}_{\aleph_0} \times \mathrm{LO}_{\aleph_0}$ be the relation such that $B(\mathcal{L}, \mathcal{A})$ holds if and only if $\mathcal{L} = \mathcal{L}_{S_2(\mathcal{A})}$.

**Theorem 2.9.** *The relation $B$ is first-order definable in $\mathbb{LO}^+_{\aleph_0}$. Thus $\mathbb{LO}^+_{\aleph_0}$ is bi-interpretable with second-order arithmetic via the interpretation within Definition 2.4.*

As preparation to proving Theorem 2.9, we exhibit a condition which is equivalent to $\mathcal{L} \cong \mathcal{L}_A$ when both have a least element.

**Lemma 2.10.** *Fix $\mathcal{L} \in LO_{\aleph_0}$ and a set $A \subseteq \mathbb{N}$ coding a linear ordering $\mathcal{L}_A$ with least element 0. Then $\mathcal{L} \cong \mathcal{L}_A$ if and only if there is a set*

$$C \subseteq \{\mathcal{B} : \mathcal{B} \trianglelefteq \mathcal{L}\} \times dom(A) \times (dom(A) \cup +\infty)$$

*such that:*

1. *$(\mathcal{L}, 0, +\infty) \in C$.*
2. *If $(\mathcal{B}, a_1, a_2) \in C$ with $a_1 \neq a_2$, then $\mathcal{B}$ has a least element.*
3. *If $(\mathcal{B}, a_1, a_3) \in C$ and $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$ with $\mathcal{B}_2$ either empty or having a least element, then there exists $a_2 \in dom(A)$ with $a_1 \leq_A a_2 \leq_A a_3$ such that $(\mathcal{B}_1, a_1, a_2) \in C$ and $(\mathcal{B}_2, a_2, a_3) \in C$.*
4. *If $(\mathcal{B}, a_1, a_3) \in C$ and $a_2 \in dom(A)$ with $a_1 \leq_A a_2 \leq_A a_3$, then there exist $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$ and $(\mathcal{B}_1, a_1, a_2) \in C$ and $(\mathcal{B}_2, a_2, a_3) \in C$.*
5. *If $(\mathcal{B}, a, a) \in C$ then $\mathcal{B} = \mathbf{0}$.*
6. *If $(\mathbf{0}, a_1, a_2) \in C$, then $a_1 = a_2$.*

*Proof.* The idea is that $(\mathcal{B}, a_1, a_2)$ is in $C$ if and only if the order type of $A$ restricted to the interval $[a_1, a_2)$ is the linear ordering $\mathcal{B}$.

If $\mathcal{L} \cong \mathcal{L}_A$, then it suffices to take $C$ to be the set of all

$$(\mathcal{B}, a_1, a_2) \in \{\mathcal{B} : \mathcal{B} \trianglelefteq \mathcal{L}\} \times dom(A) \times (dom(A) \cup +\infty)$$

such that $\mathcal{B}$ is isomorphic to the interval $[a_1, a_2)$ of $\mathcal{L}_A$. This is readily verified to satisfy the enumerated conditions.

Conversely, we construct an isomorphism between $\mathcal{L}$ and $\mathcal{L}_A$ using a back-and-forth construction. Assume such a set $C$ exists. Let $V$ be the set of pairs of tuples $((x_1, \dots, x_n), (a_1, \dots, a_n)) \in \mathcal{L}^{<\omega} \times \mathcal{L}_A^{<\omega}$ such that:

(1) for all $1 \le i, j \le n$, $x_i \le_{\mathcal{L}} x_j \iff a_i \le_A a_j$, and
(2) for all $1 \le i, j \le n$ with $x_i \le_{\mathcal{L}} x_j$, we have $(\mathcal{L}_{[x_i,x_j)}, a_i, a_j) \in C$, $(\mathcal{L}_{<x_j}, 0, a_j) \in C$, and $(\mathcal{L}_{\ge x_i}, a_i, +\infty) \in C$.

Suppose that $((x_1, \dots, x_n), (a_1, \dots, a_n)) \in V$. Using the definition of $C$, we get that

$$\forall x \in \mathcal{L} \ \ \exists a \in dom(A) \ \ [((x_1, \dots, x_n, x), (a_1, \dots, a_n, a)) \in V], \qquad \text{and}$$
$$\forall a \in dom(A) \ \ \exists x \in \mathcal{L} \ \ [((x_1, \dots, x_n, x), (a_1, \dots, a_n, a)) \in V].$$

Noting that $(\varepsilon, \varepsilon) \in V$, where $\varepsilon$ is the empty tuple, starts the recursion. Thus, a back-and-forth construction yields an isomorphism $\mathcal{L} \cong \mathcal{L}_A$. □

We modify the coding of triples of natural numbers as in Definition 2.4 to code triples that involve linear orderings.

**Definition 2.11.** Fix an ordinal $\alpha$ and a linear order $\mathcal{C} \in \mathrm{LO}_\kappa$. Let $Triple_\alpha(\mathcal{C})$ be the set of all triples $(\mathcal{B}, a_1, a_2)$ in $\mathrm{LO}_\kappa \times \mathbb{N} \times \mathbb{N}$ such that

$$\alpha \cdot \mathbf{2} + \mathcal{B} + \alpha^* + \alpha + \mathbf{a_1} + \alpha^* + \alpha + \mathbf{a_2} + \alpha^* + \alpha + \alpha^* \cdot \mathbf{2} \trianglelefteq \mathcal{C}.$$

**Lemma 2.12.** *Given an order type $\mathcal{L} \in LO_\kappa$, let $\alpha$ be an additively indecomposable ordinal of cardinality $\kappa$ such that $\alpha \not\trianglelefteq \mathcal{L}$ and $\alpha^* \not\trianglelefteq \mathcal{L}$. Then for every countable set $C \subseteq \{\mathcal{B} : \mathcal{B} \trianglelefteq \mathcal{L}\} \times \mathbb{N}^2$, there is a $\mathcal{C} \in LO_\kappa$ such that $C = Triple_\alpha(\mathcal{C})$.*

*Proof.* The linear ordering

$$\mathcal{C} := \sum_{(\mathcal{B}, a_1, a_2) \in C} (\alpha \cdot \mathbf{2} + \mathcal{B} + \alpha^* + \alpha + \mathbf{a_1} + \alpha^* + \alpha + \mathbf{a_2} + \alpha^* + \alpha + \alpha^* \cdot \mathbf{2})$$

suffices. We note that $\mathcal{C}$ has cardinality $\kappa$, being a sum of linear orderings of size $\kappa$.

The only segments of $\mathcal{C}$ of the form $\alpha \cdot \mathbf{2}$ or $\alpha^* \cdot \mathbf{2}$ are the ones shown. Also, if

$$\begin{aligned} &\alpha \cdot \mathbf{2} + \mathcal{B} + \alpha^* + \alpha + \mathbf{a_1} + \alpha^* + \alpha + \mathbf{a_2} + \alpha^* + \alpha + \alpha^* \cdot \mathbf{2} = \\ &\quad \alpha \cdot \mathbf{2} + \mathcal{B}' + \alpha^* + \alpha + \mathbf{a_1'} + \alpha^* + \alpha + \mathbf{a_2'} + \alpha^* + \alpha + \alpha^* \cdot \mathbf{2}, \end{aligned}$$

with $\mathcal{B}, \mathcal{B}' \trianglelefteq \mathcal{L}$, then necessarily $\mathcal{B} = \mathcal{B}'$, $a_1 = a_1'$ and $a_2 = a_2'$, because the only segments isomorphic to $\alpha$ or $\alpha^*$ are the ones shown. □

*Proof of Theorem 2.9.* We are now ready to define the relation $B$. Fixing linear orderings $\mathcal{L}, \mathcal{A} \in \mathrm{LO}_{\aleph_0}$, we (in a first-order manner) determine whether $\mathcal{L}$ and $\mathcal{L}_{S_2(\mathcal{A})}$ both have a least element. If not, we consider $\mathbf{1} + \mathcal{L}$ and $\mathbf{1} + \mathcal{L}_{S_2(\mathcal{A})}$, noting that $\mathcal{L} \cong \mathcal{L}_{S_2(\mathcal{A})}$ if and only if $\mathbf{1} + \mathcal{L} \cong \mathbf{1} + \mathcal{L}_{S_2(\mathcal{A})}$.

