NOTES ON THE JUMP OF A STRUCTURES

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Abstract. We introduce the notions of a *complete set of computably infinitary Π^0_n relations* on a structure, of the *jump of a structure*, and of *admitting nth jump inversion*.

Introduction

This paper is part of the study of the interactions between structural properties of a structure and computational properties of its presentations. We concentrate on the Turing jumps of the presentations of a structure, and the main notion defined in this paper is the one of the jump of a structure. Even if the definitions and theorems of this paper are all new, many of the ideas were already in the air, but they were not concretely formulated.

We start by defining what it means for a set of relations in a structure to be a complete set of computably infinitary Π^0_n relations. The idea is that a set of relations is computably infinitary Π^0_n complete if it captures the whole Π^0_n structural information of the structure. In the second section we look at classes of structures which have a finite complete set of computably infinitary Π^0_n relations. In the first section, we also define the nth jump of a structure to be the structure together with a complete set of computably infinitary Π^0_n relations. In the last section we look at the degree spectrum of the jump of a structure. We also define the notion of a structure $\mathcal{A}$ admitting nth jump inversion and prove it is equivalent to saying whenever $\mathcal{A}$ has an X-low$_n$ copy for some X, then it has an X-computable copy.

Throughout this paper we use $\mathcal{L}$ to denote a computable first order language. We use Π^c_n to denote the set of computably infinitary Π^0_n $\mathcal{L}$-formulas (see [AK00, Ch 7] for background on this language), and we use $\Pi^{c,Z}_n$ to denote the class of Z-computably infinitary Π^0_n $\mathcal{L}$-formulas.

1. Main Definitions

Definition 1.1. Let $\mathcal{A}$ be an $\mathcal{L}$-structure. Let $\{P_0, P_1, ...\}$ be a finite or infinite set of uniformly Π^c_n relations on $\mathcal{A}$. That is, there is a c.e. list of Π^c_n-formulas defining the relations P_i on $\mathcal{A}$. We say that $\{P_0, P_1, ...\}$ is a *complete set of* Π^c_n *relations on* $\mathcal{A}$ if every Π^c_n $\mathcal{L}$-formula $\psi(\bar{x})$ is equivalent to a $\Sigma^{c,0^{(n)}}_1$ $(\mathcal{L} \cup \{P_0, ...\})$-formula, and there is a computable procedure to find this equivalent formula. In other words, $\{P_0, P_1, ...\}$ is a *complete set of* Π^c_n *relations on* $\mathcal{A}$ if for every Π^c_n $\mathcal{L}$-formula $\psi(\bar{x})$ we can uniformly produce a $0^{(n)}$-computable list $\varphi_0(\bar{x}), \varphi_1(\bar{x}), \varphi_2(\bar{x}), ...$ of finitary existential $\mathcal{L}$-formulas that may mention the relations $P_0, P_1, ...$ such that

$$\mathcal{A} \models \psi(\bar{x}) \iff \bigvee_i \varphi_i(\bar{x}).$$

Observe that if $\{P_0, P_1, ...\}$ is a complete set of Π^c_n relations on $\mathcal{A}$ then every Σ^c_{n+1} $\mathcal{L}$-formula is also equivalent to a $\Sigma^{c,0^{(n)}}_1$ $(\mathcal{L} \cup \{P_0, ..., P_k\})$-formula.

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Definition 1.2. If $\{P_0, P_1, ...\}$ is a complete set of Π^c_n relations on $\mathcal{A}$, we say that $(\mathcal{A}, P_0, P_1, ...)$ is an *nth jump of* $\mathcal{A}$ and write

$$A^{(n)} = (\mathcal{A}, P_0, P_1, ...).$$

Note that being the nth jump of a structure is a property of isomorphism types of structures and not of presentations of structures.

The second thing to notice is that every structure always has an nth jump: Just let $\{P_0, P_1, ...\}$ be a computable list of all the Π^c_n relations on $\mathcal{A}$.

Observation 1.3. If $\mathcal{A}$ has a copy $\leq_T X$, then $\mathcal{A}^{(n)}$ has a copy $\leq_T X^{(n)}$.

Even though a structure might have many nth jumps, the following lemma show that, from a complexity viewpoint, all these jumps are the same.

