The least Σ-jump inversion theorem for n-families

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Abstract: Studying the jump of structures [Montalbán 2009], [Puzarenko 2009], [Stukachev 2009] that for each set X that computes the halting problem $\emptyset'$ there is a countable family of sets which is Σ-definable precisely in the admissible sets $\mathbb{A}$ whose jumps compute X. Moreover, for every countable family of sets which computes $\emptyset'$ there is a family of families of sets which is Σ-definable precisely in the admissible sets $\mathbb{A}$ whose jumps compute X. These results, in fact, hold for the hierarchy of n-families (families of families of families of ...).

Key Words: jump of structure, enumeration jump, Σ-jump, Σ-reducibility, countable family, n-family
Category: F.1.1., F.1.2., F.4.1.

1 Introduction

The study of computational properties of families was started in [Kalimullin and Puzarenko 2009] and [Kalimullin and Faizrahmanov 2016 (a)].

Definition 1. A 0-family is a subset of ω. For an integer $n > 0$, an n-family is a countable set of $(n-1)$-families.

According to [Kalimullin and Faizrahmanov 2016 (a)] the definition of computably enumerable n-families is inductive: an n-family $\mathcal{F}$ is computably enumerable if it's elements, $(n-1)$-families, are uniformly computably enumerable. More precisely we give this definition generalized to an arbitrary admissible sets (see [Ershov 1996]):

Definition 2. [Kalimullin and Faizrahmanov 2016 (a)] A Σ-formula Φ (possibly with parameters) *defines* a 0-family $X \subset \omega$ in an admissible set $\mathbb{A}$ if it defines the the predicate $x \in A$. A Σ-formula Φ containing at least one parameter x *defines* an $(n+1)$-family $\mathcal{F}$, if there is a Σ-definable subset $E \subseteq \mathbb{A}$ such that the formulae $\Phi(x)$, $x \in E$, define all elements of $\mathcal{F}$ and only them.

This definition extends the definition given in [Kalimullin and Puzarenko 2009].

We will see below that for the n-families it is enough to consider only special cases of admissible sets: the hereditary finite structures $\mathbb{HF}(\mathfrak{M})$, where $\mathfrak{M}$ is some algebraic structure. Let M be the domain of $\mathfrak{M}$ and let σ be the language of $\mathfrak{M}$. The domain of $\mathbb{HF}(\mathfrak{M})$ is the class of $HF(M)$ of hereditarily finite sets over the M is defined by induction as follows:

- $H_0(M) = \{\emptyset\}$;
- $H_{n+1}(M) = H_n(M) \cup \mathcal{P}_\omega(H_n(M) \cup M)$;
- $HF(M) = \bigcup_{n<\omega} H_n(M) \cup M$

(where $P_\omega(X)$ denotes the set of all finite subsets of X). The structure $\mathbb{HF}(\mathfrak{M})$ is defined in a signature $\sigma \cup \{U^{(1)}, \in^{(2)}, \emptyset\}$ (called a *hereditarily finite superstructure over* $\mathfrak{M}$), so that $U^{\mathbb{HF}(\mathfrak{M})} = M$, $\in^{\mathbb{HF}(\mathfrak{M})} \subseteq (HF(M)) \times (HF(M) \setminus M)$ is the membership relation on $\mathbb{HF}(\mathfrak{M})$, the constant symbol $\emptyset$ is interpreted as the empty "set", and symbols in the signature σ are interpreted in the same way as on $\mathfrak{M}$.

For example we can code every n-family $\mathcal{F}$ into the admissible superstructure $\mathbb{HF}(\mathfrak{M}_\mathcal{F})$ over the special structure $\mathfrak{M}_\mathcal{F}$ defined as follows.

- Let A be an arbitrary 0-family. A structure $\mathfrak{M}_A$ of signature $\sigma = \{r, I^1, R^2\}$ is defined by following:

  the domain of the structure is representable as a disjoint union $\omega \cup X$, where $X = \{x_n : x \in A\}$;

  $R^{\mathfrak{M}_A} = \{\langle n, n+1\rangle : n \in \omega\} \cup \{\langle x_n, n\rangle : n \in A\}$, $r^{\mathfrak{M}_A} = 0$ and $I^{\mathfrak{M}_A} = \{r^{\mathfrak{M}_A}\}$.
- Let $\mathcal{F} = \{\mathcal{S}_i : i \in \omega\}$ be an n-family, $n > 0$. Following [Kalimullin and Puzarenko 2009] we can code $\mathcal{F}$ into a structure $\mathfrak{M}_\mathcal{F}$ of signature σ fix an element $r^{\mathfrak{M}_\mathcal{F}}$ and consider a disjoint structures $\mathfrak{M}_i^k$ of signature σ such that for all $k, i \in \omega$:
  1. $\mathfrak{M}_i^k \cong \mathfrak{M}_{\mathcal{S}_i}$ (the parameter $k \in \omega$ guarantees that each $\mathcal{S}_i$ is repeated infinitely many times);

2. $r^{\mathfrak{M}_\mathcal{F}} \notin |\mathfrak{M}_i^k|$;

The domain of the structure is a disjoint union $\bigcup_{k,i} |\mathfrak{M}_i^k| \cup \{r^{\mathfrak{M}_\mathcal{F}}\}$.