If so, or after we have added a least element, the relation $B(\mathcal{L}, \mathcal{A})$ holds if and only if

> The set of pairs $S_2(\mathcal{A}) \subseteq \mathbb{N}^2$ coded by $\mathcal{A}$ codes a linear ordering $\mathcal{L}_{S_2(\mathcal{A})}$, and there exists $\mathcal{C} \in \mathrm{LO}_{\aleph_0}$ which codes a set of triples $C := Triple_\alpha(C)$ as in Definition 2.11 using an additively indecomposable ordinal $\alpha$ such that $\alpha \not\trianglelefteq \mathcal{L}$ and $\alpha^* \not\trianglelefteq \mathcal{L}$, and the set $C$ satisfies the condition of Lemma 2.10.

□

The following are standard consequence of bi-interpretability.

**Corollary 2.13.** *Let $\mathbb{K} \subseteq LO_{\aleph_0}$ be a definable subset in second-order arithmetic. Then $\mathbb{K}$ is definable in $\mathbb{LO}^+_{\aleph_0}$.*

*Proof.* Using the definition of $\mathbb{K}$ in second-order arithmetic together with the coding of the previous section, we can define the set of all $\mathcal{A} \in \mathrm{LO}_{\aleph_0}$ which code a set $A \subseteq \mathbb{N}^2$ representing a linear ordering in $\mathbb{K}$. Then, the set $\mathbb{K}$ consist of all linear orderings $\mathcal{L}$ such that $B(\mathcal{L}, \mathcal{A})$ holds for some such $\mathcal{A}$. □

The importance of Corollary 2.13 is that it implies the definability of several natural classes that might not seem definable otherwise. For example, it implies the definability of the subsets $\{(x, y, z) : x \cdot y = z\} \subset \mathrm{LO}^3_{\aleph_0}$, $\{x : x \text{ is scattered}\} \subset \mathrm{LO}_{\aleph_0}$ and $\{(x, y) : x \text{ has Hausdorff rank } y\} \subset \mathrm{LO}_{\aleph_0} \times ORD_{\aleph_0}$.

**Corollary 2.14.** *The structure $\mathbb{LO}^+_{\aleph_0}$ is rigid.*

*Proof.* The standard model of arithmetic is rigid. If $\mathcal{A}$ codes a set $A \subseteq \mathbb{N}^2$, then any linear ordering in the orbit of $\mathcal{A}$ codes the same set $A$.

Now, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are automorphic and $B(\mathcal{L}_1, \mathcal{A}_1)$ holds, then $B(\mathcal{L}_2, \mathcal{A}_2)$ holds where $\mathcal{A}_2$ is the image of $\mathcal{A}_1$ under this automorphism. But then $\mathcal{A}_1$ and $\mathcal{A}_2$ code the same set of pairs $A \subseteq \mathbb{N}^2$, and hence the same linear ordering. So $\mathcal{L}_1 \cong \mathcal{L}_2$. □

2.3. **The Decidability of Certain Fragments.** Though Theorem 2.9 establishes the complexity of the first-order theory of $\mathbb{LO}^+_\kappa$, it does not indicate how quickly the theory becomes complicated. Here, we establish the decidability and undecidability of certain fragments of the first-order theory of $\mathbb{LO}^+_\kappa$.

**Definition 2.15.** Let $\mathbb{LO}^{\trianglelefteq}_\kappa = (\mathrm{LO}_\kappa; \trianglelefteq)$ be the poset of linear orders of size at most $\kappa$ under the binary relation $\trianglelefteq$.

**Theorem 2.16.** *Fix an infinite cardinal $\kappa$. The $\exists$-theory of the structure $\mathbb{LO}^{\trianglelefteq}_\kappa$ is decidable. Indeed, every finite preorder is a substructure of $\mathbb{LO}^{\trianglelefteq}_\kappa$.*

*Proof.* It suffices to show that every $n$ element preorder is a substructure of $\mathbb{LO}_\kappa^{\trianglelefteq}$. We do so by constructing a (finite) subset $T_k \subset \mathrm{LO}_\kappa$ that is universal for partial orders with $n$-many elements. We then expand $T_n$ to a (finite) subset $S_n \subset \mathrm{LO}_\kappa$ that is universal for preorders with $n$-many elements.

We construct $T_n$ as the union of sets $T_n(i)$ for $0 \leq i \leq n$, defined recursively in $i$. Let $T_n(0)$ be the set

$$T_n(0) := \left\{ \sum_{m \in \omega} (\eta + \mathbf{k}) : n \leq k < 2n \right\}.$$

Having defined the (finite) sets $T_n(j)$ for $j < i$, let $T_n(i)$ be the set

$$T_n(i) := \left\{ \sum_{m \in \omega} \sum_{\tau \in U} \tau : U \subseteq \bigcup_{j<i} T_n(j) \right\}$$

where the subset $U$ of $\bigcup_{j<i} T_n(j)$ is an ordered subset. From $T_n := \bigcup_{0 \leq i \leq n} T_n(i)$, we define $S_n$ by

$$S_n := \{\mathbf{j} + \tau : \tau \in T_n, 0 \leq j \leq n\}.$$

In order to show that $T_n$ is universal for partial orders with $n$-many elements, we analyze the structure of $T_n$. The elements of $T_n(0)$ form an antichain of size $n$. For $i > 0$, an element $\sum_{m \in \omega} \sum_{\tau \in U} \tau \in T_n(i)$ is $\trianglelefteq$-above an element in $T_n(j)$ for $j < i$ if and only if a summand $\tau \in U$ is $\trianglelefteq$-above the element of $T_n(j)$. These observations make it clear that $T_n$ is universal for partial orders with $n$-many elements.

In order to show that $S_n$ is universal for preorders with $n$-many elements, we show that $\mathbf{j_1} + \tau$ and $\mathbf{j_2} + \tau$ satisfy $\mathbf{j_1} + \tau \trianglelefteq \mathbf{j_2} + \tau$ for any $\tau \in T_n$ and $j_1, j_2 \in \{0, \ldots, n\}$. This is immediate for $\tau \in T_n(0)$ as

$$\begin{aligned} \mathbf{j_1} + \sum_{m \in \omega} (\eta + \mathbf{k}) &\trianglelefteq_E (\eta + (\mathbf{k} - \mathbf{j_1}) + \mathbf{j_1}) + \sum_{m \in \omega} (\eta + \mathbf{k}) \\ &= \sum_{m \in \omega} (\eta + \mathbf{k}) \\ &\trianglelefteq_E \mathbf{j_2} + \sum_{m \in \omega} (\eta + \mathbf{k}). \end{aligned}$$

Of course, we are using that $j_1 \leq k$ as a result of $j_1 \leq n \leq k$. As a consequence of the recursive construction of $T_n(i)$ for $i > 0$, all $\tau \in T_n$ have an initial segment that is in $T_n(0)$. From this, it follows $\mathbf{j_1} + \tau \trianglelefteq \mathbf{j_2} + \tau$ for all $\tau \in T_n$. Thus, for every element $x$ in $T_n$, the set $S_n$ contains $n$-many distinct elements $y$ with $x \trianglelefteq y$ and $y \trianglelefteq x$. It follows that the set $S_n$ is universal for preorders with $n$-many elements. $\square$

Of course, it would be desirable to know whether or not the existential theory of $\mathbb{LO}_\kappa^+$ was decidable. The existential theory of $\mathbb{LO}_\kappa^+$ is, intuitively if not computationally, more complicated than the existential theory of $\mathbb{BA}_\kappa^+$ as a consequence of results such as the following.