Lemma 1.4. *Let $P_0, P_1, ...$ and $R_0, R_1, ...$ be complete sets of Π^c_n relations on $\mathcal{A}$. For every $Y \geq_T 0^{(n)}$ we have that*

$$(\mathcal{A}, R_0, R_1, ...) \text{ has copy } \leq_T Y \iff (\mathcal{A}, P_0, P_1, ...) \text{ has copy } \leq_T Y.$$

Proof. Fix a presentation of $\mathcal{A}$ and assume that Y can compute $\mathcal{A}$ and all the relations R_i uniformly. We will show that Y uniformly computes all the relations P_i. Since each P_i is defined by a Π^c_n formula, and this formula is equivalent to a $\Sigma^{c,0^{(n)}}_1$ $(\mathcal{L} \cup \{R_0, ...\})$-formula, we have that P_i is c.e. in $Y \oplus 0^{(n)} \equiv_T Y$. The complement of P_i is Σ^c_n, and in particular Σ^c_{n+1}. Hence it is also equivalent to a $\Sigma^{c,0^{(n)}}_1$ $(\mathcal{L} \cup \{R_0, ...\})$-formula and is also c.e. in Y. So P_i is computable in Y, uniformly in i. □

Recall that for non-trivial structures (using Knight's theorem)

$$degSp(\mathcal{A}) = \{X \in \mathcal{D} : \mathcal{A} \text{ has a copy } \leq_T X\},$$

where $\mathcal{D}$ is the set of Turing degrees. So, the lemma above can be restated as

$$degSp(\mathcal{A}, R_0, R_1, ...) \cap \mathcal{D}_{(\geq_T 0^{(n)})} = degSp(\mathcal{A}, P_0, P_1, ...) \cap \mathcal{D}_{(\geq_T 0^{(n)})}$$

(where $\mathcal{D}_{(\geq_T 0^{(n)})}$ is the set of Turing degrees above $0^{(n)}$). Therefore, we have that the degree spectrum of $\mathcal{A}^{(n)}$ on the degrees $\geq_T 0^{(n)}$ is independent of the possible choices of $\mathcal{A}^{(n)}$.

2. Examples

We now turn into looking at examples of jumps of structures. We start with linear orderings.

Lemma 2.1. *Let $\mathcal{A} = (A, <)$ be a linear ordering and let*

$$Succ^{\mathcal{A}} = \{(a, b) \in A^2 : \nexists c \ (a < c < b)\}.$$

Then

$$\mathcal{A}' = (A, <, Succ^{\mathcal{A}}).$$

Sketch of the proof. We need to show that $Succ^{\mathcal{A}}$ is Π^c_1-complete. We will show that every Σ^c_1 formula is equivalent to a finitary universal formula that uses the predicate $Succ$, and that $0'$ can uniformly find this formula. By taking complements, we will then get that every Π^c_1 formula is equivalent to a $\Sigma^{c,0'}_1$ $(\mathcal{L} \cup \{Succ\})$-formula.

Suppose that $\mathcal{A}$ has a first and a last element called $-\infty$ and ∞; the general case is very similar. First, we note that every finitary existential sentence ψ is equivalent to a sentence that says that there are least n many different elements in $\mathcal{A}$. Let $\psi_n(x, y)$ be the formula that says that there are at least n many different elements in between x and y. Observe that $\psi_n(x,y)$ is equivalent to a finitary universal formula over the Successor predicate:

$$\psi_n(x,y) \iff \bigwedge_{j<n} \nexists z_1,...,z_j \left(x = z_1 \leq ... \leq z_j = y \wedge (\bigwedge_{i=1}^{j-1}(Succ(z_i, z_{i+1})) \right).$$

