A ROBUSTER SCOTT RANK

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Abstract. We give a new definition of Scott rank motivated by our main theorem: For every countable structure $\mathcal{A}$ and ordinal $\alpha < \omega_1$, we have that: every automorphism orbit is $\Sigma^{\mathtt{in}}_\alpha$-definable without parameters if and only if $\mathcal{A}$ has a $\Pi^{\mathtt{in}}_{\alpha+1}$ Scott sentence, if and only if $\mathcal{A}$ is uniformly boldface $\mathbf{\Delta}^0_\alpha$-categorical. As a corollary, we show that a structure is computably categorical on a cone if and only if it is the model of a countably categorical $\Sigma^{\mathtt{in}}_3$ sentence.

1. Introduction

This paper has two main objectives. One is to introduce what the author believes is the *right* definition of Scott rank. The other is to provide a simple structural characterization for the notion of *computable categoricity on a cone.*

1.1. **Scott ranks.** The Scott rank measures the complexity of a structure in terms of the complexity of the automorphism orbits of its tuples. It is a very useful tool in infinitary model theory, descriptive set theory, and computable structure theory. Its original formulation comes from Scott's proof [Sco65] that every countable structure is the unique countable model of some $\mathcal{L}_{\omega_1,\omega}$ sentence. Since then, a few non-equivalent formulations of Scott rank have been proposed. We review them in Section 3.1.

It is known that the Scott rank of a structure is connected to the complexity of its Soctt sentence, which is connected to its level of categoricity, etc. One can prove that some of these measures of complexities provide upper and lower bounds for the others. Our main theorem exposes these connections in the sharpest possible way.

Theorem 1.1. *Let $\mathcal{A}$ be a countable structure and α be a countable ordinal. The following are equivalent:*

(U1) *Every automorphism orbit is $\Sigma^{\mathtt{in}}_\alpha$-definable without parameters.*
(U2) *$\mathcal{A}$ has a $\Pi^{\mathtt{in}}_{\alpha+1}$ Scott sentence.*
(U3) *$\mathcal{A}$ is uniformly boldface $\mathbf{\Delta}^0_\alpha$-categorical.*

Let us quickly explain the terms in the theorem. The "$\mathtt{in}$" in $\Sigma^{\mathtt{in}}_\alpha$ is for "infinitary;" We refer the reader to [AK00, Section 6.4] for background on the hierarchy of $\mathcal{L}_{\omega_1,\omega}$ formulas. The *automorphism orbit* of a tuple $\bar{a}$ in a structure $\mathcal{A}$ is the set of all other tuples automorphic to $\bar{a}$. Since every definable set is a union of automorphism orbits, we have that (U1) is equivalent to saying that every $\mathcal{L}_{\omega_1,\omega}$-definable relation in $\mathcal{A}$ is $\Sigma^{\mathtt{in}}_\alpha$-definable (definability being without parameters). For (U2), recall that a *Scott sentence* for a countable structure $\mathcal{A}$ is an infinitary sentence whose only countable model is $\mathcal{A}$. Scott [Sco65] proved that such a sentences always exist. In other words, part (U2) says that $\mathcal{A}$ is the model of a *countably categorical* $\Pi^{\mathtt{in}}_{\alpha+1}$ sentence. Part (U3) refers to a different notion of categoricity, the one used in computability theory. A structure is *uniformly boldface $\mathbf{\Delta}^0_\alpha$-categorical* if it is uniformly Δ^0_α-categorical relative to some oracle; we will explain this in detail in Section 2.2.

Theorem 1.1 and its continuation below show the robustness of the statements (U1), (U2), and (U3). Motivated by the theorem, we propose yet another notion of Scott rank:

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Definition 1.2. The *categoricity Scott rank* of a structure $\mathcal{A}$ is the least ordinal α such that the automorphism orbit of each tuple in $\mathcal{A}$ is $\Sigma^{\mathtt{in}}_{\alpha}$-definable without parameters.

In other words, the Scott rank of $\mathcal{A}$ is the least α satisfying the statements of Theorem 1.1. We hope that having a robuster notion of Scott rank can help better understand it.

We continue our main theorem by adding four more properties. Depending on the background of the reader, some of these properties will sound more relevant than others. We will explain each of them briefly right after the theorem, and in more detail later in the paper.