**Theorem 2.17** (Lindenbaum [Ros82])**.** *If $x, y \in LO_\kappa$ satisfy $x \trianglelefteq_I y$ and $y \trianglelefteq_E x$, then $x = y$.*

Thus, not every commutative semigroup embeds as $x = x + x + x$ and $x \neq x + x$ is impossible. Despite this, the authors conjecture the decidability of the existential fragment.

*Conjecture* 2.18. Fix an infinite cardinal $\kappa$. The first-order existential theory of the structure $\mathbb{LO}_\kappa^+$ is decidable.

It seems, though, that any possible proof of this would necessarily be rather involved.

2.4. **Changing the Cardinal $\kappa$.** When showing $\mathbb{LO}_\kappa^+$ interprets second-order arithmetic, the cardinal $\kappa$ played no significant role in the analysis (provided it was infinite). Here, we study the dependence of the first-order theory of $\mathbb{LO}_\kappa^+$ on the (infinite) cardinal $\kappa$.

**Theorem 2.19.** *Fix $\kappa > \aleph_0$. Then the first-order theories of $\mathbb{LO}_{\aleph_0}^+$ and $\mathbb{LO}_\kappa^+$ are distinct.*

*Proof.* The distinction in the theories we exploit is the number of dense linear orders without endpoints. In $\mathrm{LO}_{\aleph_0}$, there is exactly one dense linear order without endpoints, namely the order type $\eta$ of the rationals. In $\mathrm{LO}_\kappa$, there are multiple dense linear orders without endpoints, namely the order type $\eta$ of the rationals and the order type of a suborder of the reals of size $\aleph_1$ containing the rationals. Since

$$\psi_{DLOWE}(x) := (\forall y)(\forall z)\,[x \neq y + \mathbf{2} + z] \wedge (\forall y)\,[x \neq \mathbf{1} + y] \wedge (\forall y)\,[x \neq y + \mathbf{1}]$$

defines the dense linear orders without endpoints, this distinction witnesses that the first-order theories of $\mathbb{LO}_{\aleph_0}^+$ and $\mathbb{LO}_\kappa^+$ are distinct. □

As there are at most $2^{\aleph_0}$ distinct first-order theories in a finite language, the first-order theories of $\mathbb{LO}_{\kappa_1}^+$ and $\mathbb{LO}_{\kappa_2}^+$ cannot be distinct for all infinite cardinals $\kappa_1$ and $\kappa_2$. We leave the general question open.

**Question 2.20.** For which uncountable cardinals $\kappa_1$ and $\kappa_2$ are the first-order theories of $\mathbb{LO}_{\kappa_1}^+$ and $\mathbb{LO}_{\kappa_2}^+$ distinct?

Alternately, it would be interesting and perhaps easier to determine whether the first-order theory of $\mathbb{LO}_\kappa^+$ is eventually constant: Is there a cardinal $\lambda$ such that the first-order theories of $\mathbb{LO}_{\kappa_1}^+$ and $\mathbb{LO}_{\kappa_2}^+$ are the same if $\kappa_1, \kappa_2 > \lambda$?

2.5. **Computable Linear Orders.** Just as it might seem that the class of uncountable linear orderings is richer than the class of countable linear orderings, it might seem that the class of countable linear orderings is richer than the class of computable linear orderings. We show that this is not the case as the theory of the class of computable linear orderings under concatenation is as complicated as possible.

**Definition 2.21.** Define $LO_{rec}$ to be the set of all isomorphism types of computable linear orderings and $\mathbb{LO}^+_{rec} = (LO_{rec}; +)$ to be the monoid of computable linear orderings under concatenation.

**Theorem 2.22.** *The first-order theory of $\mathbb{LO}^+_{rec}$ is 1-equivalent to the $\omega$-jump of Kleene's $\mathcal{O}$.*

*Proof.* We start by showing that the first-order theory of $\mathbb{LO}^+_{rec}$ is 1-reducible from $\mathcal{O}^{(\omega)}$. Let $X$ be the set of indices $e \in \mathbb{N}$ of total computable functions coding linear orderings, noting that $X$ is computable in $\emptyset^{(2)}$. Also, there is a total computable function $f \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that, if $e_1, e_2 \in X$, then $f(e_1, e_2)$ is in $X$ and has the order type of the sum of the linear orders coded by $e_1$ and $e_2$.

The issue is that a linear ordering will have many different indices. Let $I$ be the set of pairs $(e_1, e_2) \in X \times X$ such that $e_1$ and $e_2$ are indices for isomorphic linear orderings. Observe that $I$ is computable in $\mathcal{O}$, as the isomorphism problem for computable linear orderings is $\Sigma^1_1$. Using Kleene's $\mathcal{O}$, we can therefore compute a presentation of the monoid $\mathbb{LO}^+_{rec}$. Hence, within $\omega$-jumps, we get the first-order theory of $\mathbb{LO}^+_{rec}$.

The interesting direction is the reverse direction. We will code a model of first-order arithmetic with a predicate $\mathcal{O}$ in $\mathbb{LO}^+_{rec}$, where $\mathcal{O}$ is the set of all indices $e$ for computable well-orderings: that is, the number $e$ is an index for a total computable function that is the characteristic function of a set of pairs $A$ representing a linear ordering which is well-ordered. We already defined a model of first-order arithmetic within $\mathbb{LO}^+_{rec}$ in Section 3.1, noting that the parameter used there to code multiplication is (can be taken to be) a computable linear ordering.

Thus, we need to define Kleene's $\mathcal{O}$. We have that $A \subseteq \mathbb{N}^2$ represents a computable well-ordering if and only if there is a set of pairs $C \subseteq LO_{rec} \times \mathbb{N}$ such that

(1) If $(\mathcal{B}, a) \in C$, then $\mathcal{B}$ is an (right) additively indecomposable infinite ordinal and $a \in dom(A)$.
(2) For every $a \in dom(A)$, there exists a $\mathcal{B} \in \mathrm{LO}_{rec}$ such that $(\mathcal{B}, a) \in C$.
(3) If $(\mathcal{B}_1, a_1) \in C$ and $(\mathcal{B}_2, a_2) \in C$, then $a_1 \leq_A a_2$ if and only if $\mathcal{B}_1 \trianglelefteq \mathcal{B}_2$.

This equivalence exploits that, for example, if $\alpha$ is a computable ordinal, then so is $\omega^\alpha$.

Moreover, we note that for every computable $A \subseteq \mathbb{N}^2$ representing a computable well-order, there exists a set $C$ satisfying the conditions above and a $\mathcal{C} \in LO_{rec}$ such that

$$(\mathcal{B}, a) \in C \iff \mathcal{B} + a + \omega^* \trianglelefteq \mathcal{C}$$

for all (right) additively indecomposable infinite ordinals $\mathcal{B}$ and $a \in \mathbb{N}$. Indeed, for each $a \in dom(A)$, let $\mathcal{L}_a := \omega^{\mathcal{A}_{\leq a}}$. Then

$$\mathcal{C} := \sum_{a \in dom(A)} \mathcal{L}_a + a + \omega^*$$

suffices, noting that $\mathcal{C} \in LO_{rec}$ as $\mathcal{L}_a$ is uniformly computable in $a$.

Hence, we have the definability of Kleene's $\mathcal{O}$ within $\mathbb{LO}^+_{rec}$. □

## 3. Boolean Algebras Under Direct Sum

Though the class of Boolean algebras and linear orders share many similarities, an important distinction quickly arises. Whereas linear orders can contain "local information" (information encoded within a subinterval that is not reflected elsewhere), Boolean algebras contain only "global information." This distinction makes the requisite encoding more sophisticated. It also is, essentially, the reason we are not able to demonstrate the bi-interpretability of second-order arithmetic in $\mathbb{BA}^{\oplus}_{\aleph_0}$.

**Definition 3.1.** Fix an infinite cardinal $\kappa$. Define $\mathrm{BA}_\kappa$ to be the set of all isomorphism types of Boolean algebras of size at most $\kappa$ and $\mathbb{BA}^{\oplus}_{\kappa} = (\mathrm{BA}_\kappa; \oplus)$ to be the commutative monoid of Boolean algebras of size at most $\kappa$ under direct sum.

