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INDECOMPOSABLE LINEAR ORDERINGS AND HYPERARITHMETIC ANALYSIS

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Abstract. A *statement of hyperarithmetic analysis* is a sentence of second order arithmetic S such that for every $Y \subseteq \omega$, the minimum ω-model containing Y of $\mathsf{RCA}_0+\mathsf{S}$ is $HYP(\mathrm{Y})$, the ω-model consisting of the sets hyperarithmetic in Y. We provide an example of a mathematical theorem which is a statement of hyperarithmetic analysis. This statement, that we call INDEC, is due to Jullien [Jul69]. To the author's knowledge, no other already published, purely mathematical statement has been found with this property until now. We also prove that, over RCA_0, INDEC is implied by Δ^1_1-CA_0 and implies ACA_0, but of course, neither ACA_0, nor $\mathsf{ACA}_0{}^+$ imply it.

We introduce five other statements of hyperarithmetic analysis and study the relations among them. Four of them are related to finitely-terminating games. The fifth one, related to iterations of the Turing jump, is strictly weaker than all the other statements that we study in this paper, as we prove using Steel's method of forcing with tagged trees.

1. Introduction

This paper is part of an ongoing project of analyzing the subsystems of second order arithmetic. This program is called Reverse Mathematics, and its main theme is the following: Given a theorem of ordinary mathematics, determine the weakest natural subsystem of second order arithmetic in which the theorem is provable. (The basic reference on Reverse Mathematics is Simpson's book [Sim99].) Surprisingly, it often happens that this question has a precise answer, and moreover, it is usually the case that the answer is one of five specific systems. This systems are RCA_0, WKL_0, ACA_0, ATR_0, and Π^1_1-ACA_0, listed in increasing order of proof-theoretic strength. (See [Sim99, p. 32]. We will describe the systems we will use in Subsection 1.6 below.) The system RCA_0, of Recursive Comprehension, is usually used as a base system; when we say that for some particular theorems the question above has a specific answer, we mean that, if RCA_0 is assumed, it can be proved that the theorem is equivalent to one of those five systems. RCA_0 resembles Computable Mathematics in the sense that, when working in RCA_0, all the sets we can assume exists are the ones that are computable from the ones we already know exist. It can be proved that the ω-models of RCA_0 are exactly the ones whose second order part is closed under Turing reduction and disjoint union, where the *disjoint union of two sets* $X, Y \subseteq \omega$ is the set $X \oplus Y = \{2n : n \in X\} \cup \{2n+1 : n \in Y\}$. The models of second order arithmetic whose first order part is the standard one $(\omega, 0, 1, +, \times)$, are called ω-models. We will identify these models with their second order parts. The

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system of Arithmetic Comprehension, ACA_0, has a similar behavior, but with respect to arithmetic reducibility. The ω-models of ACA_0 are exactly the ones whose second order part is closed under arithmetic reduction and disjoint union. As are the classes of recursive sets and of arithmetic sets, the class of hyperarithmetic sets is a very natural one and enjoys many closure properties. This is the class that will concern us in this paper. For more information on hyperarithmetic reductions, see Subsection 1.5 below.

We say that an ω-model is *hyperarithmetically closed* if it is closed under disjoint union and for every set $X, Y \subseteq \omega$, if X is hyperarithmetically reducible to Y and Y is in the model, then X is in the model too.

Definition 1.1. A system of axioms of second order arithmetic T is a *theory of hyperarithmetic analysis* if

- it holds in $HYP(\mathrm{Y})$ for every $Y \subseteq \omega$, where $HYP(\mathrm{Y})$ is the ω-model consisting of the sets hyperarithmetic in Y ; and
- all its ω-models are hyperarithmetically closed.

Note that this is equivalent to saying that every for every set $Y \subseteq \omega$, $HYP(\mathrm{Y})$ is the minimum ω-model of T which contains Y, and that every ω-model of T is closed under disjoint unions.

In [Ste78, Section 5], Steel defines "theories of hyperarithmetic analysis" as the ones which have $HYP{=}HYP(\emptyset)$ as their minimum ω-model. People were interested in these theories because they characterize the class HYP. Our definition is a relativized version of the previous one, and it characterizes not only HYP, but also the relation of hyperarithmetic reduction: When T is a theory of hyperarithmetic analysis, a set X is hyperarithmetically reducible to a set Y if and only if every ω-model of T which contains Y, also contains X.

The bad news is that there is no theory whose ω-models are exactly the ones that are hyperarithmetically closed. This follows from a more general result of Van Wesep [Van77, 2.2.2]: For every theory T whose ω-models are all hyperarithmetically closed, there is another theory T' whose models are all also hyperarithmetically closed and which has more ω-models than T does. So, there might not be a natural theory of hyperarithmetic analysis. Indeed, there are many. Examples of known theories of hyperarithmetic analysis are the following schemes: Σ^1_1-dependent choice (Σ^1_1-DC_0), Σ^1_1-choice (Σ^1_1-AC_0), Δ^1_1-comprehension (Δ^1_1-CA_0), and weak-Σ^1_1-choice (weak-Σ^1_1-AC_0). The unrelativized versions of these results were proved by Harrison [Har68], Kreisel [Kre62], [Kle59] and [Sim99, Theorem VIII.4.16]. (See Subsection 1.6 below for definitions of these statements.) As listed, these statements go from strongest to weakest, they all imply ACA_0, and, except for Σ^1_1-DC_0, they are implied by ATR_0 (see [Sim99, VIII.3 and VIII.4]). Moreover, the implications Σ^1_1-$\mathsf{DC}_0 \implies \Sigma^1_1$-$\mathsf{AC}_0$, Σ^1_1-$\mathsf{AC}_0 \implies \Delta^1_1$-$\mathsf{CA}_0$, and Δ^1_1-$\mathsf{CA}_0 \implies$ weak-Σ^1_1-AC_0 can not be reversed as proved by Friedman [Fri67], Steel [Ste78] and van Wesep [Van77], respectively.

We say that a sentence S is a *sentence of hyperarithmetic analysis* if $\mathsf{RCA}_0{+}\mathsf{S}$ is a theory of hyperarithmetic analysis. In [Fri75, Section II], Friedman mentions two sentences related to hyperarithmetic analysis. These sentences, ABW (arithmetic Bolzano-Weierstrass) and SL (sequential limit systems), use the concept of arithmetic set of reals, which is not used outside logic. Another previously known sentence of hyperarithmetic analysis is Game-AC studied by Van Wesep [Van77].

He studied it in a more general context than second order arithmetic. But if we restrict it to second order arithmetic, it essentially says that if we have a sequence of open games such that player *II* has a winning strategy in each of them, then there exists a sequence of strategies for all of them. He proved that, when restricted to second order arithmetic, Game-AC is equivalent to Σ^1_1-AC_0. (An open game is a game like the ones we describe in Subsection 1.2 below, with the difference that it might last for ω many steps, and if it does, player *II* is the winner.)

We will introduce five new statements of hyperarithmetic analysis: CDG-CA, CDG-AC, DG-CA, DG-ACand JI. The first four statements are related to finitely terminating games. These are perfect-information games between two players, where at each turn a player might have infinitely many possible moves but every run of the game ends in finitely many steps. All these four statements are easily stated, and obviously true. So obviously true that they would not even be considered theorems, although they are proof-theoretically strong. The same happens with the statement Game-AC. The reason is that all these statements about games have the form either of comprehension axioms or of choice axioms. The last statement JI is the weakest statement of hyperarithmetic analysis we study and has to do with the iteration of the Turing jump along ordinals. We use Steel's method of forcing with tagged trees to prove that it is strictly weaker than the other statements. This is the same method used by Steel [Ste78] and by Van Wesep [Van77] to prove that the implications Σ^1_1-$\mathsf{AC}_0 \implies \Delta^1_1$-$\mathsf{CA}_0$, and Δ^1_1-$\mathsf{CA}_0 \implies$ weak-Σ^1_1-AC_0 cannot be reversed.

However, to the author's knowledge, no previously published mathematical theorem, which does not mention concepts from logic, has been proved a statement of hyperarithmetic analysis. In this paper we present an example of such a theorem. This theorem, that we call INDEC, was first proved by Pierre Jullien in his Ph.D. thesis [Jul69, Theorem IV.3.3]. INDEC is published in English in, for example, [Fra00, 6.3.4(3)] and [Ros82, Lemma 10.3]. We prove not only that INDEC is a statement of hyperarithmetic analysis, but also that, over RCA_0, INDEC is implied by Δ^1_1-CA_0 and implies ACA_0. Note that since *HYP* is the minimum ω-model of INDEC, neither ACA_0, nor $\mathsf{ACA}_0{}^+$ can imply it.

Another interesting fact about INDEC is that is incomparable over ACA_0 to other natural statements of mathematics. This is probably the first example of previously published purely mathematical statements which are incomparable and are between ACA_0 and ATR_0. The statements we have in mind are the following: The existence of elementary equivalence invariants for Boolean Algebras, and Ramsey Theorem. The former statement was studied by Shore [Sho06]. He first analyzed how to work with the statement in second order arithmetic and then proved that it is equivalent to $\mathsf{ACA}_0{}^+$ over RCA_0. ($\mathsf{ACA}_0{}^+$ is equivalent to ACA_0 plus the sentence $\forall X(X^{(\omega)}$ exists), where $X^{(\omega)}$ is the ωth Turing jump of X.) The latter statement, Ramsey's Theorem, has been extensively studied in the context of reverse mathematics (see [Sim99, III.7], [CJS01], or [Mil04, Chapter 7]). It is known that it slightly stronger than ACA_0. The reason why these statements are incomparable with INDEC is the following one. Barwise and Schlipf [BS75] proved that Σ^1_1-AC_0 (and hence also Δ^1_1-CA_0 and INDEC) is conservative over ACA_0 for Π^1_2 formulas. In other words, any Π^1_2 formula sentence which is provable in Σ^1_1-AC_0 is already provable in ACA_0. Then, since $\mathsf{ACA}_0{}^+$ can be axiomatized by a Π^1_2 sentece over ACA_0, and is strictly stonger than ACA_0, it is not implied by Σ^1_1-AC_0, and hence it is not implied by RCA_0+INDEC either. The same argument is true about Ramsey's theorem.