Theorem (1.1 continued)**.** *The following are also equivalent to (U1), (U2) and (U3):*

- (U4) *The set of presentations of $\mathcal{A}$ is $\mathbf{\Pi}^0_{\alpha+1}$ in the Borel hierarchy.*
- (U5) *Every $\Pi^{\mathtt{in}}_{\alpha}$-type realized in $\mathcal{A}$ is $\Sigma^{\mathtt{in}}_{\alpha}$-supported within $\mathcal{A}$.*
- (U6) *There is a $\Pi^{\mathtt{in}}_{\alpha+1}$-sentence φ true of $\mathcal{A}$ such that if $\mathcal{B} \models \varphi$, then $\mathcal{B} \equiv_{\alpha+1} \mathcal{A}$.*
- (U7) *No tuple in $\mathcal{A}$ is α-free.*

In (U4), we are referring to the set of presentations of $\mathcal{A}$ as a set of reals, and looking at its complexity from the viewpoint of descriptive set theory. The equivalence between (U4) and (U2) follows from Lopez-Escobar's theorem [LE65], that says that a class of presentations of structures, closed under isomorphism, is $\mathbf{\Pi}^0_{\beta}$ in the Borel hierarchy if and only if it is $\Pi^{\mathtt{in}}_{\beta}$-axiomatizable.

For part (U5), a type $p(\bar{x})$ is $\Sigma^{\mathtt{in}}_{\alpha}$*-supported* within $\mathcal{A}$ if there is a $\Sigma^{\mathtt{in}}_{\alpha}$-formula $\varphi(\bar{x})$ that implies, within $\mathcal{A}$, all the formulas in $p(\bar{x})$, and of course that is realized in $\mathcal{A}$. Part (U5) follows trivially from (U1), as the $\Sigma^{\mathtt{in}}_{\alpha}$-formula defining the orbit clearly implies the $\Pi^{\mathtt{in}}_{\alpha}$-type of the elements in the orbit. The reason we introduce this weakening of (U1) is that it will be very useful in our proof as a pivotal point to prove the rest of the statements.

In (U6), $\equiv_{\alpha+1}$ refers to $\Sigma^{\mathtt{in}}_{\alpha+1}$-elementary equivalence, also known as $(\alpha+1)$-back-and-forth equivalence. Part (U6) follows trivially from (U2), as the Scott sentence of $\mathcal{A}$ serves as φ for (U6). The interesting feature about (U6) is that it implies that $\mathcal{A}$ has maximal under $\leq_{\alpha+1}$, which is a useful fact if one is trying to find the structures that satisfy the theorem. (See Subsection 2.3 for a definition of the $(\alpha+1)$-back-and-forth relations $\leq_{\alpha+1}$.) When $\mathbb{K}$ has countably many $\Pi^{\mathtt{in}}_{\alpha}$-types (i.e. it is $\Sigma^{\mathtt{in}}_{\alpha}$-small–see [Mon]), (U6) is equivalent to $\mathcal{A}$ being maximal under $\leq_{\alpha+1}$ (as it follows from [Mon10, Lemma 2.2]). Thus, in classes where we have a good understanding of the $\leq_n$-back-and-forth types, like Boolean algebras [HM12] for instance, we can find the structures which satisfy Theorem 1.1 by searching for the $\leq_{\alpha+1}$-maximal ones.

Part (U7) is a very combinatorial property sometimes useful in proofs and constructions. The notion of α-freeness was introduced by Ash and Knight and was used to give a characterization of Δ^0_{α}-categoricity for α-friendly structures. We will get back to this in Section 2.3.

The proof of most of the equivalences in Theorem 1.1 are not particularly difficult, and require putting together variations of various known results. The most interesting implication is (U2) $\Rightarrow$ (U5), which uses a sharper version of the usual omitting types theorem for infinitary logic. Once set up correctly, the proof of this omitting types theorem is very similar to the standard one. To avoid introducing the notions of consistency property, fragment, and other notions of infinitary logic, we prove a variation better suited for our purposes.

2. Computable categoricity

This section should be viewed as the second part of the introduction: we talk about our second objective of giving a simple structural characterization for computable categoricity on a cone. The reader only interested in the infinitary model theory part may skip this section and move on to Section 3.

2.1. **Computable categoricity on a cone.** The notion of computable categoricity, originally introduced by Mal'cev [Mal62], has been intensively studied in computability theory in the past decades. Many of the properties one considers in computable structure theory are not invariant under isomorphisms; that is, structures may have isomorphic computable presentations with different computational properties. For instance, there are computable presentations of the countable, infinite-dimensional $\mathbb{Q}$-vector space, $\mathbb{Q}^\infty$, where all the finite-dimensional subspaces are computable, and computable presentations of $\mathbb{Q}^\infty$ where no finite-dimensional subspace is computable (see [DHK$^+$07]). Computably categorical structures are exactly the ones where this does not happen:

Definition 2.1. A computable structure $\mathcal{A}$ is *computably categorical* if there is a computable isomorphism between any two computable copies of $\mathcal{A}$.