Second, if we have a formula with free variables $\psi(x_1,...,x_k)$, we can write ψ as a disjunction over all the permutations $(\tau_1,...,\tau_k)$ of $(1,...,k)$ of the formulas $\left(\bigwedge_{j<k} x_{\tau_j} < x_{\tau_{j+1}}\right) \wedge \psi(x_1,...,x_k)$. Third, for every finitary existential formula $\psi(x_1,...,x_k)$, we have that $x_1 < x_2 < ... < x_k \wedge \psi(x_1,...,x_k)$ is equivalent to a finite disjunction of formulas of the form

$$x_1 < x_2 < ... < x_k \wedge \psi_{n_0}(-\infty, x_{\tau_1}) \wedge \psi_{n_1}(x_{\tau_1}, x_{\tau_2}) \wedge ... \wedge \psi_{n_k}(x_{\tau_k}, \infty),$$

for some $n_0,...,n_k \in \omega$. Therefore, we have that any Σ^c_1 formula $\psi(x_1,...,x_k)$ is equivalent to the disjunction over all the permutations $(\tau_1,...,\tau_k)$ of $(1,...,k)$ of formulas of the form

$$x_{\tau_1} < ... < x_{\tau_k} \wedge \left(\bigvee_j \left(\psi_{n^j_0}(-\infty, x_{\tau_1}) \wedge \psi_{n^j_1}(x_{\tau_1}, x_{\tau_2}) \wedge ... \wedge \psi_{n^j_k}(x_{\tau_k}, \infty) \right) \right).$$

Fourth, it can be shown that in the infinite disjunction in the formula above all but finitely many of the disjuncts are redundant. Furthermore, $0'$ can find these finitely many disjuncts. □

Notice that in all the linear orderings the same relation $Succ^{\mathcal{A}}$ that is Π^c_1 complete. This motivates the following definition.

Definition 2.2. Let $\mathcal{K}$ be a class of $\mathcal{L}$-structures. A c.e. set $\varphi_0, \varphi_1, ...$ of Π^c_n formulas is *a complete set of* Π^c_n *formulas for* $\mathcal{K}$, if for each structure $\mathcal{A} \in \mathcal{K}$ we have that $\{\varphi_0^{\mathcal{A}}, \varphi_1^{\mathcal{A}}, ...\}$ is a complete set of Π^c_n formulas on $\mathcal{A}$.

The Boolean algebra predicates considered by Downey, Jockusch [DJ94], Thurber [Thu95], Knight and Stob [KS00] are exactly the ones needed to define the first four jumps of a Boolean algebra.

Lemma 2.3 (Harris, Montalbán [HM])**.** *Let* $\mathcal{B}$ *be a Boolean algebra.*

- $\mathcal{B}' = (\mathcal{B}, atom^{\mathcal{B}})$,
- $\mathcal{B}'' = (\mathcal{B}, atom^{\mathcal{B}}, inf^{\mathcal{B}}, atomless^{\mathcal{B}})$.
- $\mathcal{B}''' = (\mathcal{B}, atom^{\mathcal{B}}, inf^{\mathcal{B}}, atomless^{\mathcal{B}}, atomic^{\mathcal{B}}, 1\text{-}atom^{\mathcal{B}}, atominf^{\mathcal{B}})$.
- $\mathcal{B}^{(4)} = (\mathcal{B}, atom^{\mathcal{B}}, inf^{\mathcal{B}}, atomless^{\mathcal{B}}, atomic^{\mathcal{B}}, 1\text{-}atom^{\mathcal{B}}, atominf^{\mathcal{B}}, \sim\text{-}inf^{\mathcal{B}}, Int(\omega+\eta)^{\mathcal{B}}, infatomicless^{\mathcal{B}}, 1\text{-}atomless^{\mathcal{B}}, nomaxatomless^{\mathcal{B}})$.

Furthermore, for every n *there is a finite set of* Π^c_n *formulas which are* Π^c_n *complete for the class of Boolean algebras. (Definitions of the relations above can be found in* [KS00] *and* [HM]*.)*

We note that not all the predicates in the four items above are Π^c_n for the corresponding n, but they are Boolean combinations of Π^c_n predicates. There is no problem relaxing our definition of nth jump to allow the predicates to be Boolean combinations of Π^c_n predicates so long they still generate all other Π^c_n predicates.