We now formally introduce all these statements of hyperarithmetic analysis.

1.1. **Indecomposability Statement.** We start with the most natural of all these statements. As we said in the introduction, INDEC is due to Jullien [Jul69].

Definition 1.2. Given a linear ordering $\mathcal{A} = \langle A, \leq \rangle$, a *cut in* $\mathcal{A}$ is a pair of sets $\langle L, R \rangle$ such that $L = A \smallsetminus R$ is an initial segment of $\mathcal{A}$. We say that $\mathcal{A}$ is *indecomposable* if for every cut $\langle L, R \rangle$, $\mathcal{A}$ embeds either into L or into R. (Here we are thinking of L and R as sub-orderings of $\mathcal{A}$.) We say that $\mathcal{A}$ is *indecomposable to the right* if for every cut $\langle L, R \rangle$ with $R \neq \emptyset$, we have that $\mathcal{A}$ embeds in R. Analogously we define *indecomposable to the left*. A linear ordering is *scattered* if η, the order type of the rational numbers, does not embed in it.

Statement 1.3. We let INDEC be the statement

> Every scattered indecomposable linear ordering is either indecomposable to the right or indecomposable to the left.

Indecomposable linear orderings are very useful when studying properties of linear orderings. Every scattered linear ordering can be written as a finite sum of indecomposable linear orderings, so they are in some sense the building blocks for the class of scattered linear orderings. Countable indecomposable linear orderings can be written as ω- or ω^*-sums of smaller indecomposable linear orderings. These facts are due to Laver [Lav71]; see also [Ros82, Chapter 10]. In the same paper, Laver proved Fraïssé's conjecture which says that there is no infinite descending sequence or infinite antichain in the quasi-ordering formed by the countable linear orderings ordered by embeddablity. These structure theorems together allow us to prove properties about linear orderings by transfinite induction. For example, Jullien [Jul69, Chapter V] used these structure theorems for scattered linear orderings to classify the countable extendible linear orderings. (A linear ordering is *extendible* if every countable partial ordering which does not embed it has a linearization which does not embed it either.) The author used them to prove that every hyperarithmetic linear ordering is equimorphic to a recursive one in [Mon05b], and to analyze the proof theoretic strength of Jullien's theorem and Fraïssé's conjecture in [Mon06].

We prove in section 2 that, over RCA_0, INDEC is implied by Δ^1_1-CA_0 and that it implies ACA_0. The former proof is not very complicated. The latter one is more interesting and has some ideas that will be used in Section 3 to prove that INDEC is a theory of hyperarithmetic analysis. To prove that the ω-models of INDEC are hyperarithmetically closed, we start by considering an ω-model $\mathcal{M}$ of INDEC. Of course, we think of $\mathcal{M}$ as set of subsets of ω. Then, we prove that for every computable increasing sequence of ordinals $\{\alpha_n\}_{n\in\omega}$, converging to a computable ordinal α, we have that if $(\forall n) 0^{(\alpha_n)} \in \mathcal{M}$, then $0^{(\alpha)} \in \mathcal{M}$. To prove this we use Ash and Knight's machinery to construct a specific linear ordering such that when we apply INDEC to it, we can deduce that $0^{(\alpha)} \in \mathcal{M}$. Then we relativize and use effective transfinite induction to prove that for every set $X \in \mathcal{M}$ and every X-computable ordinal $\alpha \in \mathcal{M}$, $X^{(\alpha)} \in \mathcal{M}$. When we refer to Ash and Knight's machinery we refer to the results that Ash and Knight derived from Ash's $0^{(\alpha)}$-priority arguments (see [AK00]).

1.2. **Game statements.** Before introducing the game statements, let us quickly review our notation for trees. We write $\mathcal{N}^{<\omega}$ for the set of finite strings of natural numbers, and $2^{<\omega}$ for the set of finite strings of zeros and ones, ordered by inclusion. A *tree* is a downward closed subset of $\mathcal{N}^{<\omega}$ and a *binary tree* is a downward subset of $2^{<\omega}$. Given a tree T and $\sigma \in T$, we let $T_\sigma = \{\tau : \sigma^\frown\tau \in T\}$, where $\sigma^\frown\tau$ is the string obtained by concatenating σ and τ. We use $\emptyset$ for the empty string. Given a string σ, we let $|\sigma|$ be its length, $\sigma\restriction n$ be the initial substring of σ of length n, and $\sigma^- = \sigma\restriction(|\sigma|-1)$.

Definition 1.4. To each well founded tree T, we associate a game $G(T)$ which is played as follows. Player I starts by playing a number $a_0 \in \mathcal{N}$ such that $\langle a_0\rangle \in T$. Then player II plays $a_1 \in \mathcal{N}$ such that $\langle a_0, a_1\rangle \in T$, and then player I plays $a_2 \in \mathcal{N}$ such that $\langle a_0, a_1, a_2\rangle \in T$. They continue like this until they get stuck. The first one who cannot play *loses*. Equivalently, the first one that reaches an end node of T *wins*. We call the sequence $\langle a_0, ..., a_k\rangle$ obtained at the end of the game, a *run* of the game, and any sequence obtained any time along the game, a *partial run*. Note that since T is well founded, the game cannot last forever. We call the games which are of the form $G(T)$ *finitely terminating games*.

Remark 1.5. Finitely terminating games are in one to one correspondence with clopen games. A *clopen game* is played over the full tree $\mathcal{N}^{<\omega}$ and runs of the game go for ω many steps. At the end of time, the players are left with an infinite sequence $X \in \mathcal{N}^\omega$, and player I *wins* if that sequence belongs to a previously chosen clopen set $\mathcal{A} \subset \mathcal{N}^\omega$. Otherwise II wins. This defines the game $G(\mathcal{A})$. A *clopen* set is a set which is closed and open. Every clopen set $\mathcal{A}$ is determined by a well founded tree $T_{\mathcal{A}}$ and a subset A of the set of end nodes of T. It is determined in the sense that $X \in \mathcal{A}$ if and only if the initial segment of X which is an end node of T belongs to A. It is not hard to see that for every clopen game $G(\mathcal{A})$, using $T_{\mathcal{A}}$ and A, one can construct a well-founded tree T such that $G(\mathcal{A})$ and $G(T)$ are in some sense equivalent. Also, given a well-founded tree T, it is not hard to construct a clopen set $\mathcal{A}$ that will induce an equivalent game.

Definition 1.6. Let $T_I = \{\sigma \in T : |\sigma| \text{ is even}\}$ and $T_{\mathit{II}} = \{\sigma \in T : |\sigma| \text{ is odd}\}$. So, T_I is the set of partial runs σ of $G(T)$ such that if σ has been played so far in a game, then it is I's turn to play. Similarly with T_{II}. Let P be either I or II. A *strategy for* P in a tree game $G(T)$ is a function $s\colon T_P \to \mathcal{N}$. We say that a partial run $\sigma \in T$ *follows a strategy* s if for every $\tau \subset \sigma$, $\tau \in T_P \implies \sigma(|\tau|) = s(\tau)$. A strategy s for P is a *winning strategy* if for every run σ of T which follows s, $\sigma \notin T_P$. In other words, s is a winning strategy for P if whenever P plays following s, he is ensured to win despite what the other player plays. A game $G(T)$ is *determined* if there is a winning strategy for one of the two players. We say that a game is *completely determined* if there is a map $d\colon T \to \{\mathtt{W}, \mathtt{L}\}$ such that for every $\sigma \in T$, if $d(s) = \mathtt{W}$, then I has a winning strategy in the game T_σ, and if $d(s) = \mathtt{L}$, then II has a winning strategy in the game $G(T_\sigma)$. We call such a d, a *winning function* for $G(T)$. We cal a tree T *determined (completely determined)* if $G(T)$ is determined (completely determined).

It is clear that I and II cannot both have winning strategies, so winning functions, if they exists, have to be unique. (This can be proved in RCA_0.) On the other hand, winning strategies do not need to be unique. This is because the value of a strategy at a node which does not follow it is not relevant at all.

Theorem 1.7. *The following are equivalent over* RCA_0.

(1) ATR_0;
(2) *Every finitely terminating game is determined;*
(3) *Every finitely terminating game is completely determined.*

Proof. The equivalence between (1) and (2) is proved in Steel's thesis [Ste76]. (See [Sim99, Theorem V.8.7]). The fact that (1) implies (3) follows from the uniformity in the proof of (1) $\Longrightarrow$ (3). It is clear that (3) implies (2). □

Now, we introduce four statements about finitely terminating games that we will later prove are statements of hyperarithmetic analysis. But, first, we need the following definition.

Definition 1.8. Given a sequence $\{T_n : n \in \mathcal{N}\}$, we let $\sum_n T_n$ be the tree S such that for each $n \in \mathcal{N}$, $S_{\langle n\rangle} = T_n$. So, we can think of the game $G(\sum_n T_n)$ as a game in which player I starts by choosing a game from $\{G(T_n) : n \in \mathcal{N}\}$, and then players I and II play it starting with player II. Whoever wins the chosen game, *wins* $G(\sum_n T_n)$.

Given a game G, let G^* be the game that is played exactly as G but players I and II are interchanged. So, for instance, if $G = G(T)$, we can assume that $G^* = G(T^*)$, where $T^*_n = \{0^\frown\sigma : \sigma \in T\}$.

Statement 1.9.