Throughout, we operate under the convention that the set $\mathrm{BA}_\kappa$ includes the *trivial algebra* (where $0 = 1$). Being the identity element of the monoid, we denote the trivial algebra by $\mathbf{0}$.

**Definition 3.2.** For $u, v \in \mathrm{BA}_\kappa$, we write $u \trianglelefteq v$ if $(\exists w)\,[v = u \oplus w]$, that is, if $u$ is a *relative algebra* of $v$. We write $u \triangleleft v$ if both $u \trianglelefteq v$ and $v \ntrianglelefteq u$.

We emphasize the relation $\trianglelefteq$ is not a partial order as there exist distinct $a, b \in \mathrm{BA}_\kappa$ with $a \trianglelefteq b$ and $b \trianglelefteq a$. On the other hand, it is immediate the relation $\trianglelefteq$ is reflexive and transitive, so a preorder.

### 3.1. Interpreting Second-Order Arithmetic.

As preparation to interpreting second-order arithmetic in $\mathbb{BA}^{\oplus}_{\aleph_0}$, we develop a small repertoire of definable subsets of $\mathrm{BA}_{\aleph_0}$. Though the ideas are similar to Section 2.1, the encoding is slightly more subtle. A bit more care is required for $\mathbb{BA}^{\oplus}_{\aleph_0}$ than for $\mathbb{LO}^+_\kappa$ as any local structure within a Boolean algebra appears globally. Thus, it seems impossible to have all elements of the universe $U$ coding $\mathbb{N}$ be comparable under the $\trianglelefteq$ relation as we did with linear orders.

**Lemma 3.3.** *Each of the following subsets of $BA_{\aleph_0}$ is first-order definable in $\mathbb{BA}^{\oplus}_{\aleph_0}$:*

(1) *TPO (the set of elements whose relative algebras are a totally ordered under $\trianglelefteq$)*
(2) *SA (the set of all superatomic algebras)*
*CON (the set of all algebras of the form IntAlg($\omega^\alpha \cdot (1+\eta)$))*
(3) *PI (the set of pseudo-indecomposable algebras)*
(4) *NA (the set $\{IntAlg(\omega^n) : n \in \mathbb{N}\}$)*
(5) *NCON (the set $\{IntAlg(\omega^n \cdot (1+\eta)) : n \in \mathbb{N}\}$)*

*Proof.* We exhibit a first-order formula witnessing the definability of each subset.

(1) The formula

$$\psi_{TPO}(x) := (\forall u \trianglelefteq x)(\forall v \trianglelefteq x)\,[u \trianglelefteq v \quad \vee \quad v \trianglelefteq u]\,.$$

is easily seen to define the set *TPO*.

(2) The formulas

$$\psi_{SA}(x) := \psi_{TPO}(x) \wedge (x \neq x \oplus x)$$

and

$$\psi_{CON}(x) := \psi_{TPO}(x) \wedge (x = x \oplus x)$$

define the sets *SA* and *CON*, respectively as every element of *SA* is not idempotent and every element of *CON* is idempotent.

This relies on the equality $TPO = SA \cup CON$. The inclusion $SA \cup CON \subseteq TPO$ is a consequence of the fact that the countable superatomic algebras are linearly ordered by $\trianglelefteq$. The inclusion $TPO \subseteq SA \cup CON$ is a bit more delicate. Suppose $\mathcal{B} \in TPO$, and suppose $\mathcal{B}$ is not superatomic. Since non-superatomic algebras are not relative algebras of superatomic algebras, we have that every superatomic $y \trianglelefteq \mathcal{B}$ is a relative algebra of every non-superatomic $z \trianglelefteq \mathcal{B}$. Let $I \subseteq \mathcal{B}$ be the set all $b \in \mathcal{B}$ whose downward algebra, $\mathcal{B} \restriction b$, is superatomic. Notice that the quotient $\mathcal{B}/I$ is atomless, and that every $a \notin I$ bounds the same types of superatomic Boolean algebras that $\mathcal{B}$ bounds. A back-and-forth argument can be then used to show that $\mathcal{B} \restriction a$ and $\mathcal{B} \restriction b$ are isomorphic if and only if both $a$ and $b$ are not in $I$, or both are in $I$ and have the same Cantor-Bendixson rank and degree. The same argument then shows that $\mathcal{B}$ has to be isomorphic to $\mathrm{IntAlg}(\omega^\alpha \cdot (1+\eta))$, where $\alpha$ is the least such that $\mathrm{IntAlg}(\omega^\alpha) \ntrianglelefteq \mathcal{B}$.

Alternatively, in the language of Ketonen [Ket78], the equality $TPO = SA \cup CON$ follows from the fact that $x \in TPO$ if and only if every relative algebra of $x$ is superatomic or uniform.

(3) The formula

$$\psi_{PI}(x) := (\forall y)(\forall z)\,[x = y \oplus z \implies x = y \vee x = z]$$

defines the set of pseudo-indecomposable elements as the pseudo-indecomposable algebras are defined by this property.

(4) We write $\psi_{SA,PI}(x)$ for $\psi_{SA}(x) \wedge \psi_{PI}(x)$, and we write $\psi_{SA}(x,y)$ for $\psi_{SA}(x) \wedge \psi_{SA}(y)$. Among the Boolean algebras which are superatomic and pseudo-indecomposable, there is a successor-like operation:

$$\psi_{succ}(y,z) := \psi_{SA,PI}(y,z) \ \wedge \ y \lhd z \ \wedge \ (\forall w)[y \lhd w \lhd z \implies \neg\psi_{PI}(w)].$$

The formula

$$\psi_{NA}(x) := \psi_{SA,PI}(x) \ \wedge \ (\forall z \unlhd x)\,[\psi_{SA,PI}(z) \ \wedge \ z \neq \mathbf{0} \implies (\exists y)\,[\psi_{succ}(y,z)]]$$

defines the set $\{\mathrm{IntAlg}(\omega^n) : n \in \mathbb{N}\}$. The reason is that no superatomic algebra of rank $\omega$ or greater satisfies the second conjunct. This is because, taking $\mathrm{IntAlg}(\omega^\omega)$ for $z$, we have no "predecessor" $y$.

(5) The formula

$$\psi_{NCON}(x) := \psi_{CON}(x) \wedge (\exists z)\,[\psi_{NA}(z) \wedge z \not\trianglelefteq x]$$

defines the set $\{\mathrm{IntAlg}(\omega^n \cdot (1+\eta)) : n \in \mathbb{N}\}$ as the second conjunct ensures the rank of $x$ is strictly smaller than $\omega$.

Hence, the enumerated subsets are first-order definable in $\mathbb{BA}^{\oplus}_{\aleph_0}$. □

These definable subsets will be exploited in our encoding of the standard model of arithmetic into $\mathbb{BA}^{\oplus}_{\aleph_0}$. Indeed, we will encode the natural number $n \in \mathbb{N}$ by the algebra $\mathrm{IntAlg}(\omega^n \cdot (1+\eta))$. Thus, the set of natural numbers is definable by Lemma 3.3(5). Further, the order on the natural numbers is definable as $m \leq n$ if and only if the set of superatomic relative algebras of $\mathrm{IntAlg}(\omega^m \cdot (1+\eta))$ is a subset of the set of superatomic relative algebras of $\mathrm{IntAlg}(\omega^n \cdot (1+\eta))$.

**Definition 3.4.** If $(n_1, \ldots, n_k) \in [\mathbb{N}]^k$ is an unordered $k$-tuple, let $t_k(n_1, \ldots, n_k)$ be the algebra

$$t_k(n_1, \ldots, n_k) := \sum_{i \in 1+\eta} \big(\mathrm{IntAlg}(\omega^{n_1} \cdot (1+\eta)) \oplus \cdots \oplus \mathrm{IntAlg}(\omega^{n_k} \cdot (1+\eta))\big).$$

If $z \in \mathrm{BA}_\kappa$ and $k \in \mathbb{N}$, let $S_k(z)$ be the subset

$$S_k(z) := \Big\{(n_1, \ldots, n_k) \in [\mathbb{N}]^k : t_k(n_1, \ldots, n_k) \unlhd z\Big\}$$

and say that $z$ *codes* the set $S_k(z)$.