There has been a lot of work characterizing the computably categorical structures within certain classes of structures. Precise characterizations have been found for linear orders (Dzgoev and Goncharov [GD80]), Boolean algebras (Goncharov, and independently La Roche [LR78]), ordered abelian groups (Goncharov, Lempp, and Solomon [GLS03]), torsion-free abelian groups (Nurtazin [Nur74]), p-groups (Goncharov [Gon80] and Smith [Smi81]), trees of finite height (Lempp, McCoy, R. Miller, and Solomon [LMMS05]), etc.

On the other hand, there are many classes where we do not expect such characterization are even possible. Downey, Kach, Lempp, Lewis-Pye, Montalbán and Turetsky [DKL$^+$] recently showed that there is no structural characterization for the notion of computable categoricity by showing that the index set of the computably categorical structures is Π^1_1-complete. In contrast, the relativized version of computable categoricity is relatively well-behaved, as proved by Goncharov [Gon75] long ago.

Definition 2.2. Given $X \in 2^\omega$, an X-computable structure $\mathcal{A}$ is *X-computably categorical* if there is an X-computable isomorphism between any two X-computable copies of $\mathcal{A}$. A structure $\mathcal{A}$ is *relatively computably categorical* if it is X-computably categorical for all $X \in 2^\omega$.

Goncharov [Gon75] showed that a structure $\mathcal{A}$ is relative computably categorical if and only if it has a c.e. Scott family of $\exists$-formulas. In this paper, we look at a different variation of the definition of computable categoricity that has an even nicer structural classification.

Definition 2.3. A structure $\mathcal{A}$ is *computably categorical on a cone* if there is a $Y \in 2^\omega$ such that $\mathcal{A}$ is X-computably categorical for all $X \geq_T Y$.

We remark that, when we are looking at natural examples, most properties relativize. Thus, for natural structures, the three notions, computable categoricity, relative computable categoricity and computable categoricity on a cone, coincide. As a corollary of Theorem 1.1, we get our second main result.

Theorem 2.4. *Let $\mathcal{A}$ be a countable structure. The following are equivalent:*

(1) *$\mathcal{A}$ is computably categorical on a cone.*
(2) *$\mathcal{A}$ has a $\Sigma^{\mathtt{in}}_3$ Scott sentence.*

Let us note that $\mathcal{A}$ has a $\Sigma^{\mathtt{in}}_3$ Scott sentence if and only if there exists a tuple $\bar{a} \in A^{<\omega}$ such that $(\mathcal{A}, \bar{a})$ has a $\Pi^{\mathtt{in}}_2$-Scott sentence, which is equivalent to $(\mathcal{A}, \bar{a})$ having categoricity Scott rank 1. This generalizes through the transfinite in the expected way:

Theorem 2.5. *Let $\mathcal{A}$ be a countable structure and α a countable ordinal. The following are equivalent:*

(C1) *$\mathcal{A}$ is Δ^0_α-categorical on a cone.*
(C2) *$\mathcal{A}$ has a $\Sigma^{\mathtt{in}}_{\alpha+2}$ Scott sentence.*
(C3) *There is a tuple $\bar{a} \in A^{<\omega}$ such that $(\mathcal{A}, \bar{a})$ has categoricity Scott rank at most α.*

This theorem motivates the definition of the *parametrized categoricity Scott rank* of a structure $\mathcal{A}$, $pSR(\mathcal{A})$, as the least ordinal α for which, for some tuple $\bar{a} \in A^{<\omega}$, $(\mathcal{A}, \bar{a})$ has categoricity Scott rank α. It follows from (U2) and (C2) that, depending on the structure $\mathcal{A}$, either $cSR(\mathcal{A}) = pSR(\mathcal{A})$ or $cSR(\mathcal{A}) = pSR(\mathcal{A}) + 1$, where $cSR(\mathcal{A})$ is the categoricity Scott rank of $\mathcal{A}$. Depending on the application, one rank might be more useful than the other.