For each $x, y \in |\mathfrak{M}_\mathcal{F}|$ we define

$$R(x,y) \Leftrightarrow x = (\exists k, i)\,[x = r^{\mathfrak{M}_\mathcal{F}} \,\&\, y = r^{\mathfrak{M}_i^k} \;\vee\; R^{\mathfrak{M}_i^k}(x,y)].$$

Let $I^{\mathfrak{M}_\mathcal{F}} = \bigcup_{k,i} I^{\mathfrak{M}_i^k}$. By this inductive definition the elements of $I^{\mathfrak{M}_\mathcal{F}}$ were appeared originally as $r^{\mathfrak{M}_A}$ for sets (0-families) $A \in \dots \in \mathcal{F}$. For $i \in I^{\mathfrak{M}_\mathcal{F}}$ we denote the corresponding such set via A_i.

It is easy to check that every n-family $\mathcal{F}$ is Σ-definable in $\mathbb{HF}(\mathfrak{M}_\mathcal{F})$. For example, if $n = 0$ then a 0-family $A \subseteq \omega$ is defined by the formula saying that there is a sequence

$$n_0 = r, n_1, n_2, \dots, n_x, p, q,$$

such that $R(n_i, n_{i+1})$ for all $i < x$, and $R(n_x, p)$, $R(n_x, q)$. Moreover, it follows from [Kalimullin and Puzarenko 2009] that the Σ-definability of $\mathcal{F}$ is equivalent to the Σ-definability of $\mathfrak{M}_\mathcal{F}$ itself.

Proposition 3. *[Kalimullin and Puzarenko 2009] An n-family $\mathcal{F}$ is Σ-definable in a countable admissible set $\mathbb{A}$ iff the structure $\mathfrak{M}_\mathcal{F}$ (and, therefore, $\mathbb{HF}(\mathfrak{M}_\mathcal{F})$) is Σ-interpretable in $\mathbb{A}$.*

Under Σ-interpretation of a structure $\mathfrak{M}$ in a language σ we understand a Σ-definable structure $\mathfrak{N}$ in the language $\sigma \cup \{\sim\}$, where $\sim$ is a new congruence relation on $\mathfrak{N}$ such that $\mathfrak{N}/\sim\,\cong \mathfrak{M}$.

Definition 4. Let $\mathcal{F}$ be an n-family and $\mathfrak{M}$ be a structure. We say that $\mathcal{F}$ is *Σ-reducible* to $\mathfrak{M}$ (written $\mathcal{F} \leqslant_\Sigma \mathfrak{M}$) if *$\Sigma$-definable* in $\mathbb{HF}(\mathfrak{M})$. Similarly, $\mathfrak{M} \leqslant_\Sigma \mathcal{F}$ if $\mathfrak{M}$ is Σ-interpretable in $\mathbb{HF}(\mathfrak{M}_\mathcal{F})$. If $\mathcal{F}$ and $\mathcal{S}$ are n- and m-families correspondingly we say that $\mathcal{F}$ is *Σ-reducible* to $\mathcal{S}$ if $\mathcal{F} \leqslant \mathfrak{M}_\mathcal{S}$. As usual, the relation $\equiv_\Sigma$ holds in the case of Σ-reductions from the left to the right and from the right to the left.

Note that for an n-family $\mathcal{F}$ and the $(n+1)$-family $\{\mathcal{F}\}$ we have $\{\mathcal{F}\} \equiv_\Sigma \mathcal{F}$. By this reason we can look on the n-family $\mathcal{F}$ as to an m-family for $m > n$.