A pair of elements $(a, m) \in \mathrm{BA}_{\aleph_0} \times \mathrm{BA}_{\aleph_0}$ is a code for $\mathcal{N}_1$ in $\mathrm{BA}^{\oplus}_{\aleph_0}$ if, with

- $y_1 + y_2 = y_3$ if and only if $y_1, y_2 \leq y_3$ and $(y_1, y_2, y_3) \in S_3(a)$, and
- $y_1 \cdot y_2 = y_3$ if and only if either $y_3 = 0 \wedge (y_1 = 0 \vee y_2 = 0)$ or $0 < y_1, y_2 \leq y_3$ and $(y_1, y_2, y_3) \in S_3(m)$,

the structure $\mathbb{BA}^{\oplus}_{\aleph_0}$ satisfies the sentence that says $S_3(a)$ and $S_3(m)$ define functions $+ \colon \mathbb{N}^2 \to \mathbb{N}$ and $\cdot \colon \mathbb{N}^2 \to \mathbb{N}$ with $a+0=a$, $a+(b+1)=(a+b)+1$, $a \cdot 0 = 0$, and $a \cdot (b+1) = a \cdot b + b$ for all $a, b \in \mathbb{N}$.

The important point is that the function $x_1, \dots, x_k \mapsto t_k(n_1, \dots, n_k)$ is definable, namely by the formula:

$$\psi_{t_k}(x_1, \dots, x_k, x) \quad := \quad \psi_{PI}(x) \;\wedge\; x = x + x \;\wedge\; \bigwedge_{i=1}^{i=k} [\psi_{NCON}(x_i) \;\wedge\; x_i \trianglelefteq x]$$
$$\wedge\; (\forall y \triangleleft x) \left[ y \trianglelefteq \bigoplus_{i=1}^{i=k} x_i \right].$$

Clearly for all $(n_1, \dots, n_k) \in [\mathbb{N}^k]$,

$$\psi_{t_k}(\mathrm{IntAlg}(\omega^{n_1} \cdot (1+\eta)), \dots, \mathrm{IntAlg}(\omega^{n_k} \cdot (1+\eta)), t_k(n_1, \dots, n_k))$$

holds. For the other direction suppose that $\psi_{t_k}(\mathcal{B}_1, \dots, \mathcal{B}_k, \mathcal{B})$ holds. By the third conjunct, there are integers $n_i$ such that $\mathcal{B}_i \cong \mathrm{IntAlg}(\omega^{n_i} \cdot (1+\eta))$. Let $\mathcal{C} := t_k(n_1, \dots, n_k)$. Using a back-and-forth argument, one can show that, given $b \in \mathcal{B}$ and $c \in \mathcal{C}$, $\mathcal{B} \restriction b$ and $\mathcal{C} \restriction c$ are isomorphic if and only if either $\mathcal{B} \restriction b \cong \mathcal{B}$ and $\mathcal{C} \restriction c \cong \mathcal{C}$, or they are both isomorphic to the same relative algebra of $\bigoplus_{i=1}^{i=k} \mathcal{B}_i$. It follows that $\mathcal{B} \cong \mathcal{C}$.

Further inspection of Definition 3.4 shows that the property of being a code for $\mathcal{N}_1$ is first-order definable in $\mathbb{BA}^{\oplus}_{\aleph_0}$.

**Definition 3.5.** Fix a set $X = \{x_i\}_{i \in I} \subseteq \mathbb{N}$. The *code for* $X$ is the algebra $t_I(X)$ given by

$$t_I(X) := \sum_{i \in I} \mathrm{IntAlg}(\omega^{x_i} \cdot (1+\eta))$$

As $t_I(X)$ is countable for any $X \subseteq \mathbb{N}$, every subset $X \subseteq \mathbb{N}$ has a code in $\mathrm{BA}_{\aleph_0}$.

**Theorem 3.6.** *That* $Th(\mathcal{N}_2) \leq_1 Th(\mathbb{BA}^{\oplus}_{\aleph_0})$.

*Proof.* Let $\varphi$ be a sentence in the language of $\mathcal{N}_2$. Let $\psi(a, m)$ be the formula with two free variables in the language of $\mathbb{BA}^{\oplus}_{\aleph_0}$ obtained from $\varphi$ by replaces instances of

- $x \leq y$ with $(\forall z \in SA)\,[z \trianglelefteq x \implies z \trianglelefteq y]$
- $x + y = z$ with $t_3(x, y, z) \trianglelefteq a$,
- $x \cdot y = z$ with $t_3(x, y, z) \trianglelefteq m$,
- $x \in X$ with $x \trianglelefteq v_X$,
- $\exists x$ with $\exists x \in NCON$, and
- $\exists X$ with $\exists v_X$.

Let $\chi$ be the sentence stating there is a first-order code $(a, m)$ for $\mathcal{N}_1$ and $\psi(a, m)$.

Then $\mathcal{N}_2 \models \varphi$ if and only if $\mathbb{BA}^{\oplus}_{\aleph_0} \models \chi$ as a code for $\mathcal{N}_1$ codes a structure isomorphic to $\mathcal{N}_1$. □

It is worth noting that the encoding within this subsection relied on the ambient structure being $\mathbb{BA}^{\oplus}_{\aleph_0}$ rather than $\mathbb{BA}^{\oplus}_{\kappa}$ for some uncountable $\kappa$. Though necessary for our analysis, this assumption seems unnecessary.

*Conjecture* 3.7. Fix an uncountable cardinal $\kappa$. The structure $\mathbb{BA}_\kappa^\oplus$ interprets second-order arithmetic.

We also wonder whether the structure $\mathbb{BA}_{\aleph_0}^\oplus$ is bi-interpretable with second-order arithmetic.

**Question 3.8.** Is $\mathbb{BA}_{\aleph_0}^\oplus$ bi-interpretable with second-order arithmetic via the interpretation in Definition 3.4?

We finish by noting that, like with linear orders, the theories of the monoids $\mathbb{BA}_{\aleph_0}^\oplus$ and $\mathbb{BA}_\kappa^\oplus$ are distinct if $\kappa > \aleph_0$. Perhaps the simplest distinction is the number of atomless Boolean algebras.

## 4. Groups Under Direct Product with the Subgroup Relation

By analogy with our study of linear orders and Boolean algebras, our study of groups should involve only the direct product operation. Unfortunately, the language of direct products seemingly offers no "local structure" in which to do encoding. Consequently, we also work with the subgroup relation.

**Definition 4.1.** Fix an infinite cardinal $\kappa$. Define $\mathrm{GR}_\kappa$ to be the set of all isomorphism types of groups of size at most $\kappa$ and $\mathbb{GR}_\kappa^{\times,\leq} = (\mathrm{GR}_\kappa; \times, \leq)$ to be the partially ordered commutative monoid of groups of size at most $\kappa$ under direct product with the subgroup relation.

Throughout, we operate under the convention that the set $\mathrm{GR}_\kappa$ includes the trivial group. Being the identity element of the monoid, we denote the trivial group by $\mathbf{0}$.

### 4.1. Interpreting Second-Order Arithmetic.

As preparation to interpreting second-order arithmetic in $\mathbb{GR}_\kappa^{\times,\leq}$, we develop a small repertoire of definable subsets of $\mathrm{GR}_\kappa$. The encoding is not too different, though it is again more subtle as a consequence of the inability to define singleton elements.

**Lemma 4.2.** *Each of the following subsets of $GR_\kappa$ is first-order definable in $\mathbb{GR}_\kappa^{\times,\leq}$ (allowing subscripts as parameters):*

1. *MIN (the set of nontrivial elements containing no proper subgroup)* *$MIN_{y_1,\dots,y_k}$ (the set MIN without $y_1, \dots, y_k$)*
2. *TPO (the set of elements whose ideals are a total preorder under $\leq$)*
3. *$POW_y$ for $y \in MIN$ (the set of elements $\{y^n : n \in \mathbb{N}\}$)*

*Proof.* We exhibit a first-order formula witnessing the definability of each subset.