2.2. **Uniform computable categoricity.** A structure is said to be *uniformly computably categorical* if there is a computable operator Θ such that, for every presentation $\mathcal{B}$ of $\mathcal{A}$, we have that $\Theta(\mathcal{B})$ is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Ventsov [Ven92] proved that a structure is relatively computably categorical if and only if it is uniformly computably categorical after adding some parameters. A simple proof of this result is given in [DHK03, Theorem 2.5]. Here is the "on a cone" version of this notion:

Definition 2.6. A structure $\mathcal{A}$ is *uniformly continuously categorical* if there exists a continuous operator Θ such that, for every presentation $\mathcal{B}$ of $\mathcal{A}$, we have that $\Theta(\mathcal{B})$ is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

From Ventsov's result, we get that $\mathcal{A}$ is computably categorical on a cone if and only if it is uniformly continuously categorical after adding some parameters. From Theorem 1.1, we get that $\mathcal{A}$ is uniformly continuously categorical if and only if it has categoricity Scott rank 1. All this extends naturally through the transfinite.

Definition 2.7. A structure $\mathcal{A}$ is *uniformly boldface $\mathbf{\Delta}^0_\alpha$ categorical* if there exists a $\mathbf{\Delta}^0_\alpha$ operator Θ such that, for every presentation $\mathcal{B}$ of $\mathcal{A}$, we have that $\Theta(\mathcal{B})$ is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

We recall that a function $\Theta\colon 2^\omega \to 2^\omega$ is a $\mathbf{\Delta}^0_\alpha$ *operator* if the set $\{(X, n, i) \in 2^\omega \times \omega \times 2 : \Theta(X)(n) = i\}$ is $\mathbf{\Delta}^0_\alpha$; or equivalently, if there is a continuous function $\tilde{\Theta}$ such that $\Theta(X) = \tilde{\Theta}(X^{(\alpha)})$ for infinite α and $\Theta(X) = \tilde{\Theta}(X^{(\alpha-1)})$ for finite α.

It is known that $\mathcal{A}$ is relatively Δ^0_α categorical if and only if it is uniformly lightface Δ^0_α categorical after adding parameters, if and only if $\mathcal{A}$ has a $\Sigma^{\mathtt{c}}_\alpha$-Scott family with parameters. This is due to Ash [Ash87]. A close inspection of Ash's proof as given in [AK00, Theorem 10.14] using Ash–Knight–Manasse–Slaman's forcing [AKMS89], one also gets that $\mathcal{A}$ is uniformly lightface Δ^0_α categorical if and only if it has $\Sigma^{\mathtt{c}}_a$-Scott family without parameters. By considering this equivalence on a cone, we get (U3) $\Leftrightarrow$ (U1).

2.3. **α-freeness.** We need to recall the asymmetric back-and-forth relations as in [AK00, Chapter 15]. Given a countable structure $\mathcal{A}$, tuples $\bar{a}, \bar{b} \in A^{<\omega}$ of the same length, and an ordinal α, the α-back-and-forth relation, $\leq_\alpha$, can be defined in a couple of different ways:

$$\begin{aligned} \bar{a} \leq_\alpha \bar{b} \qquad &\iff \qquad \Pi^{\mathtt{in}}_\alpha\text{-tp}_{\mathcal{A}}(\bar{a}) \subseteq \Pi^{\mathtt{in}}_\alpha\text{-tp}_{\mathcal{A}}(\bar{b}) \\ &\iff \qquad (\forall \beta < \alpha)(\forall \bar{d} \in A^{<\omega})(\exists \bar{c} \in A^{<\omega})\ \bar{a}\bar{c} \geq_\beta \bar{b}\bar{d}, \end{aligned}$$

where $\Pi^{\mathtt{in}}_\alpha\text{-tp}_{\mathcal{A}}(\bar{a}) = \{\psi(\bar{x}) \in \Pi^{\mathtt{in}}_\alpha : \mathcal{A} \models \psi(\bar{a})\}$ is the $\Pi^{\mathtt{in}}_\alpha$-type of $\bar{a}$ in $\mathcal{A}$.

The notion of α-freeness was introduced by Ash and Knight in [AK00, page 269] as a useful combinatorial tool to describe Δ^0_α-categoricity.

Definition 2.8. A tuple $\bar{a} \in A$ is *α-free* if

$$(\forall \beta < \alpha)(\forall \bar{d} \in A^{<\omega})(\exists \bar{a}', \bar{d}' \in A^{<\omega})\ \bar{a}\bar{d} \leq_\beta \bar{a}'\bar{d}' \text{ and } \bar{a} \not\leq_\alpha \bar{a}'.$$

The lightface version, with parameters, of the equivalence (U1) ⇔ (U7) appears in [AK00, Proposition 17.6 and Theorem 17.7] under the assumption that $\mathcal{A}$ satisfies an effectiveness condition called α-friendliness. To show that (U1) ⇔ (U7), all we need to do is observe that every structure is α-friendly relative to a large enough oracle, and that (U1) is equivalent to saying that $\mathcal{A}$ has a $\Sigma^{\mathtt{c}}_{\alpha}$-Scott family on a cone. We also need to observe that the parameter-free version of [AK00, Proposition 17.6 and Theorem 17.7] holds via almost the same proofs.