- $\mathsf{CDG\text{-}CA}$: Given a sequence $\{T_n : n \in \mathcal{N}\}$ of completely determined trees, there exists a set X such that $n \in X$ iff I has a winning strategy for $G(T_n)$.
- $\mathsf{CDG\text{-}AC}$: Given a sequence $\{T_n : n \in \mathcal{N}\}$ of completely determined trees, $\sum_n T_n$ is also completely determined.
- $\mathsf{DG\text{-}CA}$: Given a sequence $\{T_n : n \in \mathcal{N}\}$ of determined trees, there exists a set X such that $n \in X$ iff I has a winning strategy for T_n.
- $\mathsf{DG\text{-}AC}$: Given a sequence $\{T_n : n \in \mathcal{N}\}$ of determined trees, $\sum_n T_n$ is also determined.

Remark 1.10. The statement $\mathsf{DG\text{-}AC}$is equivalent to a clopen-game version of the statement Game-AC mentioned in the introduction.

1.3. **The Jump Iteration statement.** This statement will be useful when trying to prove that a sentence is one of hyperarithmetic analysis.

Statement 1.11. The *Jump Iteration* statement, JI, is the following:

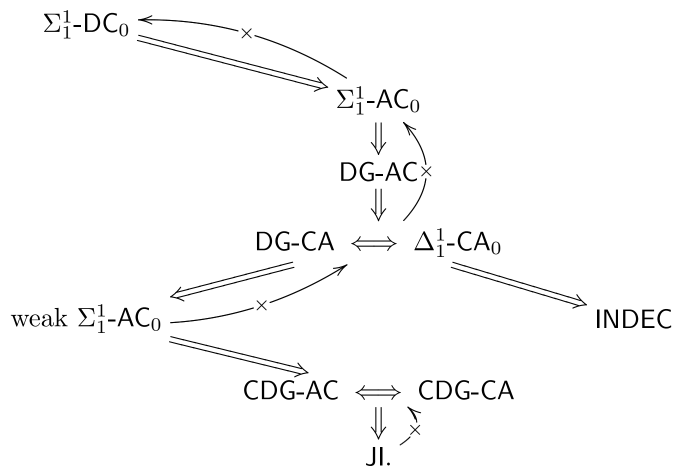
For every set X and every ordinal α,
if, for every $\beta < \alpha$, $X^{(\beta)}$ exists, then $X^{(\alpha)}$ exists.

Let us prove that the ω-models of RCA_0+JI are hyperarithmetically closed. Consider a ω-model $\mathcal{M}$ of RCA_0+JI. If it were not hyperarithmetically closed, there would be a set $X \in \mathcal{M}$ and a least X-computable ordinal α such that $X^{(\alpha)} \notin \mathcal{M}$. But this contradicts JI. So JI is a sentence of hyperarithmetic analysis.

A similar argument is used to prove that the ω-models of INDEC are hyperarithmetically closed in Section 3. We believe that by formalizing the ideas in that Section one can prove that $\mathsf{INDEC} \Longrightarrow \mathsf{JI}$.

1.4. **Summary of results.** The following theorem contains all the implications we know how to prove between the different statements of hyperarithmetic analysis that we study in this paper.

Theorem 1.12. *All the theories and statements mentioned in the diagram below are ones of hyperarithmetic analysis. The implications and the non-implications in the diagram hold over RCA_0. Moreover, all the non-implications are witnessed by ω-models.*



Since all the statements in the diagram follow from Σ^1_1-DC_0, for every $Y \subseteq \omega$, $HYP(Y)$ is a model of them [Sim99, Theorem VIII.4.16]. We will prove that the ω-models of INDEC are hyperarithmetically closed in Section 3. Since all the other statements imply JI, every ω-model of each of them is hyperarithmetically closed.

The part of the diagram which only mentions Σ^1_1-DC_0, Σ^1_1-AC_0, Δ^1_1-CA_0 and weak-Σ^1_1-AC_0, was already mentioned in the introduction. That Δ^1_1-CA_0 implies INDEC will be proved in Section 2. All the other implications are proved in Section 4. The fact that JI does not imply $\mathsf{CDG\text{-}CA}$ is proved in Section 5.

Many arrows are missing from the diagram. For instance, we do not know whether $\mathsf{DG\text{-}AC}$ is strictly in between Σ^1_1-AC_0 and Δ^1_1-CA_0, or is equivalent to one of them. We would also like to know more about how INDEC relates to the other statements. We conjecture that it implies $\mathsf{CDG\text{-}CA}$, but we have not even proved that it implies JI. Another interesting question is whether $\mathsf{CDG\text{-}CA}$ and $\mathsf{CDG\text{-}AC}$ are equivalent to weak-Σ^1_1-AC_0.

1.5. **Hyperarithmetic Theory.** Standard references for Hyperarithmetic Theory are [AK00] and [Sac90].

Let $\mathcal{L} = \langle L, \leq_L \rangle$ be a presentation of a linear ordering (i.e., $\mathcal{L}$ is a linear ordering whose domain L is a subset of ω) which has a least element 0. Given $X, Y \subseteq \omega$, we say that Y is an *$H(X, \mathcal{L})$-set* if $Y^{[0]} = X$ and for every $l \in L \setminus \{0\}$

$$Y^{[l]} = \bigoplus_{k <_L l} (Y^{[k]})'$$

where $Y^{[j]} = \{n : \langle j, n \rangle \in Y\}$ and $\bigoplus_{k \in A} B_k = \{\langle k, n \rangle : k \in A, n \in B_k\}$. When $\mathcal{L}$ is an ordinal it is not hard to prove by transfinite induction that there exists a unique $H(X, \mathcal{L})$-set. We denote that set by $X^{(\mathcal{L})}$. But if we consider another isomorphic presentation of $\mathcal{L}$, even the Turing degree of $X^{(\mathcal{L})}$ may change. Although, there are some cases when we know it does not change. Given an countable ordinal α and a set X, we say that α is an X-computable ordinal if there is a presentation of α

recursive in X. When α is an X-computable ordinal, all the $H(X,\mathcal{L})$-sets, where $\mathcal{L}$ is an X-computable presentations of α, are Turing equivalent; this result is due to Spector [Spe55]. The least non-X-computable ordinal is denoted by ω_1^X. Note that the set of X-computable ordinals is closed downward. We use ω_1^{CK} to denote $\omega_1^{\emptyset}$, where CK stands for Church-Kleene.

Theorem 1.13. [Kle55, Ash86] *Given sets $X, Y \subseteq \omega$, the following are equivalent*

1. $(\exists \alpha < \omega_1^Y) X \leq_T Y^{(\alpha)}$, *where $\leq_T$ means "is computable in".*
2. $X \in \Delta_1^1(Y)$, *that is, there exists Σ_1^1 formulas ψ and φ such that* $(\forall n) n \in X \iff \psi(n, Y) \iff \neg\varphi(n, Y)$.
3. *There is a Y-computable infinitary formula φ such that* $X = \{n : \varphi(n)\}$.

If $\omega_1^Y = \omega_1^{CK}$ we also have:

4. *there is a computable infinitary formula φ such that* $X = \{n : \varphi(n, Y)\}$.

(A computable infinitary formula is a formula where infinite disjunctions and infinite conjunctions are allowed, so long as they are taken over computably enumerable sets of computable infinitary formulas. See [AK00, Chapter 7] for more information on these formulas.)

Definition 1.14. When sets $X, Y \subseteq \omega$ satisfy any of the first three conditions in the theorem above we say that X *is hyperarithmetically reducible to* Y and write $X \leq_H Y$. We let $HYP(Y) = \{X \subseteq \omega : X \leq_H Y\}$ and $HYP = HYP(\emptyset)$.

1.6. **Subsystems of second order arithmetic.** We only review the subsystems of second order arithmetic that we will be using. We refer the reader to [Sim99] for more information. We use the same notation as in [Sim99], and, for instance, we use capital letters for set variables and lower case letters for number variables.

As our basic system we will use RCA_0. It consists of the axioms for semi-rings plus Δ_1^0-comprehension and Σ_1^0-induction. ACA_0 consists of RCA_0 plus the axiom scheme of arithmetic comprehension. Δ_1^1-CA_0 also includes Δ_1^1-comprehension. It is known that Δ_1^1-CA_0 is a theory of hyperarithmetic analysis.

Σ_1^1-dependent choice is the following scheme, where φ ranges over the Σ_1^1 formulas:

$$\forall Y \exists Z(\varphi(Y, Z)) \implies \exists X \forall n(\varphi(X^{[n]}, X^{[n+1]})).$$

Σ_1^1-choice is the following scheme, where φ ranges over the Σ_1^1 formulas:

$$\forall n \exists X(\varphi(n, X)) \implies \exists X \forall n(\varphi(n, X^{[n]})).$$

The scheme weak-Σ_1^1-choice is the following, where φ ranges over the arithmetic formulas:

$$\forall n \exists ! X(\varphi(n, X)) \implies \exists X \forall n(\varphi(n, X^{[n]})),$$

where $\exists !$ stands for "there exists a unique". Together with RCA_0 they form the systems Σ_1^1-DC_0, Σ_1^1-AC_0 and weak-Σ_1^1-AC_0 respectively.

1.7. **Linear orderings.** We use $\mathcal{N}$ for the set of all the natural numbers and ω for the linear ordering $\omega = \langle \mathcal{N}, \leq_{\mathcal{N}} \rangle$. (We also use ω, at the meta-level, as the standard first order model of arithmetic.)

A linear ordering $\mathcal{A} = \langle A, \leq_{\mathcal{A}} \rangle$ is said to be *recursive* if both A and $\leq_{\mathcal{A}}$ are recursive. Some recursive linear orderings we will be dealing with are: $\mathbf{1}$, the linear ordering with one element; $\mathbf{m}$, the linear ordering with m elements; ω, the ordering of the natural numbers; ζ, the ordering of the integers; η, the ordering of the rationals; ω^n, the ordering of the n-tuples of natural numbers ordered lexicographically; and ω^{n*}, the reverse linear ordering of ω^n.