(1) The formula

$$\psi_{MIN}(x) := x \neq \mathbf{0} \ \wedge \ (\forall y)\,[y \leq x \ \wedge \ y \neq x \implies y = \mathbf{0}]$$

is easily seen to define the set *MIN*. We note that *MIN* consists of precisely the cyclic groups $\mathbb{Z}_p$ of prime order and the additive group $\mathbb{Z}$ of the integers.

It follows that the formula

$$\psi_{MIN_{y_1,\ldots,y_k}}(x) := \psi_{MIN}(x) \ \wedge \ \bigwedge_{i=1}^{i=k} x \neq y_i$$

defines the set $MIN_{y_1,\ldots,y_k}$.

(2) The formula

$$\psi_{TPO}(x) := (\forall u \leq x)(\forall v \leq x)\,[u \leq v \ \vee \ v \leq u]$$

is easily seen to define the set *TPO*.

(3) The formula

$$\begin{aligned} \psi_{POW_y}(x) \quad := \quad & (\forall z \leq x)[z \neq \mathbf{0} \implies (\exists w)[z = y \times w]] \\ & \wedge (\forall u \leq x)[y \times u \neq u] \end{aligned}$$

defines the set $POW_y$. We reason as follows.

If $x \in POW_y$, then $x = y^n$ for some $n \in \mathbb{N}$. The subgroups of $x$ are precisely the groups $y^k$ for $0 \leq k \leq n$. All the conjuncts are clearly satisfied.

Conversely, fixing a nonzero $x$ satisfying $\psi_{POW_y}(x)$, we show $x \in POW_y$. The first conjunct implies that there is a group $w_1$ such that $x = y \times w_1$. If $w_1$ is the trivial group, then $x = y$ and $x \in POW_y$. Otherwise, by choice of $w_1$ for $z$, there is a group $w_2$ such that $w_1 = y \times w_2$. Continuing in this fashion, if at some point $w_n$ is trivial, we have $x = y^n \in POW_y$. Otherwise, one can show that $\bigotimes_{i\in\omega}^{weak} y$ is a subgroup of $x$. But this contradict the second conjunct, taking $u = \bigotimes_{i\in\omega}^{weak} y$.

Hence, the enumerated subsets are first-order definable in $\mathbb{GR}_\kappa^{\times,\leq}$. □

These definable subsets will be exploited in our encoding of the standard model of arithmetic into $\mathbb{GR}_\kappa^{\times,\leq}$. Hereout, we fix an element $w \in MIN$, so $w$ is abelian being either $\mathbb{Z}_p$ for some prime $p$ or $\mathbb{Z}$. We will encode the natural number $n \in \mathbb{N}$ by $w^n$. Thus, with $w$ as a parameter, the set $\mathbb{N}$ of natural numbers is definable as $POW_w$ by Lemma 4.2(3). Further, the order on the natural numbers is definable as $m \leq n$ if and only if $w^m \leq w^n$, and addition is definable as $w^{m+n} = w^m \times w^n$. To define multiplication we will need to be able to code arbitrary sets of triples. The coding of triples will use different copies of $\mathbb{N}$ built from minimal elements other than $w$.

Fix $k \in \mathbb{N}$ and distinct $w_1, \ldots, w_k \in MIN$, with $w_1 = w$.

**Definition 4.3.** If $(n_1, \ldots, n_k) \in \mathbb{N}^k$ is an ordered $k$-tuple and $y \in MIN_{w_1,\ldots,w_k}$, let $t_{k,y}(n_1, \ldots, n_k)$ be the group

$$t_{k,y}(n_1, \ldots, n_k) := w_1^{n_1} \times \cdots \times w_k^{n_k} \times y.$$

Now we want to use these groups to code sets $X \subseteq \mathbb{N}^k$.

**Definition 4.4.** Fix an injective enumeration $\{y_i\}_{i\in\mathbb{N}}$ of $MIN_{w_1,\dots,w_k}$. Fix a set $X = \{\bar{n}_i\}_{i\in I} \subseteq \mathbb{N}^k$, where $\bar{n}_i = (n_1^i, \dots, n_k^i)$. The *code for* $X$ is the group $t_I(X)$ given by

$$t_I(X) := \prod_{i\in I}^{*} t_{k,y_i}(n_1^i, \dots, n_k^i),$$

i.e., the group $t_I(X)$ is the free product of the groups $t_{k,y_i}(n_1^i, \dots, n_k^i)$ for $i \in I$.

To decode $t_I(X)$ we will use the following theorem.

**Theorem 4.5** (Kurosch's theorem)**.** *A subgroup $H$ of a free product $\prod_j^{*} A_j$ is itself a free product*

$$H = F * \prod_k^{*} x_k^{-1} U_k x_k,$$

*where $F$ is a free group and each $x_k^{-1} U_i x_k$ is the conjugate of a subgroup $U_k$ of one of the factors $A_j$ by an element of the free group $\prod_j^* A_j$.*

As a corollary we obtain that an abelian subgroup $H$ of a free product $\prod_j^* A_j$ is either $\mathbb{Z}$ or a conjugate of a subgroup $U$ of one of the factors $A_j$. This is because a nontrivial free product is never abelian, and the only abelian free group is $\mathbb{Z}$.

It follows that $(n_1, \dots, n_k) \in X$ if and only if there is a $y \in MIN_{w_1,\dots,w_k}$ such that $t_{k,y}(n_1, \dots, n_k) \leq t_I(X)$. Also, if $t_{k,y}(n'_1, \dots, n'_k) \leq t_I(X)$, then $t_{k,y}(n'_1, \dots, n'_k)$ is a subgroup of $t_{k,y}(n_1, \dots, n_k)$ because $t_{k,y}(n_1, \dots, n_k)$ is the only factor in the free product that contains $y$.

**Definition 4.6.** If $z \in \mathrm{GR}_\kappa$ and $k \in \mathbb{N}$, let $S_k(z)$ be the subset

$$\begin{aligned} S_k(z) := \big\{(n_1, \dots, n_k) \in \mathbb{N}^k : (\exists y \in MIN_{w_1,\dots,w_k})[t_{k,y}(n_1, \dots, n_k) \leq z \\ \wedge (\forall n'_1, \dots, n'_k \in \mathbb{N}^k)[t_{k,y}(n'_1, \dots, n'_k) \leq z \implies \bigwedge_{i=1}^{i=k} n'_i \leq n_i]]\big\} \end{aligned}$$

and say that $z$ *codes* the set $S_k(z)$.

The discussion above shows that $X = S_k(t_I(X))$.

In practice, we want to use $S_k(z)$ as a set of tuples in $POW_{w_1} \times \cdots \times POW_{w_k}$. The definitions are essentially the same:

$$\psi_{S_k}(z_1,\ldots,z_k,z) := \bigwedge_{i=1}^{i=k} \psi_{POW_{w_i}}(z_i) \ \wedge \ (\exists y \in MIN_{w_1,\ldots,w_k})(\forall z_1',\ldots,z_k')$$

$$\left[ z_1 \times \cdots \times z_k \times y \leq z \ \ \wedge \ \ \left[ \bigwedge_{i=1}^{i=k} \psi_{POW_{w_i}}(z_i') \ \wedge \ z_1' \times \cdots \times z_k' \times y \leq z \implies \bigwedge_{i=1}^{i=k} z_i' \leq z_i \right]\right].$$

It is not hard to see that

$$(n_1,\ldots,n_k) \in S_k(z) \iff \psi_{S_k}(w_1^{n_1},\ldots,w_k^{n_k},z).$$

The issue is that we are using different copies of the natural numbers $POW_{w_1},\ldots,POW_{w_k}$. We therefore define bijections between them.