3. Infinitary logic

In this section we prove the parts of our main theorem that only have to do with infinitary logic, and do not involve computability theory or back-and-forth relations.

We refer the reader to [Kei71] and [AK00, Chapter 6] for background on infinitary logic. Recall that we use the notation $\Sigma^{\mathtt{in}}_{\alpha}$ for the infinitary Σ_α formulas, and $\Sigma^{\mathtt{c}}_{\alpha}$ for the computably infinitary Σ_a formulas.

3.1. **Previously used notions of Scott rank.** For historical purposes, let us quickly review the previous definitions of Scott rank. This subsection is not necessary for the rest of the paper, and the reader who is only interested in the proof of Theorem 1.1 may skip to 3.2.

First, we need to define the *symmetric back-and-forth* relations $\sim_\alpha$. Given tuples $\bar{a}, \bar{b}$ in a structure $\mathcal{A}$, and of the same length, we define:

- $\bar{a} \sim_0 \bar{b}$ if they satisfy the same atomic formulas.
- $\bar{a} \sim_{\alpha+1} \bar{b}$ if for every $d \in A$, there exists $c \in A$ such that $\bar{a}c \sim_\alpha \bar{b}d$ and, for every $c \in A$, there exists $d \in A$ such that $\bar{a}c \sim_\alpha \bar{b}d$.
- For limit α, $\bar{a} \sim_\alpha \bar{b}$ if for some $\beta < \alpha$, $\bar{a} \sim_\beta \bar{b}$.

Scott showed that if $\bar{a} \sim_\alpha \bar{b}$ for all $\alpha < \omega_1$, then $\bar{a}$ and $\bar{b}$ are automorphic, assuming the underlying structure $\mathcal{A}$ is countable. For each tuple $\bar{a}$, there is a least ordinal $\rho(\bar{a})$ such that, for any other tuple $\bar{b}$, if $\bar{a} \sim_{\rho(\bar{a})} \bar{b}$, then $\bar{a}$ and $\bar{b}$ are automorphic. The most common version of Scott rank, derived from Scott [Sco65] is defined as follows:

$$SR(\mathcal{A}) = \sup\{\rho(\bar{a}) + 1 : \bar{a} \in A^{<\omega}\}.$$

Another version sometimes used is $sr(\mathcal{A}) = \sup\{\rho(\bar{a}) : \bar{a} \in A^{<\omega}\}$, which is the least α such that $\sim_\alpha$ coincides with the automorphism equivalence relation.

Another rank that is not far off from this one is the *game rank*; they are at most ω apart (see Gao [Gao07]).

A version of Scott rank better suited when one wants to use notions from finitary first-order logic in infinitary logic is the one used by Sacks and his students (see for instance [Sac07, Section 2]). To define it, we first need a hierarchy of fragments of infinitary logic defined as follows: Let $\mathcal{L}_0(\mathcal{A})$ the the finitary first-order logic in the vocabulary of $\mathcal{A}$. Let $\mathcal{L}_{\alpha+1}(\mathcal{A})$ be defined by adding to $\mathcal{L}_\alpha(\mathcal{A})$ the formulas $\bigwedge_{\varphi\in p}\varphi(\bar{x})$ for each $\mathcal{L}_\alpha(\mathcal{A})$-type $p(\bar{x})$ realized in $\mathcal{A}$, and closing under $\vee$, $\wedge$, $\neg$, $\exists$ and $\forall$, and take unions at limit ordinals. The Scott rank of $\mathcal{A}$ is then defined as the least α such that $\mathcal{A}$ is an atomic $\mathcal{L}_\alpha(\mathcal{A})$-model. This rank and the previous Scott rank, SR, coincide at the multiples of ω^2. On computable structures, they also agree at ω_1^{CK} and $\omega_1^{CK}+1$.

In Ash and Knight's book [AK00, § 6.7], there are two another ranks, r and R, which they say they prefer over sr and SR because they are closer to the complexity of formulas. We should remark that sr and SR are closer to the *quantifier rank* of formulas when it is defined by counting individual quantifiers rather than alternating blocks of quantifiers. But if we want to work in parallel with the hyperarithmetic hierarchy and the Borel hierarchy, we need to count alternations of quantifiers rather than single quantifiers. This is essentially the same motivation we have for introducing our version of Scott rank. They define $R(\mathcal{A})$ as the least

α such that all orbits are $\Pi^{\mathtt{in}}_{<\alpha}$ definable. Thus R is very close to our categoricity Scott rank; depending on the structure, they are either equal or off by 1. They coincide at limit ordinals, and, on computable structures, they coincide at ω_1^{CK} and $\omega_1^{CK}+1$.