Now we define some recursive operations on linear orderings. Let $\mathcal{A} = \langle A, \leq_{\mathcal{A}} \rangle$ be a linear ordering. Given $a \in A$, define $\mathcal{A}_{(<a)}$ to be the restriction of $\mathcal{A}$ to $\{x \in A : x <_{\mathcal{A}} a\}$. Analogously define $\mathcal{A}_{(\leq a)}$, $\mathcal{A}_{(>a)}$, and $\mathcal{A}_{(\geq a)}$. Given $a, b \in A$, let $[a, b]_{\mathcal{A}}$ be the restriction of $\mathcal{A}$ to $\{x \in A : a \leq_{\mathcal{A}} x \leq_{\mathcal{A}} b\}$. Let $(a, b)_{\mathcal{A}}$ be the restriction of $\mathcal{A}$ to $\{x \in A : a <_{\mathcal{A}} x <_{\mathcal{A}} b\}$. The *reverse* linear ordering of $\mathcal{A} = \langle A, \leq_{\mathcal{A}} \rangle$ is $\mathcal{A}^* = \langle A, \geq_{\mathcal{A}} \rangle$. Let $\mathcal{B} = \langle B, \leq_{\mathcal{B}} \rangle$ be another linear ordering. The *product*, $\mathcal{A} \cdot \mathcal{B}$, of two linear orderings $\mathcal{A}$ and $\mathcal{B}$ is obtained by substituting a copy of $\mathcal{A}$ for each element of $\mathcal{B}$. That is: $\mathcal{A} \cdot \mathcal{B} = \langle A \times B, \leq_{\mathcal{A} \cdot \mathcal{B}} \rangle$ where $\langle x, y \rangle \leq_{\mathcal{A} \cdot \mathcal{B}} \langle x', y' \rangle$ iff $y <_{\mathcal{B}} y'$ or $y = y'$ and $x \leq_{\mathcal{A}} x'$. The *sum*, $\sum_{i \in \mathcal{A}} \mathcal{B}_i$, of a set of linear orderings $\{\mathcal{B}_i\}_{i \in A}$ indexed by another linear ordering $\mathcal{A}$, is constructed by substituting a copy of $\mathcal{B}_i$ for each element $i \in A$. So, for example, $\mathcal{A} \cdot \mathcal{B} = \sum_{i \in \mathcal{B}} \mathcal{A}$. When $\mathcal{A} = \mathbf{m}$, we sometimes write $\mathcal{B}_0 + ... + \mathcal{B}_{m-1}$ or $\sum_{i=0}^{m-1} \mathcal{B}_i$ instead of $\sum_{i \in \mathbf{m}} \mathcal{B}_i$. Let $\mathcal{A}^{\mathcal{B}}$ be the linear ordering whose domain consist of finite strings $\sigma = \langle \langle a_0, b_0 \rangle, ..., \langle a_k, b_k \rangle \rangle \in (\mathcal{A} \times \mathcal{B})^{<\omega}$ such that $b_0 \leq_{\mathcal{B}} b_1 \leq_{\mathcal{B}} ... \leq_{\mathcal{B}} b_k$. Given $\sigma, \tau \in \mathcal{A}^{\mathcal{B}}$, let $\sigma \leq_{\mathcal{A}^{\mathcal{B}}} \tau$ if either $\sigma \subseteq \tau$ or for the least i such that $\sigma(i) \neq \tau(i)$ we have $\sigma(i) \leq_{\mathcal{A} \cdot \mathcal{B}} \tau(i)$. Observe that $\mathcal{A}^{(\mathcal{B}+\mathcal{C})} \cong \mathcal{A}^{B} \cdot \mathcal{A}^{C}$. Other recursive linear orderings that we will use are $\omega^\omega = \sum_{n \in \omega} \omega^n$ and $\omega^{\omega *} = (\omega^\omega)^*$.

If $\mathcal{A}$ can be embedded in $\mathcal{B}$, we write $\mathcal{A} \preccurlyeq \mathcal{B}$. $\mathcal{A}$ is *scattered* if $\eta \not\preccurlyeq \mathcal{A}$.

Lemma 1.15. *(RCA_0) If $\mathcal{Z}$ is a scattered linear ordering and $\{\mathcal{B}_z : z \in \mathcal{Z}\}$ is a family of scattered linear orderings, then $\sum_{z \in \mathcal{Z}} \mathcal{B}_z$ is also scattered. In particular, the product of scattered linear orderings is scattered.*

Proof. Suppose that f is an embedding $\eta \hookrightarrow \sum_{z \in \mathcal{Z}} \mathcal{B}_z$. If for every $q \in \eta$, $f(q)$ belongs to a different summand $\mathcal{B}_{z_q}$, then the map $q \mapsto z_q$ would be an embedding of η into $\mathcal{Z}$. So, there has to be a pair $p, q \in \eta$ and a $z \in \mathcal{Z}$, such that both $f(p)$ and $f(q)$ are in $\mathcal{B}_z$. But then $\eta \preccurlyeq [p, q]_\eta \preccurlyeq \mathcal{B}_z$. □

Trees will be an important tool in this paper. We started introducing basic notation for trees at the beginning of Subsection 1.2. We can linearly order the nodes of a tree in various ways. One is the *Kleene-Brouwer* ordering of $\mathcal{N}^{<\omega}$ defined as follows:

$$\sigma \leq_{KB} \tau \iff \sigma \supseteq \tau \vee \exists i (\sigma(i) \leq \tau(i) \ \& \ \forall j < i (\sigma(j) = \tau(j))).$$

Given a tree $T \subseteq \mathcal{N}^{<\omega}$, we let $KB(T)$ be the Kleene-Brouwer ordering restricted to T. ACA_0 can prove that if a tree T is well-founded, then $KB(T)$ is well-ordered [Sim99, Lemma V.1.3] (see [Hir94] for the reversal). Even though RCA_0 cannot prove this, it can prove the following.

Lemma 1.16. *(RCA_0) If T is well founded, then $KB(T)$ is scattered.*

Proof. Suppose that f is an embedding of η into $KB(T)$. By recursion, we construct two sequences $\langle p_n : n \in \mathcal{N} \rangle$, and $\langle q_n : n \in \mathcal{N} \rangle$ of elements of η such that for each n, $p_n \leq_\eta p_{n+1} <_\eta q_{n+1} \leq_\eta q_n$, $|f(p_n)| \geq n$ and $f(p_n) \upharpoonright n = f(q_n) \upharpoonright n$. Just define p_{n+1} and q_{n+1} as the least pair (in some enumeration of η^2) which satisfies the conditions above. Such a pair has to exist because f restricted to $(p_n, q_n)_\eta \cong \eta$ is a map into $KB(T_{f(p_n) \upharpoonright n}) \cong \sum_{m \in \omega} KB(T_{(f(p_n) \upharpoonright n)^\frown m}) + \mathbf{1}$. Finally, $\cup_{n \in \mathcal{N}} f(p_n) \upharpoonright n$ is a path though T contradicting its well-foundedness. □

On $2^{<\omega}$ we also have the *Left-to-right ordering*, $\leq_{LR}$. It coincides with the Kleene-Brouwer on incompatible strings, but when $\sigma \subset \tau$ we let $\sigma \leq_{LR} \tau$ if $\tau(|\sigma|) = 1$ and $\sigma \geq_{LR} \tau$ if $\tau(|\sigma|) = 0$. Observe that $\langle 2^{<\omega}, \leq_{LR}\rangle$ has order type η.

Lemma 1.17. *(RCA_0) If $\mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A}$, then $\eta \preccurlyeq \mathcal{A}$.*

Proof. Assume $\mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A}$. Observe that then $\mathcal{A} + \mathbf{1} + \mathcal{A} \preccurlyeq \mathcal{A} + \mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A}$. So, there exist two embeddings $f_0, f_1 \colon A \hookrightarrow A$ and an $a \in \mathcal{A}$ such that $\forall x, y \in A(f_0(x) <_A a <_A f_1(y))$. Now, given $\sigma \in 2^{<\omega}$ define

$$f(\sigma) = f_{\sigma(0)}(f_{\sigma(1)}(...(f_{\sigma(|\sigma|-1)}(a))...)).$$

f is an embedding of $\langle 2^{<\omega}, \leq_{LR}\rangle \cong \eta$ into $\mathcal{A}$. $\square$

2. Between ACA_0 and Δ^1_1-CA_0

In this section we prove that INDEC implies ACA_0 and is implied by Δ^1_1-CA_0 over RCA_0. From the latter of these implications we get that $HYP(Y)$ is a model of INDEC for every $Y \subseteq \omega$. That ω-models of INDEC are closed under hyperarithmetic reduction will be proved in the next section.

Definition 2.1. We say that a linear ordering $\mathcal{A}$ is *weakly indecomposable* if for every $a \in \mathcal{A}$, either $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leq a)}$ or $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$.

Note that an indecomposable linear ordering is weakly indecomposable. Also note that, by Lemma 1.17, if $\mathcal{A}$ is scattered then for no $a \in \mathcal{A}$ do we have both $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leq a)}$ and $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$.

Theorem 2.2. *The following are equivalent over RCA_0, and they are both implied by Δ^1_1-CA_0.*

(1) *INDEC*
(2) *If $\mathcal{A}$ is a scattered, weakly indecomposable linear ordering, then there exists a cut $\langle L, R\rangle$ of A such that*

$$L = \{a \in \mathcal{A} : \mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}\} \text{ and } R = \{a \in \mathcal{A} : \mathcal{A} \preccurlyeq \mathcal{A}_{(\leq a)}\} \tag{2.1}$$

We call the cut $\langle L, R\rangle$ satisfying (2.1), the *middle cut* of $\mathcal{A}$.

Proof. We first prove that (1) and (2) are equivalent.