If Y is arbitrary set and $\mathcal{F}$ is an n-family, $n > 0$, then we define by induction the *join* of Y and $\mathcal{F}$ by letting

$$Y \oplus \mathcal{F} = \{Y \oplus \mathcal{S} : \mathcal{S} \in \mathcal{F}\}.$$

Recall that for the case $n = 0$ the standard notation is

$$Y \oplus A = \{2x : x \in Y\} \cup \{2x+1 : x \in A\}.$$

For an n-family $\mathcal{F}$ and an integer k denote by $\mathcal{F}^k$ the n-family $\{k\} \oplus \mathcal{F}$. Clearly that for every integer k and n-family $\mathcal{F}$, we have $\mathcal{F} \equiv_\Sigma \mathcal{F}^k$. For n-families $\mathcal{F}, \mathcal{G}$ define the n-family

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F}^0 \cup \mathcal{G}^1.$$

It is easy to see that $\mathcal{F} \leq_\Sigma \mathcal{F} \oplus \mathcal{G}$, $\mathcal{G} \leq_\Sigma \mathcal{F} \oplus \mathcal{G}$, and

$$\mathcal{F} \leq_\Sigma \mathfrak{M}, \mathcal{G} \leq_\Sigma \mathfrak{M} \implies \mathcal{F} \oplus \mathcal{G} \leq_\Sigma \mathfrak{M}$$

for every structure $\mathfrak{M}$.

2 Jump and jump inversion on n-families

Definition 5. [Montalbán 2009], [Puzarenko 2009], [Stukachev 2009]. For any structure $\mathfrak{M}$ the structure $\mathfrak{J}(\mathfrak{M}) = (\mathbb{HF}(\mathfrak{M}), U_\Sigma)$, where where U_Σ is a ternary Σ-predicate on $\mathbb{HF}(\mathfrak{M})$ universal for the class of all binary Σ-predicates on $\mathbb{HF}(\mathfrak{M})$, is called a *$\Sigma$-jump*.

For any n-family $\mathcal{F}$ instead of $\mathfrak{J}(\mathfrak{M}_\mathcal{F})$ we simply write $\mathfrak{J}(\mathcal{F})$. The concept of a Σ-jump with respect to Σ-reducibility does not depend on the choice of a universal Σ-predicate. Furthermore, this Σ-jump on structures having T-(e-)degrees acts in the same way as a T-(e-)jump (see [Puzarenko 2009]). As in the classical case, the Σ-jump operation satisfies the following:

1. $\mathfrak{A} \leqslant_\Sigma \mathfrak{J}(\mathfrak{A})$;
2. $\mathfrak{A} \leqslant_\Sigma \mathfrak{B} \Rightarrow \mathfrak{J}(\mathfrak{A}) \leqslant_\Sigma \mathfrak{J}(\mathfrak{B})$.

We define $\mathfrak{J}^n(\mathfrak{A})$ by induction on $n \in \omega$ as follows: $\mathfrak{J}^0(\mathfrak{A}) = \mathfrak{A}$, $\mathfrak{J}^{n+1}(\mathfrak{A}) = \mathfrak{J}(\mathfrak{J}^n(\mathfrak{A}))$. It was shown in [Puzarenko 2009] that for any structures $\mathfrak{M}$ and $\mathfrak{A}$ of a finite signature $\mathfrak{M}$ is Σ_{m+1}-definable in $\mathfrak{A}$ iff $\mathfrak{M} \leqslant_\Sigma \mathfrak{J}^m(\mathfrak{A})$.

Example 1. ([Puzarenko 2009]). For 0-familes A the jump $\mathfrak{J}(A)$ is Σ-equivalent to $\mathfrak{M}_{J(A)}$, where $J(A)$ is the the enumeration jump of A:

$$J(A) = K(A) \oplus \overline{K(A)} \text{ and } K(A) = \{n : n \in \Phi_n(A)\},$$

for the Gödel numbering of enumeration operators $\{\Phi_n\}_{n\in\omega}$.
Example 2. It is easy to check that for the family InfCE of all infinite c.e. sets we have $\mathfrak{J}(\text{InfCE}) \equiv_\Sigma J(J(\emptyset)) \equiv_e \overline{\emptyset''}$. Indeed, $\overline{\emptyset''}$ is computably isomorphic to $\{n : W_n \text{ is infinite}\}$, and a c.e. set W_n is infinite if and only if the set the (uniformly) computable set

$$V_n = \{s : W_{n,s} \neq W_{n,s+1}\}$$

is infinite, and so, if and only if $F \subseteq V_n$ for some $F \in \mathrm{InfCE}$. The predicate $F \subseteq V_n$ can be recognised by $J(F)$.