Proof. Harris and the author [HM] proved that the unary predicates R_σ for $\sigma \in \mathbf{BF}_n$ are Π^c_n complete for the class of Boolean algebras. They showed that for $n \leq 4$, the relations in the four items mentioned above are Boolean combinations of R_σ for $\sigma \in \mathbf{BF}_n$ and vice versa. □

The main lemmas in Downey, Jockusch [DJ94], Thurber [Thu95], and Knight, Stob [KS00] can now be stated as follows

Lemma 2.4. *Let X be any set, and $\mathcal{B}$ a Boolean algebra.*

1. [DJ94] *$\mathcal{B}$ has a copy $\leq_T X$ if and only if $\mathcal{B}'$ has a copy $\leq_T X'$*
2. [Thu95] *$\mathcal{B}'$ has a copy $\leq_T X$ if and only if $\mathcal{B}''$ has a copy $\leq_T X'$*
3. [KS00] *$\mathcal{B}''$ has a copy $\leq_T X$ if and only if $\mathcal{B}'''$ has a copy $\leq_T X'$*
4. [KS00] *$\mathcal{B}'''$ has a copy $\leq_T X$ if and only if $\mathcal{B}^{(4)}$ has a copy $\leq_T X'$*

Corollary 2.5 (Knight and Stob [KS00])**.** *Suppose $Y^{(4)} \leq_T X^{(4)}$. Every Y-computable Boolean algebra has a X-computable copy.*

Proof. Suppose $\mathcal{B}$ has a copy computable in Y. Then, by Observation 1.3, $\mathcal{B}^{(4)}$ has a copy computable in $Y^{(4)} \leq_T X^{(4)}$. Applying the four items of the previous lemma one at the time starting from the last one, we get that $\mathcal{B}$ has a X-computable copy. □

3. Jump inversions

The following theorem is a sort of jump inversion theorem for structures. The proof is just an applications of the ideas about generic copies structures developed by Ash, Knight, Mennasse and Slaman [AKMS89] and Chisholm [Chi90]. Essentially we prove that the jump $\mathcal{A}'$ of a structure $\mathcal{A}$ can compute a 1-generic copy of $\mathcal{A}$.

Theorem 3.1. *If $Y \geq_T 0'$ and $\mathcal{A}'$ has a copy $\leq_T Y$, then for some X with $X' \leq_T Y$, $\mathcal{A}$ has a copy computable in X.*

Proof. We will build a copy $\mathcal{B}$ of $\mathcal{A}$ with domain $B = \{b_0, b_1, ...\}$. Let $D(\mathcal{B}) \in 2^\omega$ be the diagram of $\mathcal{B}$. So, for some list of atomic formulas ψ_i with variables among $x_0, x_1, ...$ we have that $D(\mathcal{B}) = 1$ if and only if $\mathcal{B} \models \psi_i$ where x_j is interpreted as b_j. Assume that ψ_i only uses variables among $x_0, ..., x_i$. Therefore, to know the first n bits of $D(\mathcal{B})$ we only need to use the atomic relations among $b_0, ..., b_n$. For each $\sigma \in 2^n$, let $\psi_\sigma(x_0, ..., x_n)$ be the formula $\bigwedge_{i:\sigma(i)=1} \psi_i \wedge \bigwedge_{i:\sigma(i)=0} \neg\psi_i$. So, we have that $\mathcal{B} \models \psi_\sigma(b_0, ..., b_n) \iff \sigma \subseteq D(\mathcal{B})$.

We will build a bijection $F\colon B \to A$ and then define $\mathcal{B}$ by pulling back the structure of $\mathcal{A}$, and we will let $X = D(\mathcal{B})$. We need to define F computably in Y and we will also make sure that Y computes the Turing jump of $D(\mathcal{B})$. At each stage s we define a finite one-to-one partial map $p_s\colon B \to A$ with domain $\{b_0, ..., b_{n_s}\}$, and then we will let $F = \bigcup_s p_s$. Given a finite one-to-one partial map p that maps $b_0, ..., b_n$ to $a_0, ..., a_n$, let $D(p)$ be the $\sigma \in 2^n$ such that $\mathcal{A} \models \psi_\sigma(a_0, ..., a_n)$. Note that $D(\mathcal{B}) = \bigcup_n D(p_n)$.