To prove (2) from INDEC, consider a scattered, weakly indecomposable linear ordering $\mathcal{A}$. If $\mathcal{A}$ is indecomposable, then, by INDEC, it is either indecomposable to the right or to the left. In the former case we would have that $\langle L, R\rangle = \langle A, \emptyset\rangle$ satisfies (2.1) and in the latter case $\langle L, R\rangle = \langle \emptyset, A\rangle$ satisfies (2.1). In both cases a cut $\langle L, R\rangle$ as in (2.1) exists. Suppose now that $\mathcal{A}$ is not indecomposable and let $\langle L, R\rangle$ be a cut such that neither $\mathcal{A} \preccurlyeq L$ nor $\mathcal{A} \preccurlyeq R$. We claim that $\langle L, R\rangle$ is as in (2.1). We have that for every $a \in \mathcal{A}$, either $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leq a)}$ or $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$. If $a \in L$, then, since $\mathcal{A} \not\preccurlyeq L$, $\mathcal{A} \not\preccurlyeq \mathcal{A}_{(\leq a)}$, and hence $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$. On the other hand, if $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$, then a cannot be in R, because we would have that $\mathcal{A} \preccurlyeq R$. Therefore $L = \{a \in \mathcal{A} : \mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}\}$. Analogously $R = \{a \in \mathcal{A} : \mathcal{A} \preccurlyeq \mathcal{A}_{(\leq a)}\}$.

Let us now prove that (2) implies INDEC. Let $\mathcal{A}$ be a scattered indecomposable linear ordering. By (2), a cut $\langle L, R\rangle$ of $\mathcal{A}$ as in (2.1) exists. Since $\mathcal{A}$ is indecomposable, either $\mathcal{A} \preccurlyeq L$ or $\mathcal{A} \preccurlyeq R$. Without loss of generality, assume that $\mathcal{A} \preccurlyeq R$. If $L = \emptyset$ and $R = A$, then $\mathcal{A}$ is indecomposable to the left. Suppose, then, that $L \neq \emptyset$. Then, $1 + \mathcal{A} \preccurlyeq 1 + R \preccurlyeq \mathcal{A} \preccurlyeq R$. So, there exists an $a \in R$ such that $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$.

But then $\mathcal{A}+\mathcal{A} \preccurlyeq \mathcal{A}$, and by Lemma 1.17, $\eta \preccurlyeq \mathcal{A}$, contradicting the hypothesis on $\mathcal{A}$.

Finally, we prove that Δ^1_1-$\mathsf{CA} \implies$ (2). Let $\mathcal{A}$ be a scattered, weakly indecomposable linear ordering. For every $a \in \mathcal{A}$, either $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leq a)}$ or $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$, and since $\mathcal{A}$ is scattered, it cannot be that both $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leq a)}$ and $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$. So we have that $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leq a)} \iff \mathcal{A} \not\preccurlyeq \mathcal{A}_{(>a)}$. Since $\mathcal{A} \preccurlyeq \mathcal{A}_{(\leq a)}$ and $\mathcal{A} \preccurlyeq \mathcal{A}_{(>a)}$ are Σ^1_1 formulas, Δ^1_1-CA_0 implies that L and R as in (2.1) exist. □

Now we turn to proving that INDEC implies ACA_0 over RCA_0. Some ideas from the proof will be used in the next section when we prove that every ω-model of INDEC is hyperarithmetically closed. The idea of the proof of ACA_0 is to construct a recursive copy $\mathcal{C}$ of $\omega^\omega + \omega^{\omega*}$ such that its middle cut computes $0'$. We also need $\mathcal{C}$ to be recursively weakly indecomposable. We say that $\mathcal{C}$ is *recursively weakly indecomposable* if for every $c \in \mathcal{C}$, there is a recursive embedding of $\mathcal{C}$ into either $\mathcal{C}_{(\leq c)}$ or $\mathcal{C}_{(>c)}$.

Lemma 2.3. *For every $n \in \mathcal{N}$, ω^n is* recursively indecomposable to the right*. That is for every $c \in \omega^n$, there is a recursive embedding of ω^n into $\omega^n_{(>c)}$. Moreover, an index for the embedding can be found uniformly in c. Furthermore, RCA_0 proves that for every n, ω^n is indecomposable to the right.*

Proof. The proof is not hard. Just consider embeddings of the form $\langle x_0, ..., x_{n-1}\rangle \mapsto \langle x_0, ..., x_{n-1}+k\rangle$. □

Theorem 2.4. *(RCA_0) INDEC implies ACA_0.*

Proof. We will prove that INDEC implies that $K = 0'$ exists. Then, by relativizing the proof as usual, we can get that for every set X, X' exists, and hence ACA_0 holds.

We start by constructing a linear ordering $\mathcal{Z}$ such that

- For every $s \in \mathcal{Z}$ there exists $n_s \in \mathcal{N}$ such that either $\mathcal{Z}_{(<s)}$ or $\mathcal{Z}_{(>s)}$ has n_s many elements. In the former case we say that s is on the *left side*. Otherwise s is on the *right side*.
- If the set $R_{\mathcal{Z}} = \{s \in \mathcal{Z} : s \text{ is on the right side of } \mathcal{Z}\}$ exists, it computes $0'$.

Let $\{k_0, k_1, ...\}$ be a recursive enumeration of K. For each s let $K_s = \{k_0, ..., k_s\}$ and $\sigma_s = K_s \restriction k_s + 1$. Consider the following ordering of $\mathcal{N}$.

$$s <_w t \iff \sigma_s <_{KB} \sigma_t,$$

where $<_{KB}$ is the Kleene-Brouwer ordering of $2^{<\omega}$. Let $\mathcal{Z} = \langle \mathcal{N}, \leq_w\rangle$. For each s we have that either $\forall t > s(k_t > k_s)$ (in other words, s is a *true stage*), or there exists a $t > s$ such that $k_t < k_s$. In the latter case we have that $\forall t' \geq t(s <_w t')$, and hence s is on the left side of $\mathcal{Z}$, and $n_s = |\{t' < t : t' <_w s\}|$. In the former case we have that s is on the right side and $n_s = |\{t' < s : s <_w t'\}|$. Observe that ACA_0 can prove that $\mathcal{Z}$ is isomorphic $\omega + \omega^*$ but RCA_0 cannot, since $R_{\mathcal{Z}} = \{s : s \text{ is on the right side}\}$ is the set of true stages of the enumeration of K, and hence from $R_{\mathcal{Z}}$ we can compute K.

One idea to prove that K exists would be to use 2.2(2) to show that the middle cut of $\mathcal{Z}$, $\langle L_{\mathcal{Z}}, R_{\mathcal{Z}}\rangle$, has to exist. But RCA_0 cannot prove that $\mathcal{Z}$ is weakly indecomposable. (Because if s is the greatest element of $\mathcal{Z}$ and if there exists an embedding $f\colon \mathcal{Z} \to \mathcal{Z}_{(<x)}$, then we would have that $R_{\mathcal{Z}}$ is Σ^0_1: $t \in R_{\mathcal{Z}} \iff \exists n(f^n(s) <_w t)$.

But we already know that $R_{\mathcal{Z}}$ is Π^0_1, so by Δ^0_1-CA it would exists, which we cannot prove in RCA_0.) So we need to consider a more complicated linear ordering.

We construct a uniformly recursive sequence of linear orderings $\{\mathcal{P}_s\}_{s\in\mathcal{N}}$ such that

$$\mathcal{P}_s \cong \begin{cases} \omega^{n_s} & \text{if } s \text{ is on the left side} \\ \omega^{n_s *} & \text{if } s \text{ is on the right side.} \end{cases},$$

To construct $\mathcal{P}_s$ recursively, uniformly in s, start by assuming that s is a true stage and enumerating ω^{n*} where $n = |\{t' < s : s <_w t'\}|$. If at any stage $t > s$ we discover that s is not a true stage (that is, we discover that $k_t < k_s$), we change our mind and we start constructing ω^{n_s} instead. (Note that by stage t we have enumerated only finitely many elements of ω^n.) Now define

$$\mathcal{C} = \sum_{s\in\mathcal{Z}} \mathcal{P}_s.$$

Observe that $\mathcal{C}$ is isomorphic to

$$(1+\omega+\omega^2+...)+(...+\omega^{2*}+\omega^*+1) \cong \omega^\omega + \omega^{\omega *}$$

but again, we need ACA_0 to prove it. Furthermore, if the middle cut $\langle L_{\mathcal{C}}, R_{\mathcal{C}}\rangle$ existed, where $R_{\mathcal{C}} = \{y \in \mathcal{P}_s : s \text{ on the right side}\}$, then K would exist too.

Each $\mathcal{P}_s$ is scattered; this can be proven from the fact that either $\mathcal{P}_s$ or $\mathcal{P}^*_s$ is well ordered which is provable in RCA_0. So, by Lemma 1.15, $\mathcal{C}$ is scattered. Now we prove that $\mathcal{C}$ is weakly indecomposable. Consider $y \in \mathcal{C}$. First suppose that $y \in \mathcal{P}_s$ and s on the right side of $\mathcal{Z}$. So s is the n_sth true stage. We will construct an embedding f of $\mathcal{C}$ into $\mathcal{C}_{(<y)}$. Let $\tau = \langle s_1, ..., s_{n_s}\rangle$ be the tuple consisting of the first n_s true stages, where $s = s_{n_s}$. Using τ as a parameter we will construct the desired embedding recursively. Let $\mathcal{D}$ be the sum of the $\mathcal{P}_s$, for s not in τ. Note that there are recursive isomorphisms $\mathcal{C} \cong \mathcal{D} + \sum_{i=n_s}^{1} \omega^{i*} \cong \mathcal{D} + \omega^{n_s *}$. (Assume, to simplify notation, that $\mathcal{C} = \mathcal{D} + \omega^{n_s *}$.) By Lemma 2.3 there is an embedding of $\omega^{n_s *}$ into $\omega^{n_s *}_{(>y)}$. Using this embedding we can construct an embedding $\mathcal{C} \to \mathcal{C}_{(<y)}$ by leaving the elements of $\mathcal{D}$ fixed. Now suppose that $s \in L_{\mathcal{Z}}$ and hence it is not a true stage. RCA_0 can prove the existence of a sequence $\tau = \langle s_1, ..., s_{n_s}\rangle$ with $s = s_{n_s}$, such that for all t' not in the sequence we have $s_1 <_w s_2 <_w ... <_w s_{n_s} <_w t'$. (We just have to find some stage t such that $\forall t' \geq t (s <_w t')$ and then analyze $<_w$ restricted to $\{t' : t' < t\}$.) As we did in the previous case we can find an embedding of $\mathcal{C}$ into $\mathcal{C}_{(>y)}$.