The inverse reduction $\mathfrak{J}(\mathrm{InfCE}) \leq_\Sigma J(J(\emptyset))$ is obvious. Moreover, we can prove slightly different. Suppose $\mathfrak{M}_{J(J(\emptyset))} \leq_\Sigma \mathfrak{J}(\mathfrak{M})$ for some countable $\mathfrak{M}$, i.e., let $\{n : W_n \text{ is infinite}\}$ is Σ_2-definable in $\mathbb{HF}(\mathfrak{M})$. Then there is Δ_0-formula Φ such that

$$W_n \text{ is infinite} \iff \mathbb{HF}(\mathfrak{M}) \models (\exists a)(\forall b)\Phi(n, a, b)$$

Then the sequence

$$V_{n,a} = \begin{cases} W_n, & \text{if } \mathbb{HF}(\mathfrak{M}) \models (\forall b)\Phi(n, a, b); \\ \omega, & \text{otherwise,} \end{cases}$$

exhausting all infinite c.e. sets can be determined by the Σ-predicate

$$x \in V_{n,a} \iff x \in W_n \vee x \in \omega \ \& \ (\exists b)\neg\Phi(n, a, b).$$

This allows to provide the reducibility $\mathfrak{M}_{\mathrm{InfCE}} \leq_\Sigma \mathfrak{M}$ for every countable $\mathfrak{M}$ such that $J(J(\emptyset)) \leq_\Sigma \mathfrak{M}$, i.e. the 1-family InfCE is the *the least jump inversion* for the 0-family $J(J(\emptyset))$.

Let us look for such least jump inversion for any n-family $\mathcal{F}$. For each n-family $\mathcal{F}$, recursively define a finitary $(n+1)$-family $\mathcal{E}(\mathcal{F})$:

$$\mathcal{E}(\mathcal{F}) = \begin{cases} \mathcal{H}_1 \cup \{\{2x\} : x \in A\}, & \text{if } n = 0 \text{ and } \mathcal{F} = A \subseteq \omega, \\ \mathcal{H}_{n+1} \cup \{\mathcal{E}(\mathcal{S}) : \mathcal{S} \in \mathcal{F}^0\}, & \text{if } n > 0, \end{cases}$$

where $\mathcal{H}_1 = \{\{2n, 2n+1\} : n \in \omega\}$ and $H_{n+1} = \{\mathcal{H}_n\}$. This is very similar to a definitions in [Kalimullin and Puzarenko 2009] and [Faizrahmanov and Kalimullin 2016 (b), (c)].

According to the following theorem we will call $\mathcal{E}(\mathcal{F})$ as the *least Σ-jump inversion for* $\mathcal{F}$ (meaning that in fact it is an inversion of $J(\emptyset) \oplus \mathcal{F}$).

Theorem 6. *For any n-family $\mathcal{F}$ the $(n+1)$-family $\mathcal{E}(\mathcal{F})$ is a least jump inversion of $\mathcal{F}$. Namely,*

1) $\mathcal{F} \leqslant_\Sigma \mathfrak{J}(\mathcal{E}(\mathcal{F}))$;

2) for each countable structure $\mathfrak{B}$ of a finite signature $\mathcal{E}(\mathcal{F}) \leqslant_\Sigma \mathfrak{B}$ if $\mathcal{F} \leqslant_\Sigma \mathfrak{J}(\mathfrak{B})$.

3) $\mathfrak{J}(\mathcal{E}(\mathcal{F})) \leqslant_\Sigma J(\emptyset) \oplus \mathcal{F}$.

Proof. 1) To show that $\mathcal{F} \leqslant_\Sigma \mathfrak{J}(\mathcal{E}(\mathcal{F}))$ fix a structure $\mathfrak{A} \cong \mathfrak{M}_{\mathcal{E}(\mathcal{F})}$ and define a Σ_2-subset M of $\mathbb{HF}(\mathfrak{A})$, constant $r^{\mathfrak{M}} \in M$ and Δ_2-predicates $I^{\mathfrak{M}}$, $R^{\mathfrak{M}}$ on M such that the structure $\mathfrak{M} = \langle M; r^{\mathfrak{M}}, I^{\mathfrak{M}}, R^{\mathfrak{M}}\rangle$ is isomorphic to $\mathfrak{M}_{\mathcal{F}^0}$. Let C be the set of all $x \in |\mathfrak{A}|$ for which there exists a finite sequence $x_0, x_1, \ldots, x_{k+1}$ such that $x_0 = x$, $I^{\mathfrak{A}}(x_{k+1})$, $R^{\mathfrak{A}}(x_i, x_{i+1})$ for every $i \leqslant k$ and for some $n \in \omega$ the singleton $\{2n\}$ is encoded under x_{k+1}. Denote by D the set of all end vertices in C, i.e. such elements $x \in C$ that $\neg R^{\mathfrak{A}}(x, y)$ for every $y \in C$. Consider a binary relation G on $\mathbb{HF}(\mathfrak{A})$ consisting of all pairs $\langle x, n\rangle \in D \times \omega$ for which there is an $y \in |\mathfrak{A}|$ such that $I^{\mathfrak{A}}(y)$, $R^{\mathfrak{A}}(x, y)$ and the singleton $\{2n\}$ is encoded under y. By the definition of $\mathcal{E}(\mathcal{F})$ the relation G is Σ_2-predicate on $\mathbb{HF}(\mathfrak{A})$. Note that if we put under every element $x \in D$ a copy of structure $\mathfrak{M}_{A_x}$, where $A_x = \{n : G(x, n)\}$, then the structure $\bigcup_{x\in D} \mathfrak{M}_{A_x} \cup (\mathfrak{A} \restriction C)$ will be isomorphic to $\mathfrak{M}_{\mathcal{F}^0}$. To formalize this we define