**Lemma 4.7.** *For $w_1, w_2 \in MIN$, the set $BIJ_{w_1,w_2} := \{(w_1^n, w_2^n) : n \in \mathbb{N}\} \subseteq GR_\kappa^2$ is definable in $\mathbb{GR}_\kappa^{\times,\leq}$ (with $w_1$ and $w_2$ as parameters).*

*Proof.* We let $\psi_{BIJ_{w_1,w_2}}(z_1,z_2)$ be the formula that says that there exists an element $z \in \mathrm{GR}_\kappa$ such that $\psi_{S_2}(\cdot,\cdot,z)$ defines a one-to-one, onto, order-preserving function between $POW_{w_1}$ and $POW_{w_2}$ and that $\psi_{S_2}(z_1,z_2,z)$ holds. □

We can now modify the decoding functions $S_k$ to code sets of tuples in $POW_w^k$ (recall that $w = w_1$):

$$\begin{aligned}\psi_{S_k'}(z_1,\ldots,z_k,z) \ := \ & \bigwedge_{i=1}^{i=k} \psi_{POW_w}(z_i) \\ & \wedge (\exists y_2,\ldots,y_k)\left[\bigwedge_{i=2}^{i=k} \psi_{BIJ_{w,w_i}}(z_i,y_i) \wedge \ \psi_{S_k}(z_1,y_2,\ldots,y_k,z)\right].\end{aligned}$$

**Definition 4.8.** An element $m \in \mathrm{GR}_\kappa$ is a code for $\mathcal{N}_1$ if the operation $\cdot : POW_w^2 \to POW_w$ defined by

- $z_1 \cdot z_2 = z_3$ if and only if $\psi_{S_3'}(z_1,z_2,z_3,m)$,

satisfies the sentence that defines multiplication recursively from addition in the structure $(POW_w; \times, \cdot, \leq)$. (Recall that addition of numbers is interpreted as product of groups.)

Careful inspection of Definition 4.3 shows that the property of being a code for $\mathcal{N}_1$ is first-order definable in $\mathbb{GR}_\kappa^{\times,\leq}$ with parameters $w$, $w_2$, and $w_3$.

**Theorem 4.9.** *Fix $\kappa \geq \aleph_0$. Then $Th(\mathcal{N}_2) \leq_1 Th(\mathbb{GR}_\kappa^{\times,\leq})$.*

*Proof.* Let $\varphi$ be a sentence in the language of $\mathcal{N}_2$. The atomic subformulas of $\varphi$ have the forms $x = y$, $x \leq y$, $x + y = z$, $x \times y = z$, and $x \in X$. Let $\psi(m, w, w_2, w_3)$ be the formula with free variables shown in the language of $\mathbb{GR}_\kappa^{\times,\leq}$ obtained from $\varphi$ by replacing instances of

- $x \leq y$ with $x \leq y$

- $x + y = z$ with $x \times y = z$,
- $x \times y = z$ with $\psi_{S_3'}(x, y, z, m)$ (with parameters $w$, $w_2$, and $w_3$),
- $x \in X$ with $\psi_{S_1}(x, v_X)$, (with parameter $w$)
- $\exists x$ with $\exists x \in POW_w$, and
- $\exists X$ with $\exists v_X$.

Let $\chi$ be the sentence stating that there are $w, w_2, w_3 \in MIN$ and a code $m$ for $\mathcal{N}_1$ and $\psi(m, w, w_2, w_3)$.

Then $\mathcal{N}_2 \models \varphi$ if and only if $\mathbb{GR}_\kappa^{\times,\leq} \models \chi$ as a code for $\mathcal{N}_1$ codes a structure isomorphic to $\mathcal{N}_1$. □

## 5. Decidable Theories of Structures

Given the decidability of Presburger Arithmetic and the simplicity of infinite cardinal addition, it is not surprising that the theory of cardinal numbers under addition is decidable. This decidability has implications for the decidability of vector spaces over $\mathbb{Q}$ under direct sums and the decidability of equivalence structures under disjoint union.

**Definition 5.1.** Fix an ordinal $\alpha$. Define $\mathrm{CARD}_\alpha$ to be the set of all cardinals strictly less than $\aleph_\alpha$ and $\mathbb{CARD}_\alpha^+ = (\mathrm{CARD}_\alpha; +)$ to be the commutative monoid of cardinals strictly less than $\aleph_\alpha$ under cardinal addition.

**Definition 5.2.** For $u, v \in \mathrm{CARD}_\alpha$, we write $u \leq v$ if $(\exists w)\,[v = u + w]$. We write $u < v$ if $u \leq v$ and $u \neq v$.

**Lemma 5.3** (Presburger [Pre91])**.** *The first-order theory of* $\mathbb{CARD}_0^+$*, i.e., the theory of Presburger Arithmetic* $(\mathbb{N}; +)$*, is decidable.*

**Lemma 5.4** (Feferman and Vaught [FV59])**.** *Fix an ordinal* $\alpha$*. The first-order theory of* $(\alpha; \max)$ *is decidable, where* $\max\{\alpha_1, \alpha_2\}$ *is the maximum of* $\alpha_1$ *and* $\alpha_2$*.*

**Theorem 5.5** (Feferman and Vaught [FV59])**.** *The first-order theory of* $\mathbb{CARD}_\alpha^+$ *is decidable for all ordinals* $\alpha$*.*

*Proof.* The idea is to exploit that $\mathrm{CARD}_0$ is a definable subset of $\mathrm{CARD}_\alpha$, being exactly the set of non-idempotent elements. Indeed, we transform any first-order formula $\varphi$ into a logically equivalent formula $\varphi^T$ for which the variables are known to be either finite or infinite cardinals. The decidability of $\mathbb{CARD}_\alpha^+$ is then a consequence of Lemma 5.3 and Lemma 5.4.

By induction on the complexity of a first-order formula $\varphi$, we define a formula $\varphi^T$ logically equivalent to $\varphi$. In order to simplify the induction, we assume the logical symbols are negation, conjunction, and the existential quantifier. If $\varphi$ is atomic, we define $\varphi^T := \varphi$. If $\varphi = \varphi_1 \wedge \varphi_2$, we define $\varphi^T := \varphi_1^T \wedge \varphi_2^T$. If $\varphi = \neg\varphi_1$, we define $\varphi^T := \neg\varphi_1^T$. If $\varphi = (\exists x)\,[\psi(x)]$, we define $\varphi^T := (\exists x)\,\left[x = x + x \wedge \psi(x)^T\right] \vee (\exists x)\,\left[x \neq x + x \wedge \psi(x)^T\right]$.

The benefit of $\varphi^T$ over $\varphi$ is that, in any atomic subformula, every variable is scoped to be either a finite or infinite cardinal. As "finite + finite

= finite", "finite + infinite = infinite", and "infinite + infinite = infinite", any atomic subformula can be effectively converted to a logically equivalent atomic subformula consisting of only variables scoped to be finite, a subformula consisting of only variables scoped to be infinite, or TRUE or FALSE.

The decidability of $\mathbb{CARD}_\alpha^+$ is then a consequence of Lemma 5.3 and Lemma 5.4. □

**Theorem 5.6** (Feferman and Vaught [FV59], Tarski)**.** *Fix ordinals $\alpha_1$ and $\alpha_2$ with $0 \leq \alpha_1 < \alpha_2 < \omega^\omega \cdot 2$. The first-order theories of $\mathbb{CARD}_{\alpha_1}^+$ and $\mathbb{CARD}_{\alpha_2}^+$ are distinct.*

*Moreover, if $\alpha_1 = \omega^\omega \cdot \zeta_1 + \beta_1$ and $\alpha_2 = \omega^\omega \cdot \zeta_2 + \beta_2$ are any ordinals with $\beta_1, \beta_2 < \omega^\omega \cdot 2$ chosen maximally with this property, then the theories of $\mathbb{CARD}_{\alpha_1}^+$ and $\mathbb{CARD}_{\alpha_2}^+$ are identical if and only if $\beta_1 = \beta_2$.*

The idea for the proof is to exploit that, if $\aleph_\delta$ is any infinite cardinal, then the cardinal $\aleph_{\delta+\omega^k \cdot n_k + \cdots + \omega \cdot n_1 + n_0}$ is a definable singleton of $\mathbb{CARD}_\alpha^+$ (presuming it exists in $\mathrm{CARD}_\alpha$) using $\aleph_\delta$ as a parameter.