Since $\mathcal{C}$ is scattered and weakly indecomposable, by 2.2(2), the middle cut $\langle L_{\mathcal{C}}, R_{\mathcal{C}}\rangle$ exists. Therefore, $\langle L_{\mathcal{Z}}, R_{\mathcal{Z}}\rangle$ and $0'$ exist too. □

3. Models of INDEC

In this section we prove that INDEC is a theory of hyperarithmetic analysis. We already know that $HYP(Y) \models \mathsf{INDEC}$ for every $Y \subseteq \omega$. This is because we know that Δ^1_1-CA_0 implies INDEC, and that for every Y, $HYP(Y) \models \Delta^1_1$-$\mathsf{CA}_0$.

Theorem 3.1. *The ω-models of INDEC are closed under hyperarithmetic reducibility.*

Let $\mathcal{M}$ be an ω-model of INDEC. To prove that $\mathcal{M}$ is closed under hyperarithmetic reducibility we have to prove that for every $X \in \mathcal{M}$ and any X-recursive ordinal α, $X^{(\alpha)} \in \mathcal{M}$. We will actually prove that for every $\alpha < \omega_1^{CK}$, $0^{(\alpha)} \in \mathcal{M}$.

Then, a relativization of the proof will give the desired result. Since INDEC implies ACA_0, we have that, if for some recursive α, $0^{(\alpha)} \in \mathcal{M}$, then $0^{(\alpha+1)} \in \mathcal{M}$ too. We will prove that if $\{\alpha_n\}_{n\in\mathcal{N}}$ is a recursive increasing sequence of recursive ordinals with limit α and $0^{(\alpha_n)} \in \mathcal{M}$ for every n, then $0^{(\alpha)} \in \mathcal{M}$ too. This implies, using transfinite induction, that for every $\alpha < \omega_1^{CK}$, $0^{(\alpha)} \in \mathcal{M}$. Fix such a sequence $\{\alpha_n\}_{n\in\mathcal{N}}$.

We will construct a recursive scattered linear ordering $\mathcal{Y}$ (of the form $\zeta^{\alpha\cdot\omega} \cdot \mathcal{Z}$ for some other recursive linear ordering $\mathcal{Z}$), and a recursive linear ordering $\mathcal{C}$ such that

(C1) $\mathcal{C} \cong \mathcal{Y} \cdot (\omega^\omega + \omega^{\omega *})$;
(C2) The cut $\langle L_\mathcal{C}, R_\mathcal{C}\rangle$ of order type $\langle \mathcal{Y}\cdot\omega^\omega, \mathcal{Y}\cdot\omega^{\omega*}\rangle$ has Turing degree $0^{(\alpha)}$;
(C3) For each $n \in \mathcal{N}$ there exist an $m_n \in \mathcal{N}$, a recursive linear ordering $\mathcal{D}_n$ and an isomorphism $f_n \leq_T 0^{(\alpha_{m_n})}$,
$$f_n \colon \mathcal{C} \to \mathcal{Y}\cdot\omega^n + \mathcal{D}_n + \mathcal{Y}\cdot\omega^{n*}.$$

Let us first see what we can do once we have constructed such a linear ordering $\mathcal{C}$. First, we note that $\mathcal{C}$ is weakly indecomposable inside $\mathcal{M}$: Consider $a \in \mathcal{C}$ and, without loss of generality, suppose that $a \in L_\mathcal{C}$. Then, for some n, a belongs to the initial segment of $\mathcal{C}$ of order type $\mathcal{Y}\cdot\omega^n$. By (C3), this initial segment is isomorphic to the canonical recursive presentation of $\mathcal{Y}\cdot\omega^n$ via an isomorphism which is recursive in $0^{(\alpha_{m_n})}$, and hence is inside $\mathcal{M}$. Since ω^n is recursively indecomposable to the right, we can use this isomorphism to construct an embedding $\mathcal{C} \hookrightarrow \mathcal{C}_{(>a)}$ that is inside $\mathcal{M}$. Since $\mathcal{C}$ is the product of scattered linear orderings, it is scattered, and hence it is scattered inside $\mathcal{M}$. Now, since $\mathcal{M} \models \mathsf{INDEC}$, the middle cut of $\mathcal{C}$, which is $\langle L_\mathcal{C}, R_\mathcal{C}\rangle$, belongs to $\mathcal{M}$, and therefore $0^{(\alpha)} \in \mathcal{M}$.

3.1. **The construction.** In this subsection we will construct $\mathcal{C}$ and prove it is as desired. We will use Lemma 3.5 below, which we will not prove until the next subsection.

We start by constructing a linear ordering $\mathcal{Z}$ which has a cut, $\langle L_\mathcal{Z}, R_\mathcal{Z}\rangle$, of Turing degree $0^{(\alpha)}$. Then, we will construct $\mathcal{C}$ as a $\mathcal{Z}$-linear ordering (see Definition 3.3 below). Essentially, $\mathcal{C}$ is a recursive $\mathcal{Z}$-linear orderings if it can be written as a recursive sum of the form $\mathcal{C} = \sum_{x\in\mathcal{Z}} P_x(\mathcal{C})$, where $P_x(\mathcal{C})$ are uniformly recursive linear orderings. Another notion that we introduce in this section is the notion of a T-sequence. We will use T-sequences to organize the construction.

Lemma 3.2. *There exist a recursive linear ordering $\mathcal{Z}$ and a cut $\langle L_\mathcal{Z}, R_\mathcal{Z}\rangle$ in it such that:*

(Z1) $\langle L_\mathcal{Z}, R_\mathcal{Z}\rangle \equiv_T 0^{(\alpha)}$*;*
(Z2) $\mathcal{Z}$ *is scattered;*
(Z3) *There are recursive function $\psi, \varphi \colon \mathcal{Z} \to \mathcal{N}$ such for every $x \in \mathcal{Z}$,* $x \in L^\mathcal{Z} \iff \psi(x) \in 0^{(\alpha_{\varphi(x)})}$.

Proof. Let $\langle S_n : n \in \mathcal{N}\rangle$ be a recursive sequence of trees such that S_n has a unique path which is Turing equivalent to $0^{(\alpha_n)}$ uniformly in n. The fact that such a sequence exists is known. The reader can find a proof in Shore [Sho93, Theorem 2.3] that each such tree S_n exists, then, observing that Shore's proof is uniform, we get our sequence of S_n's. Consider $S = \{\sigma \in \mathcal{N}^{<\omega} : \forall \langle i,j\rangle < |\sigma| (\langle \sigma(i,0), ..., \sigma(i,j)\rangle \in S_n)\}$. Clearly S has a unique path, Y, which is Turing equivalent to $0^{(\alpha)}$. Let $\mathcal{Z} = KB(S)$. Let $L_\mathcal{Z} = \{x \in S : x \leq_{KB(S)} Y\}$ and $R_\mathcal{Z} = \mathcal{Z} \smallsetminus L_\mathcal{Z}$. Clearly

$\langle L_{\mathcal{Z}}, R_{\mathcal{Z}}\rangle \equiv_T 0^{(\alpha)}$. Given $\sigma \in S$, let $\varphi(\sigma) = \max\{i : \langle i,j\rangle < |\sigma|\} + 1$. Then, $0^{(\alpha_{\varphi(\sigma)}-1)}$ can compute a string $\tau = Y \upharpoonright |\sigma|$, and then $\sigma \in L_{\mathcal{Z}} \iff \sigma <_{KB(S)} \tau$. Let $\psi(\sigma)$ be such that $\sigma \in L_{\mathcal{Z}} \iff \psi(x) \in 0^{(\alpha_{\varphi(x)})}$.

Note that $\mathcal{Z}$ is scattered: Otherwise we could find two incomparable strings σ_1 and $\sigma_2 \in S$ such that η embeds in both $KB(S_{\sigma_1})$ and $KB(S_{\sigma_2})$. But then, neither S_{σ_1} nor S_{σ_2} would be well-founded, and S would have at least two paths. $\square$

Definition 3.3. Given a linear ordering $\mathcal{Z}$, a $\mathcal{Z}$*-linear ordering* is a first-order structure $\langle \mathcal{B}, \{P_x : x \in \mathcal{Z}\}\rangle$, where $\mathcal{B}$ is a linear ordering and the P_x are unary relation such that

- $\forall a \in \mathcal{B} \exists ! x \in \mathcal{Z}(P_x(a))$,
- $\forall x \in \mathcal{Z} \exists a \in \mathcal{B}(P_x(a))$, and
- $\forall x, y \in \mathcal{Z} \forall a, b \in \mathcal{B}(P_x(a)\ \&\ P_y(b)\ \&\ x \leq_{\mathcal{Z}} y \implies a \leq_{\mathcal{B}} b)$.

We can think of a $\mathcal{Z}$-linear ordering as a linear ordering $\mathcal{B}$, together with an order-preserving, onto map $p_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{Z}$ (defined by $p_{\mathcal{B}}(b) = x \iff P_x(b)$). We write $P_x(\mathcal{B})$ for the sub-ordering of $\mathcal{B}$ with domain $\{b \in \mathcal{B} : P_x(b)\}$. Note that $\mathcal{B} = \sum_{x\in\mathcal{Z}} P_x(\mathcal{B})$.