3.2. **Type omitting.** In this subsection, we prove the main tool for proving (U2) $\Rightarrow$ (U6). The type omitting theorem for infinitary logic is well known and its proof is similar to that of the finitary original version due to Henkin and Orley. The precise instance of the type omitting lemma we need does not follow from the versions in the literature. We need a sharp count of the alternations of quantifiers, while the versions in the literature are about fragments that are closed under quantification and hence too coarse for us. The author learned of a similar kind of variations from Julia Knight and Sy Friedman. Another difference in our version is that we omit a type that is not supported *within* a given structure, rather than in general, so we do not need to introduce extra notions from infinitary logic like that of countable fragment or constancy property. Despite these differences, once the statement is set up correctly, the idea of the proof is not new.

Definition 3.1. A set of infinitary formulas $\Phi(\bar{x})$ is $\Sigma^{\mathtt{in}}_{\alpha}$-*supported* in $\mathcal{A}$ if there exists a $\Sigma^{\mathtt{in}}_{\alpha}$ formula $\varphi(\bar{x})$ such that

$$\mathcal{A} \models \exists \bar{x}\big(\varphi(\bar{x})\big) \quad \& \quad \forall \bar{x}\big(\varphi(\bar{x}) \Rightarrow \bigwedge_{\psi\in\Phi} \psi(\bar{x})\big).$$

Lemma 3.2 (Type omitting lemma)**.** *Let $\mathcal{A}$ be a structure and φ be a $\Pi^{\mathtt{in}}_{\alpha+1}$ sentence true of $\mathcal{A}$. Let $\Phi(\bar{x})$ be a partial $\Pi^{\mathtt{in}}_{\alpha}$-type which is not $\Sigma^{\mathtt{in}}_{\alpha}$-supported in $\mathcal{A}$. Then, there exists a structure $\mathcal{B}$ which models φ and omits Φ.*

Proof. Write φ as $\bigwedge_j \forall \bar{y}_j \varphi_j(\bar{y}_j)$, where each φ_j is $\Sigma^{\mathtt{in}}_{\alpha}$. Let τ be the vocabulary of the structure $\mathcal{A}$, and let $C = \{c_0, c_1, ...\}$ be a set of fresh constants. Using a Henkin-type construction, we will build a set S of $\Sigma^{\mathtt{in}}_{\alpha}$ sentences over the vocabulary $\tau \cup C$ such that:

- **(A):** If $\bigvee \psi_i \in S$, then $\psi_i \in S$ for some i.
- **(B):** If $\exists \bar{y}\psi(\bar{y}) \in S$, then $\psi(\bar{c}) \in S$ for some tuple of constants $\bar{c}$ from C.
- **(C):** If $\bigwedge \psi_i \in S$, then $\psi_i \in S$ for all i.
- **(D):** If $\forall \bar{y}\psi(\bar{y}) \in S$, then $\psi(\bar{c}) \in S$ for all $\bar{c}$ from C.
- **(E):** For every atomic sentence ψ over $\tau \cup C$, either $\psi \in S$ or $\neg\psi \in S$, but not both.
- **(F):** For every j and every tuple $\bar{c}$ from C of length $|\bar{y}_j|$, $\varphi_j(\bar{c}) \in S$.
- **(G):** For every tuple $\bar{c}$ from C of length $|\bar{x}|$, there is a formula $\psi \in \Phi$ such that $\neg\psi(\bar{c}) \in S$.

Once we have S satisfying (A)-(E), we can build a structure $\mathcal{B}$ as usual: We let $\mathcal{B}$ have domain C and we use the atomic sentences in S to define the structure in $\mathcal{B}$. By induction on formulas, using properties (A)-(E), we get that $\mathcal{B} \models \psi$ for every $\psi \in S$. From (F), we get that $\mathcal{B} \models \varphi$ and from (G), we get that $\mathcal{B}$ omits Φ.