$$B_x = \{\langle x, 2n\rangle : x \in D, n \in \omega \setminus \{0\}\}$$

for every $x \in D$ and

$$F_x = \{\langle x, 2n+1\rangle : G(x, n), n \in \omega\}.$$

Let $M = \bigcup_{x\in D}(B_x \cup F_x) \cup C$. For every $x, y \in M$ set $R^{\mathfrak{M}}(x, y)$ iff one of the following conditions holds:

1. $x, y \in C$ and $R^{\mathfrak{A}}(x, y)$;
2. $y \in D$ and $(\exists z \in D)\,[x = \langle z, 1\rangle]$;
3. $(\exists n \in \omega)(\exists z \in D)\,[x = \langle z, 2n\rangle \,\&\, y = \langle z, 2n+2\rangle]$;
4. $(\exists n \in \omega)(\exists z \in D)\,[x = \langle z, 2n+1\rangle \,\&\, y = \langle z, 2n\rangle]$.

Finally, we define $r^{\mathfrak{M}} = r^{\mathfrak{A}} \in C$ and $I^{\mathfrak{M}}(x)$ iff $x \in D$. Clearly that M is Σ_2-subset of $\mathbb{HF}(\mathfrak{A})$ and $I^{\mathfrak{M}}$, $R^{\mathfrak{M}}$ are Δ_2-predicates on M. Therefore, $\mathcal{F} \leqslant_\Sigma \mathcal{J}(\mathcal{E}(\mathcal{F}))$.

2) Let an n-family $\mathcal{F}$ is Σ-reducible to $\mathcal{J}(\mathfrak{B})$ for some structure $\mathfrak{B}$ of a finite signature. Hence $\mathcal{F}^0 \leqslant_\Sigma \mathcal{J}(\mathfrak{B})$. Fix a Σ_2-subset A of $\mathbb{HF}(\mathfrak{B})$, constant $r^{\mathfrak{A}}$ and Δ_2-predicates $I^{\mathfrak{A}}$, $R^{\mathfrak{A}}$, η on A such that η is the congruence relation on the structure $\mathfrak{A} = (A; r^{\mathfrak{A}}, I^{\mathfrak{A}}, R^{\mathfrak{A}})$ and $\mathfrak{A}/\eta \cong \mathfrak{M}_{\mathcal{F}^0}$. Let Ψ be a Δ_0-formula such that for all $x_1, \ldots, x_n \in A$ and every $m \in \omega$

$$\mathbb{HF}(\mathfrak{B}) \models (\exists a)(\forall b)\,\Psi(a, b, x_1, \ldots, x_n, k)$$

iff $R^{\mathfrak{A}}(r^{\mathfrak{A}}, x_1)$, $R^{\mathfrak{A}}(x_i, x_{i+1})$ for every i, $1 \leqslant i < n$, and k belongs to the set which is encoded under x_n. To show that $\mathcal{E}(\mathcal{F}) \leqslant_\Sigma \mathfrak{B}$ define a Σ-subset M of $\mathbb{HF}(\mathfrak{B})$, constant $r^{\mathfrak{M}}$ and Σ-predicates $I^{\mathfrak{M}}$, $R^{\mathfrak{M}}$, θ on M such that θ is the congruence relation on the structure $\mathfrak{M} = (M; r^{\mathfrak{M}}, I^{\mathfrak{M}}, R^{\mathfrak{M}})$ and $\mathfrak{M}/\theta \cong \mathfrak{M}_{\mathcal{E}(\mathcal{F})}$. Let $M = \bigcup\limits_{i=1}^{n} M_i \cup \{\langle 0, 0\rangle\} \cup L_1 \cup L_2 \cup L_3$, where

$$M_i = \{\langle\langle x_1, \ldots, x_i\rangle, 2i\rangle : x_1, \ldots, x_i \in HF(\mathfrak{B})\},\ 1 \leqslant i \leqslant n,$$