If $\mathcal{B}$ is a $\mathcal{Z}$-linear ordering, and $\mathcal{X}$ is any linear ordering, we let $\mathcal{X} \cdot \mathcal{B}$ be the $\mathcal{Z}$-linear ordering which has $\mathcal{X} \cdot \mathcal{B}$ as its underlying linear ordering, and for each $z \in \mathcal{Z}$, $x \in \mathcal{X}$ and $a \in \mathcal{B}$, we let $\langle x, a\rangle \in P_x(\mathcal{X} \cdot \mathcal{B}) \iff a \in P_x(\mathcal{B})$.

If $a \in \mathcal{Z}$, $\mathcal{A}$ is a $\mathcal{Z}_{(\leq a)}$-linear ordering and $\mathcal{B}$ is a $\mathcal{Z}_{(\geq a)}$-linear ordering, note that we can put a $\mathcal{Z}$-linear ordering structure on $\mathcal{A} + \mathcal{B}$.

The following lemma will be the main tool in the construction of $\mathcal{C}$.

Lemma 3.4. *Given a computable sequence of ordinals $\{\beta_n\}_{n\in\mathcal{N}}$, a computable function ψ, a computable sequence of linear orderings $\{\mathcal{Z}_n : n \in \mathcal{N}\}$, and two computable sequences $\{\mathcal{A}_n : n \in \mathcal{N}\}$ and $\{\mathcal{B}_n : n \in \mathcal{N}\}$ where $\mathcal{A}_n$ and $\mathcal{B}_n$ are $\mathcal{Z}_n$-linear orderings, we can recursively construct a sequence $\{\mathcal{D}_n : n \in \mathcal{N}\}$ such that*

$$\mathcal{D}_n \cong \begin{cases} \zeta^{\beta_n+1} \cdot \mathcal{A}_n & \text{if } \psi(n) \in 0^{(\beta_n)} \\ \zeta^{\beta_n+1} \cdot \mathcal{B}_n & \text{if } \psi(n) \notin 0^{(\beta_n)}, \end{cases}$$

via an isomorphism recursive in $0^{(\beta_n)}$. Moreover, we can get an index for $\{\mathcal{D}_n : n \in \mathcal{N}\}$ recursively from indices for $\{\beta_n\}_{n\in\mathcal{N}}$, ψ, $\{\mathcal{A}_n : n \in \mathcal{N}\}$ and $\{\mathcal{B}_n : n \in \mathcal{N}\}$.

The proof of this lemma makes use of Lemma 3.5, whose proof we defer to the next subsection. Lemma 3.5 is really a corollary of the work of Ash and Knight. It can be proved using the ideas of the proof of [AJK90, Lemma 4.4]. Instead we prove it using [AK00, Theorem 18.9] and the results in [Ash91, §4] to verify the hypothesis of [AK00, Theorem 18.9] for this particular case.

Lemma 3.5. *Given a computable sequence of ordinals $\{\beta_n\}_{n\in\mathcal{N}}$, a computable function ψ and two recursive sequences of computable linear orderings $\{\mathcal{A}_n : n \in \mathcal{N}\}$ and $\{\mathcal{B}_n : n \in \mathcal{N}\}$, there is a computable sequence of computable linear orderings $\{\mathcal{D}_n : n \in \mathcal{N}\}$ such that*

$$\mathcal{D}_n \cong \begin{cases} \zeta^{\beta_n+1} \cdot \mathcal{A}_n & \text{if } \psi(n) \in 0^{(\beta_n)} \\ \zeta^{\beta_n+1} \cdot \mathcal{B}_n & \text{if } \psi(n) \notin 0^{(\beta_n)}, \end{cases}$$

via an isomorphism computable in $0^{(\beta_n)}$. Moreover, we can get an index for $\{\mathcal{D}_n : n \in \mathcal{N}\}$ computably from indices for $\{\beta_n\}_{n\in\mathcal{N}}$, ψ, $\{\mathcal{A}_n : n \in \mathcal{N}\}$ and $\{\mathcal{B}_n : n \in \mathcal{N}\}$.

Proof of Lemma 3.4 using 3.5. For each $n \in \mathcal{N}$ and $x \in \mathcal{Z}_n$, let $\mathcal{A}_{n,x} = P_x(\mathcal{A}_n) = \{a \in \mathcal{A}_n : P_x(a)\}$ and $\mathcal{B}_{n,x} = P_x(\mathcal{B}_n)$. Think of $\mathcal{A}_{n,x}$ and $\mathcal{B}_{n,x}$ as linear orderings. We use Lemma 3.5 to construct linear orderings $\{\mathcal{D}_{n,x} : n \in \mathcal{N}, x \in \mathcal{Z}\}$ such that

$$\mathcal{D}_{n,x} \cong \begin{cases} \zeta^{\beta_n+1} \cdot \mathcal{A}_{n,x} & \text{if } \psi(n) \in 0^{(\beta_n)} \\ \zeta^{\beta_n+1} \cdot \mathcal{B}_{n,x} & \text{if } \psi(n) \notin 0^{(\beta_n)}, \end{cases}$$

via an isomorphism recursive in $0^{(\beta_n)}$. Then, we just let $\mathcal{D}_n = \sum_{x \in \mathcal{Z}_n} \mathcal{D}_{n,x}$ and $P_x(\mathcal{D}_n) = \mathcal{D}_{n,x}$. □

The idea of the construction of $\mathcal{C}$ is as follows. (We will explain it in more detail later.) Fix $a \in \mathcal{Z}$ and let $m = \varphi(a)$. Let $\mathcal{Y} = \zeta^{\alpha\cdot\omega} \cdot \mathbb{Z}$, $\mathcal{Z}_0 = \mathcal{Z}_{(\leq a)}$, $\mathcal{Z}_1 = \mathcal{Z}_{(\geq a)}$. Suppose that $\mathcal{Z}$ has first and last elements a_0 and a_1. Suppose that (using the recursion theorem) we have already constructed a $\mathcal{Z}_0$-linear ordering $\mathcal{C}_0$ and a $\mathcal{Z}_1$-linear ordering $\mathcal{C}_1$. Moreover, suppose that we know that if $a \in R_{\mathcal{Z}}$, then $\mathcal{C}_0$ satisfies conditions (C1), (C2′) and (C3), where (C2′) is

> $\langle L_{\mathcal{C}}, R_{\mathcal{C}}\rangle$, where $L_{\mathcal{C}} = p_{\mathcal{C}}^{-1}(L_{\mathcal{Z}})$ and $R_{\mathcal{C}} = p_{\mathcal{C}}^{-1}(R_{\mathcal{Z}})$, is the middle cut of $\mathcal{C}$,

and if $a \in L_{\mathcal{Z}}$, the same happens for $\mathcal{C}_1$ instead of $\mathcal{C}_0$.

We want to, uniformly from $\mathcal{C}_0$ and $\mathcal{C}_1$, construct a $\mathcal{Z}$ linear ordering $\mathcal{C}$ which also satisfies conditions (C1)-(C3). We define two $\mathcal{Z}$-linear orderings $\mathcal{A}$ and $\mathcal{B}$ as follows. Let

$$\mathcal{A} = \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega + \mathcal{C}_0 + \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^*.$$

We need to define a $\mathcal{Z}$-linear ordering structure on $\mathcal{A}$. To do this, think of the first summand as a $\{a_0\}$-linear ordering (so $p_{\zeta^{\alpha\cdot\omega}\cdot\mathcal{Z}\cdot\omega}$ is the constant function equal to a_0), the second summand as a $\mathcal{Z}_0$-linear ordering and the third summand as a $\mathcal{Z}_1$-linear ordering. (We define the $\mathcal{Z}_1$-linear ordering structure of $\zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^*$ arbitrarily. For example, given $\langle z, y, v\rangle \in \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^*$, let

$$p_{\zeta^{\alpha\cdot\omega}\cdot\mathcal{Z}\cdot\omega^*}(\langle z,y,v\rangle) = \begin{cases} a & \text{if } v <_{\omega^*} 0 \vee (v = 0 \mathbin{\&} y \leq_{\mathcal{Z}} a) \\ y & \text{if } v = 0 \mathbin{\&} a \leq_{\mathcal{Z}} y.) \end{cases}$$

Let

$$\mathcal{B} = \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega + \mathcal{C}_1 + \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^*.$$

To define a $\mathcal{Z}$-linear ordering structure on $\mathcal{B}$ we think of the first summand as a $\{\mathcal{Z}_0\}$-linear ordering, the second summand as a $\mathcal{Z}_1$-linear ordering and the third summand as a $\{a_1\}$-linear ordering. Again, the $\mathcal{Z}_0$-linear ordering structure of $\zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega$ is defined arbitrarily.

Now, using Lemma 3.4, we construct a recursive $\mathcal{Z}$-linear ordering $\mathcal{C}$ such that

$$\mathcal{C} \cong \begin{cases} \zeta^{\alpha_m+1} \cdot \mathcal{A} & \text{if } a \in R_{\mathcal{Z}} \\ \zeta^{\alpha_m+1} \cdot \mathcal{B} & \text{if } a \in L_{\mathcal{Z}}, \end{cases}$$

and the isomorphism is recursive in $0^{(\alpha_m)}$. (Recall that $a \in L_{\mathcal{Z}} \iff \psi(a) \in 0^{(\alpha_m)}$.) It is not hard to see that $\mathcal{C}$ satisfies conditions (C1), (C2′) and (C3).

Now, we construct a tree that we will use to organize the construction of $\mathcal{C}$.