The construction of S is by stages as in the usual Henkin construction. At stage s, we define a finite set of sentences S_s, and we will define $S = \bigcup_{s\in\omega} S_s$ at the end. Each S_s mentions at most finitely many of the constants from C. To ensure consistency, i.e. the latter part of (E), we make sure that, at each s, there is an assignment v_s that assigns values in $\mathcal{A}$ to the constants that appear in S_s in a way that S_s holds in $\mathcal{A}$. That is, if S_s mentions the constants $c_0, ..., c_n$, and v_s maps c_i to $a_i \in \mathcal{A}$, then for each formula $\psi(c_0, ..., c_n) \in S_s$, $\mathcal{A} \models \psi(a_0, ..., a_n)$.

At each stage, we take care of a new instance of one of the requirements. Instances of the requirements (A)-(F) can all be satisfied in a straightforward way without modifying the values in the assignment v_s. For instance, suppose that at stage $s+1$, we want to satisfy requirement (B) for the sentence $\exists \bar{y}\psi(c_0, ..., c_n, \bar{y}) \in S_s$, and suppose v_s maps c_i to $a_i \in \mathcal{A}$. Since $\mathcal{A} \models \exists \bar{y}\psi(a_0, ..., a_n, \bar{y})$, we have that for some $\bar{b} \in A^{<\omega}$, $\mathcal{A} \models \psi(a_0, ..., a_n, \bar{b})$. Let $\bar{c}$

be a tuple of new constants, let v_{s+1} be the extension of v_s which maps $\bar{c}$ to $\bar{b}$, and let $S_{s+1} = S_s \cup \{\psi(\bar{c})\}$. We leave the requirements (A), (C), (D), (E) and (F) to the reader.

Requirement (G) is a standard type-omitting argument: Take a tuple $\bar{c}$ from C of the same length as $\bar{x}$, and suppose we have already built S_s. Let $\varphi(\bar{c}, \bar{d}) = \bigwedge S_s$, where $\bar{d}$ is the tuple of constants from C that occur in S_s but are not present in $\bar{c}$. So $\exists \bar{y}(\bar{x}, \bar{y})$ is a $\Sigma^{\mathtt{in}}_{\alpha}$ formula realized in $\mathcal{A}$. Since Φ is not $\Sigma^{\mathtt{in}}_{\alpha}$-supported, there is a formula $\psi(\bar{x}) \in \Phi$ such that $\mathcal{A} \models \neg\forall\bar{x}(\exists\bar{y}\varphi(\bar{x},\bar{y}) \to \psi(\bar{x}))$. That is, there are tuples $\bar{a}, \bar{b} \in \mathcal{A}^{<\omega}$ such that $\mathcal{A} \models \varphi(\bar{a}, \bar{b}) \wedge \neg\psi(\bar{a})$. Let $S_{s+1} = S_s \cup \{\neg\psi(\bar{c})\}$ and let v_{s+1} map $\bar{c}\bar{d}$ to $\bar{a}\bar{b}$. $\square$

3.3. **The implications.** We prove

$$(\text{U1}) \Rightarrow (\text{U2}) \Rightarrow (\text{U6}) \Rightarrow (\text{U5}) \Rightarrow (\text{U1}).$$

This finishes the proof of Theorem 1.1, as we have already seen that (U1) $\Leftrightarrow$ (U3) in Subsection 2.2, that (U2) $\Leftrightarrow$ (U4) right after Theorem 1.1, and that (U1) $\Leftrightarrow$ (U7) in Subsection 2.3.

Proof of (U1) $\Rightarrow$ (U2). This follows from counting the quantifiers in the standard construction of the Scott sentence out of a Scott family. Recall that if $\Phi = \{\varphi_{\bar{a}} : \bar{a} \in \mathcal{A}\}$ is a Scott family for $\mathcal{A}$, where $\varphi_{\bar{a}}$ is the formula defining the orbit of $\bar{a}$, then the following is a Scott sentence for $\mathcal{A}$ (see, for instance, [AK00, Page 97]):

$$\bigwedge_{\bar{a}\in\mathcal{A}^{<\omega}} \forall\bar{x}\left(\varphi_{\bar{a}}(\bar{x}) \quad\to\quad \left(\bigwedge_{b\in\mathcal{A}} \exists y\varphi_{\bar{a}b}(\bar{x},y)\right) \quad \& \quad \left(\forall y \bigvee_{b\in\mathcal{A}} \varphi_{\bar{a}b}(\bar{x},y)\right)\right)$$

Note that if each φ is $\Sigma^{\mathtt{in}}_{\alpha}$, then the whole sentence is $\Pi^{\mathtt{in}}_{\alpha+1}$. $\square$

That (U2) $\Rightarrow$ (U6) is straightforward.