$$L_1 = \{\langle\langle k, i, x_1, \ldots, x_n, a\rangle, 2j+1\rangle : k, i, j \in \omega,\ x_1, \ldots, x_n, a \in HF(\mathfrak{B})\},$$

$$L_2 = \{\langle\langle k, i, x_1, \dots, x_n, a, b\rangle, 2n+2\rangle : k, i \in \omega,\ x_1, \dots, x_n, a, b \in HF(\mathfrak{B})\},$$
$$L_3 = \{\langle\langle k, i, x_1, \dots, x_n, a, b\rangle, 2n+4\rangle : k, i \in \omega,\ x_1, \dots, x_n, a, b \in HF(\mathfrak{B})\}.$$

Set $r^{\mathfrak{M}} = \langle 0, 0\rangle$, $R^{\mathfrak{M}}(r^{\mathfrak{M}}, \langle x, 2\rangle)$ for every $x \in HF(\mathfrak{B})$ and

$$R^{\mathfrak{M}}(\langle\langle x_1, \dots, x_i\rangle, 2i\rangle, \langle\langle x_1, \dots, x_i, x_{i+1}\rangle, 2i+2\rangle),\ x_1, \dots, x_i, x_{i+1} \in HF(\mathfrak{B})$$

for every i, $1 \leqslant i < n$. To continue the definition of $\mathfrak{M}$ we put under every element $y = \langle\langle x_1, \dots, x_n\rangle, 2n\rangle \in M_n$ a copy of structure $\mathcal{E}(A_y)$, where A_y is the set which is encoded under element x_n in the structure $\mathfrak{A}/\eta$ if $R^{\mathfrak{A}}(x_i, x_{i+1})$ for every i, $i \leqslant i < n$, and $A_y = \emptyset$ otherwise. More precisely, define $I^{\mathfrak{M}}(\langle\langle k, i, x_1, \dots, x_n, a\rangle, 1\rangle)$,

$$R^{\mathfrak{M}}(\langle\langle x_1, \dots, x_n\rangle, 2n\rangle, \langle\langle k, i, x_1, \dots, x_n, a\rangle, 1\rangle),$$

$$R^{\mathfrak{M}}(\langle\langle k, i, x_1, \dots, x_n, a\rangle, 2j+1\rangle, \langle\langle k, i, x_1, \dots, x_n, a\rangle, 2j+3\rangle),$$
$$R^{\mathfrak{M}}(\langle\langle k, i, x_1, \dots, x_n, a, b\rangle, 2n+2\rangle, \langle\langle k, i, x_1, \dots, x_n, a\rangle, 4k+1\rangle)$$

for every $k, i, j \in \omega,\ x_1, \dots, x_n, a, b \in HF(\mathfrak{B})$. Set

$$R^{\mathfrak{M}}(\langle\langle k, i, x_1, \dots, x_n, a, b\rangle, 2n+4\rangle, \langle\langle k, i, x_1, \dots, x_n, a\rangle, 4k+1\rangle)$$

if $\mathbb{HF}(\mathfrak{B}) \models \Psi(a, b, x_1, \dots, x_n, k)$ and

$$R^{\mathfrak{M}}(\langle\langle k, i, x_1, \dots, x_n, a, b\rangle, 2n+4\rangle, \langle\langle k, i, x_1, \dots, x_n, a\rangle, 4k+3\rangle)$$

otherwise. Finally, define $x\theta y$ iff there is a z such that $R^{\mathfrak{M}}(x, z)$ and $R^{\mathfrak{M}}(y, z)$.

3) By Theorem 1 from [Stukachev 2009] there is a structure $\mathfrak{B}$ such that $J(\emptyset) \oplus \mathcal{F} \equiv_\Sigma \mathfrak{J}(\mathfrak{B})$. Since $\mathcal{F} \leqslant_\Sigma \mathfrak{J}(\mathfrak{B})$ we have $\mathcal{E}(\mathcal{F}) \leqslant_\Sigma \mathfrak{B}$. Therefore, $\mathfrak{J}(\mathcal{E}(\mathcal{F})) \leqslant_\Sigma \mathfrak{J}(\mathfrak{B}) \leqslant_\Sigma J(\emptyset) \oplus \mathcal{F}$. This ends the proof.

Corollary 7. *For every n-families $\mathcal{F}$ and $\mathcal{G}$*

1. $\mathcal{F} \leq_\Sigma \mathcal{G} \implies \mathcal{E}(\mathcal{F}) \leq_\Sigma \mathcal{E}(\mathcal{G})$;
2. $\mathcal{E}(\mathcal{F} \oplus \mathcal{G}) \equiv \mathcal{E}(\mathcal{F}) \oplus \mathcal{E}(\mathcal{G})$.