5.1. **Vector Spaces Under Direct Sum.** As the isomorphism type of a vector space over a fixed field $F$ is uniquely determined by its dimension, Theorem 5.5 has implications for the class of vector spaces.

**Definition 5.7.** Fix a countable field $F$ and an infinite cardinal $\kappa$. Define $\mathrm{VS}_\kappa$ to be the set of all vector spaces over $F$ of size less than or equal to $\kappa$ and $\mathbb{VS}_\kappa^\oplus = (\mathrm{VS}_\kappa; \oplus)$ to be the commutative monoid of vector spaces over $F$ of size less than or equal to $\kappa$ under direct sum.

It is straightforward to see that $\mathbb{VS}_\kappa^\oplus \cong \mathbb{CARD}_{\alpha+1}^+$, where $\alpha$ is such that $\kappa = \aleph_\alpha$. Consequently, our understanding of the theories $\mathbb{CARD}_\alpha^+$ yields an understanding of the theories $\mathbb{VS}_\kappa^\oplus$.

**Corollary 5.8.** *Fix an infinite cardinal $\kappa$. The first-order theory of $\mathbb{VS}_\kappa^\oplus$ is decidable.*

Moreover, Theorem 5.6 dictates when the theories of $\mathbb{VS}_{\kappa_1}^\oplus$ and $\mathbb{VS}_{\kappa_2}^\oplus$ coincide.

5.2. **Equivalence Structures Under Addition.** As the isomorphism type of an equivalence structure is uniquely determined by the number of classes of each size, Theorem 5.5 also has implications for the class of equivalence structures.

**Definition 5.9.** Fix an infinite cardinal $\kappa$. Define $\mathrm{EQ}_\kappa$ to be the set of all equivalence structures of size less than or equal to $\kappa$ and $\mathbb{EQ}_\kappa^+ = (\mathrm{EQ}_\kappa; +)$ to be the commutative monoid of equivalence structures of size less than or equal to $\kappa$ under addition (disjoint union).

It is straightforward to see that $\mathbb{EQ}_\kappa^+ \cong \prod_{\beta<\omega+\alpha} \mathbb{CARD}_{\alpha+1}^+$, where $\alpha$ is such that $\kappa = \aleph_\alpha$. The understanding of the theories $\mathbb{CARD}_\alpha^+$ yields an understanding of the theories $\mathbb{EQ}^+_\kappa$ as a consequence of work by Feferman and Vaught.

**Lemma 5.10** (Feferman and Vaught [FV59])**.** *Let $\mathcal{S} = (A; R)$ be a relational structure with a decidable theory. Let $\kappa$ be a cardinal. Then $\mathcal{S}^\kappa$, the generalized product of $\kappa$ many disjoint copies of $\mathcal{S}$, is decidable.*

**Corollary 5.11.** *Fix an infinite cardinal $\kappa$. The first-order theory of $\mathbb{EQ}^+_\kappa$ is decidable.*

Though additional work is required, the characterization of Theorem 5.6 dictates when the theories of $\mathbb{EQ}^+_{\kappa_1}$ and $\mathbb{EQ}^+_{\kappa_2}$ coincide.

5.3. **Finitely Generated Abelian Groups Under Direct Sum.** As with equivalence structures, the structure of finitely generated abelian groups under direct sum is rather straightforward.

**Theorem 5.12** (Fundamental Theorem of Finitely Generated Abelian Groups)**.** *Fix a finitely generated abelian group $\mathcal{G}$. Then there are integers $s, q_1, \dots, q_m, s_1, .., s_m$ such that*

$$\mathcal{G} \cong \mathbb{Z}^s \oplus \mathbb{Z}^{s_1}_{q_1} \oplus \cdots \oplus \mathbb{Z}^{s_m}_{q_m}.$$

*Moreover, the integers $q_i$ for $1 \leq i \leq m$ can be chosen so that all are powers of (not necessarily distinct) prime numbers.*

**Definition 5.13.** Define FGAG to be the set of all finitely generated abelian groups and $\mathbb{FGAG}^\oplus = (\text{FGAG}; \oplus)$ to be the commutative monoid of finitely generated abelian groups under direct sum.

It is straightforward to see that $\mathbb{FGAG}^\oplus \cong \bigoplus^{weak}_{\beta<\omega} \mathbb{CARD}^+_0$ as the isomorphism type of a finitely generated abelian group $\mathcal{G}$ can be specified by a function $f\colon \infty \cup (\omega \times \omega) \to \omega$ equal to zero almost everywhere, where $f(\infty)$ specifies the number of copies of $\mathbb{Z}$ and $f(m, n)$ specifies the number of copies of $\mathbb{Z}_{p^n_m}$.

**Lemma 5.14** (Feferman and Vaught [FV59])**.** *Let $\mathcal{S} = (A; R)$ be a relational structure with a decidable theory. Let $\kappa$ be a cardinal. Then $\mathcal{S}^\kappa_{FIN}$, the weak direct product of $\kappa$ many disjoint copies of $\mathcal{S}$ for which only finitely many components are nonzero, is decidable.*

**Corollary 5.15.** *The first-order theory of $\mathbb{FGAG}^\oplus$ is decidable.*

## 6. Open Questions

Though the additive operation is perhaps the most natural operation on many classes of structures, it is by far not the only possible natural choice. A study of the theory of classes of structures with a different language signature would likely yield interesting comparisons.

**Question 6.1.** Fix an infinite cardinal $\kappa$. Define $\mathbb{LO}^\preccurlyeq_\kappa = (\text{LO}_\kappa; \preccurlyeq)$ to be the class of linear orders of size at most $\kappa$ under embeddability. How complicated is the theory of $\mathbb{LO}^\preccurlyeq_\kappa$?

**Question 6.2.** Fix an infinite cardinal $\kappa$. Define $\mathbb{F}_\kappa^{\leq} = (F_\kappa\ ; \leq)$ to be the class of fields of size at most $\kappa$ under the subfield relation. How complicated is the theory of $\mathbb{F}_\kappa^{\leq}$?

## References

[FV59] S. Feferman and R. L. Vaught. The first order properties of products of algebraic systems. *Fund. Math.*, 47:57–103, 1959.

[Ket] Jussi Ketonen. On isomorphism types of countable Boolean algebras. unpublished.

[Ket78] Jussi Ketonen. The structure of countable Boolean algebras. *Ann. of Math. (2)*, 108(1):41–89, 1978.

[Pre91] Mojżesz Presburger. On the completeness of a certain system of arithmetic of whole numbers in which addition occurs as the only operation. *Hist. Philos. Logic*, 12(2):225–233, 1991. Translated from the German and with commentaries by Dale Jacquette.

[Ros82] Joseph G. Rosenstein. *Linear orderings*, volume 98 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1982.

[Sim77] Stephen G. Simpson. First-order theory of the degrees of recursive unsolvability. *Ann. of Math. (2)*, 105(1):121–139, 1977.

[Sla08] Theodore A. Slaman. Global properties of the Turing degrees and the Turing jump. In *Computational prospects of infinity. Part I. Tutorials*, volume 14 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 83–101. World Sci. Publ., Hackensack, NJ, 2008.

[Tar49] Alfred Tarski. *Cardinal Algebras. With an Appendix: Cardinal Products of Isomorphism Types, by Bjarni Jónsson and Alfred Tarski.* Oxford University Press, New York, N. Y., 1949.

Department of Mathematics, University of Chicago, 5734 S. University ave., Chicago, IL 60637, USA

*E-mail address*: kach@math.uchicago.edu

*URL*: www.math.uchicago.edu/∼kach

*E-mail address*: antonio@math.uchicago.edu

*URL*: www.math.uchicago.edu/∼antonio