Proof of (U6) $\Rightarrow$ (U5). Let φ be a $\Pi^{\mathtt{in}}_{n+1}$ sentence true of $\mathcal{A}$ as given by (U6). Suppose, towards a contradiction that there is a $\Pi^{\mathtt{in}}_{\alpha}$ type $p(\bar{x})$ realized in $\mathcal{A}$ by some tuple $\bar{a}$ which is not $\Sigma^{\mathtt{in}}_{\alpha+1}$ supported within $\mathcal{A}$. By Lemma 3.2, there is a structure $\mathcal{B}$ which models φ and omits $p(\bar{x})$. By the choice of φ, we have that $\mathcal{B} \equiv_{\alpha+1} \mathcal{A}$, which will lead us to a contradiction. Since $\mathcal{B}$ is countable, there is a countable subset $q(\bar{x}) \subseteq p(\bar{x})$ which is also omitted in $\mathcal{B}$: just take, for each tuple in $\mathcal{B}$, a formula in $p(\bar{x})$ which it does not satisfy. But then,

$$\mathcal{A} \models \exists\bar{x} \bigwedge_{\psi\in q} \psi(\bar{x}) \quad \text{ and } \quad \mathcal{B} \models \neg\exists\bar{x} \bigwedge_{\psi\in q} \psi(\bar{x}),$$

contradicting that $\mathcal{A}$ and $\mathcal{B}$ are $\Sigma^{\mathtt{in}}_{\alpha+1}$-elementary equivalent. $\square$

Proof of (U5) $\Rightarrow$ (U1). For each tuple $\bar{a}$ in $\mathcal{A}$, let $\varphi_{\bar{a}}(\bar{x})$ be a $\Sigma^{\mathtt{in}}_{\alpha}$ formula that supports $\Pi^{\mathtt{in}}_{\alpha}$-$\mathrm{tp}_{\mathcal{A}}(\bar{a})$ and is realizable in $\mathcal{A}$. First, note that $\varphi_{\bar{a}}$ is true of $\bar{a}$, as otherwise $\neg\varphi_{\bar{a}}$ would belong to $\Pi^{\mathtt{in}}_{\alpha}$-$\mathrm{tp}_{\mathcal{A}}(\bar{a})$ and be implied by $\varphi_{\bar{a}}$. Second, we need to observe that if $\mathcal{A} \models \varphi_{\bar{a}}(\bar{b})$, then $\mathcal{A} \models \varphi_{\bar{b}}(\bar{a})$ too: Suppose not, and that $\mathcal{A} \models \neg\varphi_{\bar{b}}(\bar{a})$. We would then have that $\neg\varphi_{\bar{b}}(\bar{x}) \in \Pi^{\mathtt{in}}_{\alpha}$-$\mathrm{tp}_{\mathcal{A}}(\bar{a})$, and hence that $\varphi_{\bar{a}}(\bar{x})$ implies $\neg\varphi_{\bar{b}}(\bar{x})$, which we know is not true, as $\mathcal{A} \models \varphi_{\bar{a}}(\bar{b}) \wedge \varphi_{\bar{b}}(\bar{b})$.

Let $\Phi = \{\varphi_{\bar{a}} : \bar{a} \in A^{<\omega}\}$. We claim that Φ is a Scott family for $\mathcal{A}$, i.e. that is a set of definitions for the automorphism orbits of $\mathcal{A}$. Consider the set of pairs

$$P = \{\langle\bar{a}, \bar{b}\rangle \in (\mathcal{A}^{<\omega})^2 : \mathcal{A} \models \varphi_{\bar{a}}(\bar{b})\}.$$

We claim that P has the back-and-forth property, which would imply that $\bar{a}$ and $\bar{b}$ are automorphic as wanted whenever $\langle\bar{a}, \bar{b}\rangle \in P$. Suppose $\langle\bar{a}, \bar{b}\rangle \in P$. Let $d \in A$; we want to show that there exists $c \in A$ such that $\langle\bar{a}c, \bar{b}d\rangle \in P$. Thus, we need to show that $\mathcal{A} \models \exists y\, \varphi_{\bar{b},d}(\bar{a}, y)$. Suppose not. Then $\forall y \neg\varphi_{\bar{b},d}(\bar{a}, y)$ is part of the $\Pi^{\mathtt{in}}_{\alpha}$-type of $\bar{a}$ and hence implied by $\varphi_{\bar{a}}$. But then, since $\mathcal{A} \models \varphi_{\bar{a}}(\bar{b})$, we would have $\mathcal{A} \models \forall y\neg\varphi_{\bar{b},d}(\bar{b}, y)$, contradicting that $\mathcal{A} \models \varphi_{\bar{b}d}(\bar{b}, d)$. $\square$

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