Proof. 1. Follows from $\mathcal{F} \leq_\Sigma \mathcal{G} \leq_\Sigma \mathcal{E}(\mathcal{G})$.

2. Follows from $\mathcal{E}(A \oplus B) = \mathcal{H}_1 \cup \{\{2x : x \in A \oplus B\}\} = \mathcal{H}_1 \cup \{\{4x\} : x \in A\} \cup \{\{4x+2\} : x \in B\}\} \equiv_\Sigma \{X \oplus Y : X \in \mathcal{E}(A)\ \&\ Y \in \mathcal{E}(B)\} = \mathcal{E}(A) \oplus \mathcal{E}(B)$.

By the definition of $\mathcal{E}(\cdot)$ the least double jump inversion $\mathcal{E}^2(\mathcal{F}) = \mathcal{E}(\mathcal{E}(\mathcal{F}))$ of an n-family $\mathcal{F}$ is an $(n+2)$-family. But we know from [Faizrahmanov and Kalimullin 2016 (b)] that under Turing reducibility of presentations of n-families the least double jump is an $(n+1)$-family. For example, for the case of 0-family

A the least double jump $\mathcal{E}^2(A)$ has the same Turing degrees of presentations of $\mathfrak{M}_{\mathcal{E}^2(A)}$ as the degrees of presentations of $\mathfrak{M}_{\mathcal{G}}$, where $\mathcal{G}$ is the 1-family

$$\mathcal{G} = \{F \subseteq \omega : F \text{ is finite}\} \cup \{\overline{\{x\}} : x \in A\}.$$

Below we show that for the case of Σ-reducibility we can not have an equivalence between $\mathcal{E}^2(\mathcal{F})$ and some $(n+1)$-family even for $n = 0$.

Theorem 8. *For a set A we have*

$$\mathcal{J}(\mathcal{G}) \leq_\Sigma J(\emptyset) \oplus \mathcal{E}(A) \implies \mathcal{J}(\mathcal{G}) \leq_\Sigma J(\emptyset)$$

and, therefore, for a set a set $A \notin \Sigma^0_3$ we have $\mathcal{J}(\mathcal{G}) \not\equiv_\Sigma J(\emptyset) \oplus \mathcal{E}(A)$ for every 1-family $\mathcal{G}$.

Proof. (Sketch) Let us look on the jump of $\mathcal{J}(\mathcal{G}) = \mathcal{J}(\mathfrak{M}_{\mathcal{G}})$ for 1-families $\mathcal{G}$. Due [Kalimullin and Puzarenko 2009] all Σ-predicates in $\mathfrak{M}_{\mathcal{G}}$ can be encoded in the sets

$$A_1 \oplus A_2 \oplus \cdots \oplus A_m \oplus E(\mathcal{G}),$$

where $A_i \in \mathcal{G}$ and the set $E(\mathcal{G}) = \{u : (\exists A \in \mathcal{G})\,[D_u \subseteq A]\}$ codes the $\exists$-theory of $\mathfrak{M}_{\mathcal{G}}$. But the family of jumps of these sets can not fully represent the jump of the whole $\mathcal{G}$ since we need to keep the information when a jump for a tuple $A_1, \dots, A_m$ is an extension of the jump for a tuple $A_1, \dots, A_m, A_{m+1}$. It is more easily to identify $\mathcal{J}(\mathcal{G})$ up to $\equiv_\Sigma$ with the following structure $\mathfrak{J}(\mathcal{G})$ in the language $\sigma = \{r, I^1, R^2, \circ^2\}$.

Consider the families

$$\mathcal{K}(\mathcal{G}) = \{J(A) : A \in E(\mathcal{G}) \oplus \mathcal{G}\} \text{ and}$$

$$\mathcal{M}(\mathcal{G}) = \{J(A) : A \in \langle E(\mathcal{G}) \oplus \mathcal{G}\rangle_{\oplus}\},$$

where $\langle\cdot\rangle_{\oplus}$ is the $\oplus$-closure of a class of sets.