Construction:

- Let p_0 map b_0 to a_0.
- At stage $s+1 = 2e$ extend p_s to p_{s+1} in any way so that b_e is in the domain of p_{s+1} and a_e is in the image.
- At stage $s+1 = 2e+1$ we want to decide the jump of $D(\mathcal{B})$. Suppose p_s maps $b_0, ..., b_{n_s}$ to $a_0, ..., a_{n_s}$. Using Y, decide whether there exists $q \supseteq p_s$ such that $\{e\}^{D(q)}(e)\downarrow$. Note that Y can decide this because, since it computes $\mathcal{A}'$, it knows whether
$$\mathcal{A} \models \bigvee_{\sigma \supseteq p_s, \{e\}^\sigma(e)\downarrow} \exists \bar{y}\ \psi_\sigma(a_0, ..., a_{n_s}, \bar{y}).$$
If the answer is positive, Y can search for witnesses σ and $\bar{y}$ and use them to define p_{s+1} adding $\bar{y}$ to the range of p_s. In this case Y knows that $e \in D(\mathcal{B})'$. Otherwise, we let $p_{s+1} = p_s$ and Y knows that $e \notin D(\mathcal{B})'$.

We have build $\mathcal{B}$ so that $D(\mathcal{B})' \leq_T Y$ as wanted. □

Corollary 3.2. *For every structure $\mathcal{A}$,*

$$degSp(\mathcal{A}') \cap \mathcal{D}_{(\geq_T 0')} = \{X' : X \in degSp(\mathcal{A})\}.$$

Proof. That $\{X' : X \in degSp(\mathcal{A})\} \subseteq degSp(\mathcal{A}') \cap \mathcal{D}_{(\geq_T 0')}$ follows from Observation 1.3. That $degSp(\mathcal{A}') \cap \mathcal{D}_{(\geq_T 0')} \subseteq \{X' : X \in degSp(\mathcal{A})\}$ follows from the previous theorem. □

Now we consider a stronger version of jump inversion.

Definition 3.3. We say that $\mathcal{A}$ *admits nth jump inversion* if for every set X, we have that

$$A^{(n)} \text{ has copy } \leq_T X^{(n)} \iff A \text{ has copy } \leq_T X.$$

Observation 3.4. If $\mathcal{A}$ admits jump inversion, then

$$degSp(\mathcal{A}) = \{X : X' \in degSp(\mathcal{A}')\}.$$

Structures which admit nth jump inversion have, in some sense, already been considered in the literature before. Lemma 2.3 above shows that Boolean algebras admit 4th jump inversion. In [HM], Harris and the author asked the following question (stated in a different way): does every Boolean algebra admit nth jump inversion? It follows from Theorem 3.5 below that this question is equivalent to the well-known question of whether for every X, every X-low$_n$ Boolean algebra has an X-computable copy. For linear orderings, the following results are known. Downey [DK92] proved that every linear ordering of the form $(\mathbb{Q} + 2 + \mathbb{Q}) \cdot \mathcal{A}$ admits jump inversion. Ash [Ash91] showed that linear orderings of the form $\omega^n \cdot \mathcal{A}$ admit $2n$th-jump inversion. Kach and the author [KM] then used these results to prove that all the linear orderings with finitely many descending cuts admit nth jump inversion for every n. Graphs which admit αth jump inversion have been used [GHK+05] and [CFG+] to show that there exists Δ^0_α-categorical structures which are not relatively Δ^0_α-categorical, lifting earlier results of Goncharov and others on computably categorical structures.

Theorem 3.5. *Let $\mathcal{A}$ be an $\mathcal{L}$-structure. The following are equivalent.*

(1) *$\mathcal{A}$ admits nth jump inversion;*
(2) *For every sets X, Y with $X^{(n)} \equiv_T Y^{(n)}$, we have that*

$$A \text{ has copy } \leq_T X \iff A \text{ has copy } \leq_T Y.$$

Proof. To prove that (1) implies (2), we have that if $\mathcal{A}$ has a copy computable in X, then $\mathcal{A}^{(n)}$ has a copy computable in $X^{(n)} \equiv_T Y^{(n)}$, and hence by (1) $\mathcal{A}$ has a copy computable in Y.

For the other direction, suppose that $\mathcal{A}^{(n)}$ has a $Y^{(n)}$-computable copy. Then, using n iterations of Theorem 3.1, for some X with $X^{(n)} \equiv_T Y^{(n)}$, $\mathcal{A}$ has a copy computable in X. But, then, by (2), $\mathcal{A}$ has a Y-computable copy. □

4. Questions

(1) What are other examples of structures with finite sets of complete Π^c_n relations?
(2) What are other examples of structures that admit jump inversion?
(3) Is there a structural characterization of the structures that admit jump inversion?

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