Fix a structures $\mathfrak{K} \cong \mathfrak{M}_{\mathcal{K}(\mathcal{G})}$ and $\mathfrak{M} \cong \mathfrak{M}_{\mathcal{M}(\mathcal{G})}$ such that

$$|\mathfrak{K}| \cap |\mathfrak{M}| = \{r^{\mathfrak{K}}\} = \{r^{\mathfrak{M}}\}.$$

Let $|\mathfrak{J}(\mathcal{G})| = |\mathfrak{K}| \cup |\mathfrak{M}|$, $r^{\mathfrak{J}(\mathcal{G})} = r^{\mathfrak{K}}$ and

$$I^{\mathfrak{J}(\mathcal{G})}(x) \iff I^{\mathfrak{K}}(x) \vee I^{\mathfrak{M}}(x),$$

$$R^{\mathfrak{J}(\mathcal{G})}(x,y) \iff R^{\mathfrak{K}}(x,y) \vee R^{\mathfrak{M}}(x,y),$$

$$I^{\mathfrak{J}(\mathcal{G})}(x) \iff I^{\mathfrak{K}}(x)$$

for all $x, y \in |\mathfrak{J}(\mathcal{G})|$. The binary operation $\circ$ is defined on $I^{\mathfrak{J}(\mathcal{G})}$ by such a way that $(I^{\mathfrak{J}(\mathcal{G})}, \circ)$ is a free non-associative algebra with the set of free generators $I^{\mathfrak{J}(\mathcal{G})} = I^{\mathfrak{K}}$ such that

$$J(X) = A_i \,\&\, J(Y) = A_j \implies J(X \oplus Y) = A_{i \circ j}.$$

Suppose that

$$\mathfrak{J}(\mathcal{G}) \leq_\Sigma J(\emptyset) \oplus \mathcal{E}(A) = \{J(\emptyset) \oplus \{2n, 2n+1\} : n \in \omega\} \cup \{J(\emptyset) \oplus \{2n\} : n \in A\}$$

by some Σ-formula Φ. For simplicity we assume that Φ has no parameters.

Note that the structure $\mathfrak{M}_{J(\emptyset)\oplus\mathcal{E}(A)}$ is bi-embeddable with $\mathfrak{M}_{J(\emptyset)\oplus\mathcal{H}_1} \leq_\Sigma J(\emptyset)$, where

$$J(\emptyset) \oplus \mathcal{H}_1 = \{J(\emptyset) \oplus \{2n, 2n+1\} : n \in \omega\}.$$

Moreover, they are *densely* bi-embeddable in the sense that for every finite substructure $\mathfrak{M}_0 \subseteq \mathfrak{M}_{J(\emptyset)\oplus\mathcal{E}(A)}$ there is a substructure $\mathfrak{M}_0 \subseteq \mathfrak{M}_1 \subseteq \mathfrak{M}_{J(\emptyset)\oplus\mathcal{E}(A)}$ such that $\mathfrak{M}_1 \cong \mathfrak{M}_{J(\emptyset)\oplus\mathcal{H}_1}$, and vice versa. Considering the same formula Φ in $\mathbb{HF}(\mathfrak{M}_{J(\emptyset)\oplus\mathcal{H}_1})$ we get a structure $\mathfrak{L}$ densely bi-embeddabe with $\mathfrak{J}(\mathcal{G})$. But $J(X) \subseteq J(Y)$ implies $J(X) = J(Y)$ so that this is possible only if $\mathfrak{J}(\mathcal{G}) \cong \mathfrak{L}$. Hence, $\mathfrak{J}(\mathcal{G}) \equiv_\Sigma \mathcal{J}(\mathcal{G}) \leq_\Sigma J(\emptyset)$.

In the case when Φ has parameters instead of $\mathcal{H}_1$ we should consider a 1-family in the form

$$\mathcal{H}_1 \cup \{n_1\} \cup \{n_2\} \cup \cdots \cup \{n_k\},$$

where the finite collection $n_1, \ldots, n_k \in A$ depends from these parameters to preserve the dense bi-embeddability property up to finitely many constants.

To prove the second part of the theorem suppose that $\mathcal{J}(\mathcal{G}) \equiv_\Sigma J(\emptyset) \oplus \mathcal{E}(A)$. Then by the first part $\mathcal{J}(\mathcal{G}) \leq_\Sigma \mathcal{J}(\emptyset)$. From another hand, by Theorem 6

$$A \leq_\Sigma \mathcal{J}(\mathcal{E}(A)) \leq_\Sigma \mathcal{J}^2(\mathcal{G}) \leq_\Sigma \mathcal{J}^2(\emptyset) \equiv_\Sigma J^2(\emptyset),$$

so that $A \in \Sigma^0_3$.

Since $\mathcal{J}(\mathcal{E}^2(A)) \equiv_\Sigma J(\emptyset) \oplus \mathcal{E}(A)$ by Theorem 6 we have also the following

Corollary 9. *For a set a set $A \notin \Sigma^0_3$ there is no 1-family $\mathcal{G}$ such that $\mathcal{G} \equiv_\Sigma \mathcal{E}^2(A)$, so that the least double jump inversion of a 0-family A can not be replaced by a 1-family.*

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