Lemma 3.6. *There is a recursive linear ordering $\mathcal{Z}$ satisfying the conditions of Lemma 3.2, and a recursive binary tree T such that $\mathbf{1} + LR(T) + \mathbf{1} \cong \mathcal{Z}$ via a recursive isomorphism $\sigma \mapsto a_\sigma$. Moreover, T has an path X such that $L_{\mathcal{Z}} = \{a_\sigma \in$*

$T : \sigma <_{LR(T)} X\}$ and $R_{\mathcal{Z}} = \{a_\sigma \in T : X <_{LR(T)} \sigma\}$. We can also assume that for each $\sigma \in T$, either both, $\sigma^\frown 0$ and $\sigma^\frown 1$, belong to T or both do not. Furthermore, there is a recursive family $\{\mathcal{Z}_\sigma : \sigma \in T\}$ of closed segments of $\mathcal{Z}$ such that $\mathcal{Z}_\emptyset = \mathcal{Z}$, $\mathcal{Z}_{\sigma^\frown 0} = \mathcal{Z}_{\sigma(\leq a_\sigma)}$ and $\mathcal{Z}_{\sigma^\frown 1} = \mathcal{Z}_{\sigma(\geq a_\sigma)}$ whenever $\sigma^\frown 0$ and $\sigma^\frown 1 \in T$.

Proof. Let $\mathcal{Z}^0$ be a linear ordering as in Lemma 3.2. Let $\mathcal{Z} = \mathbf{1} + \zeta \cdot \mathcal{Z}^0 + 1$. Note that $\mathcal{Z}$ still satisfies the condition of Lemma 3.2, and that in $\mathcal{Z}$ we can recursively decide whether two elements are separated by finitely many elements, and if so, we can compute how many elements there are in between.

We define T, the map $\sigma \mapsto a_\sigma$, and the family $\{\mathcal{Z}_\sigma : \sigma \in T\}$ simultaneously by induction. Along the induction we will preserve the property that for every $\sigma \in T$, $\mathcal{Z}_\sigma$, if it is finite, has an odd number of elements and at least three. Let $\mathcal{Z}_\emptyset = \mathcal{Z}$. Suppose now that $\sigma \in T$, and that we have already defined $\mathcal{Z}_\sigma$. If $\mathcal{Z}_\sigma$ has only three elements, then leave $\sigma^\frown 0$ and $\sigma^\frown 1$ outside of T and let a_σ be the middle element of $\mathcal{Z}_\sigma$. If $\mathcal{Z}_\sigma$ has at least five elements, enumerate $\sigma^\frown 0$ and $\sigma^\frown 1$ into T, and let a_σ be the $\leq_{\mathcal{N}}$-least element of $\mathcal{Z}_\sigma$ such that $\mathcal{Z}_{\sigma^\frown 0} = \mathcal{Z}_{\sigma(\leq a_\sigma)}$ and $\mathcal{Z}_{\sigma^\frown 1} = \mathcal{Z}_{\sigma(\geq a_\sigma)}$ do not have an even number of elements, and at least three. (Recall that the domain of $\mathcal{Z}$ is a subset of $\mathcal{N}$.) It is not hard to see that T, the map $\sigma \mapsto a_\sigma$, and the family $\{\mathcal{Z}_\sigma : \sigma \in T\}$ are as desired.

Let X be the leftmost path of $\{\sigma \in 2^{<\omega} : \exists \tau \in 2^{<\omega}(a_\tau \in R_{\mathcal{Z}} \;\&\; \tau \leq_{LR} \sigma)\}$. □

From now on, fix T and $\mathcal{Z}$ as in the lemma above, and we identify $\mathcal{Z}$ with $\mathbf{1} + LR(T) + \mathbf{1}$.

Definition 3.7. A *T-sequence* is a family of structures $\langle \mathcal{D}_\sigma : \sigma \in T\rangle$ such that $\mathcal{D}_\sigma$ is a $\mathcal{Z}_\sigma$-linear ordering.

We will construct a recursive functional, $\mathcal{E}$, that given (an index for) a T-sequence returns (an index for) another T-sequence. Then, we will use the recursion theorem to obtain a fixed point of this operator. Since we will be using the recursion theorem, we will want, not only that $\mathcal{E}$ maps T-sequences to T-sequences in a certain way, but also that $\mathcal{E}$ maps indices which do not correspond to T-sequences to indices which do code T-sequences. This way we ensure that any fixed point of $\mathcal{E}$ is an index of a T-sequence. For this purpose we prove the following lemma.

Lemma 3.8. *There is a recursive function that, given an index e, returns an index for a $\mathcal{Z}$-linear ordering, $\mathcal{B}_e$, such that if e codes a $\mathcal{Z}$-linear ordering $\mathcal{A}_e$, then $\mathcal{B}_e = \zeta^3 \cdot \mathcal{A}_e$.*

Proof. Being an index for a $\mathcal{Z}$-linear ordering is an arithmetic property, say Π^0_k. Then, applying Lemma 3.5, we get a family of linear orderings $\{\mathcal{D}_{e,x} : e \in \mathcal{N}, x \in \mathcal{Z}\}$ such that

$$\mathcal{D}_{e,x} = \begin{cases} \zeta^{k+1} \cdot P_x(\mathcal{A}_e) & \text{if } e \text{ codes a } \mathcal{Z}\text{-linear ordering } \mathcal{A}_e; \\ \zeta^{k+1} & \text{if } e \text{ is not the code of a } \mathcal{Z}\text{-linear ordering.} \end{cases}$$

Let $\mathcal{D}_e = \sum_{x \in \mathcal{Z}} \mathcal{D}_{e,x}$ and $P_x(\mathcal{D}_e) = \mathcal{D}_{e,x}$. Actually, if for example we code $\mathcal{Z}$-linear orderings by a linear ordering and a function p onto $\mathcal{Z}$, we get that being an index for a $\mathcal{Z}$-linear ordering is a Π^0_2 property. So, we could make $k = 2$, although this is not relevant for our purposes. □

We have now introduced all the ingredients of the construction.

Construction of $\mathcal{C}$. We construct a recursive operator $\mathcal{E}$ that, given (an index d for) a T-sequence $\bar{\mathcal{D}} = \{\bar{\mathcal{D}}_\sigma : \sigma \in T\}$, returns (an index $\mathcal{E}(d)$ for) a T-sequence $\mathcal{E}(\bar{\mathcal{D}})$. (We are abusing notation here when writing $\mathcal{E}(\bar{\mathcal{D}})$ instead of $\mathcal{E}(d)$.)

We actually want $\mathcal{E}(d)$ to be an index for a T-sequence even when d is not an index for a T-sequence. We start by constructing a T-sequence $\mathcal{D} = \{\mathcal{D}_\sigma : \sigma \in T\}$, such that if d is actually an index for a T-sequence $\bar{\mathcal{D}}$, then $\mathcal{D}_\sigma = \zeta^3 \cdot \bar{\mathcal{D}}_\sigma$ for all $\sigma \in T$. For each d and σ we can uniformly compute an index d_σ such that if d is an index for a T-sequence $\bar{\mathcal{D}}$ and $\sigma \in T$, then d_σ is an index for the $\mathcal{Z}_\sigma$-linear ordering $\bar{\mathcal{D}}_\sigma$. Now, use Lemma 3.8 to construct a T-sequence $\mathcal{D} = \{\mathcal{D}_\sigma : \sigma \in T\}$ such that, for each $\sigma \in T$, if d_σ is actually coding a $\mathcal{Z}_\sigma$-linear ordering $\bar{\mathcal{D}}_\sigma$, then $\mathcal{D}_\sigma = \zeta^3 \cdot \bar{\mathcal{D}}_\sigma$.

Now we define two T-sequences $\mathcal{A} = \{\mathcal{A}_\sigma : \sigma \in T\}$ and $\mathcal{B} = \{\mathcal{B}_\sigma : \sigma \in T\}$. If σ is an end node of T, let $\mathcal{A}_\sigma = \zeta^\alpha \cdot \mathcal{Z} \cdot \omega^{|\sigma|}$ and $\mathcal{B}_\sigma = \zeta^\alpha \cdot \mathcal{Z} \cdot \omega^{|\sigma|*}$. Define a $\mathcal{Z}_\sigma$-linear ordering structure on $\mathcal{A}_\sigma$ and $\mathcal{B}_\sigma$ arbitrarily. Suppose now that σ is not an end node of T. Let $n = |\sigma|$, a_0 be the least element of $\mathcal{Z}_\sigma$ and a_1 be the greatest one. So $\mathcal{Z}_{\sigma^\frown 0} = [a_0, a_\sigma]_\mathcal{Z}$ and $\mathcal{Z}_{\sigma^\frown 1} = [a_\sigma, a_1]_\mathcal{Z}$. Let

$$\mathcal{A}_\sigma = \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^n + \mathcal{D}_{\sigma^\frown 0} + \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^{n*}.$$

We need to define a $\mathcal{Z}_\sigma$-linear ordering structure on $\mathcal{A}_\sigma$. To do this, think of the first summand as a $\{a_0\}$-linear ordering, the second summand is a $\mathcal{Z}_{\sigma^\frown 0}$-linear ordering and the third summand as a $\mathcal{Z}_{\sigma^\frown 1}$-linear ordering. (We define the $\mathcal{Z}_{\sigma^\frown 1}$-linear ordering structure of $\zeta^\alpha \cdot \mathcal{Z} \cdot \omega^{n*}$ arbitrarily.) Let

$$\mathcal{B}_\sigma = \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^n + \mathcal{D}_{\sigma^\frown 1} + \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^{n*}$$

where the first summand is a $\mathcal{Z}_{\sigma^\frown 0}$-linear ordering, the second summand is a $\mathcal{Z}_{\sigma^\frown 1}$-linear ordering and the third summand is a $\{a_1\}$-linear ordering. Again, the $\{\mathcal{Z}_{\sigma^\frown 0}\}$-linear ordering structure of $\zeta^\alpha \cdot \mathcal{Z} \cdot \omega^n$ is defined arbitrarily.

Last, using Lemma 3.4, we construct a T-sequence $\mathcal{E}(\bar{\mathcal{D}})$ such that for each $\sigma \in T$,

$$\mathcal{E}(\bar{\mathcal{D}})_\sigma \cong \begin{cases} \zeta^{\alpha_{\varphi(\sigma)}+1} \cdot \mathcal{A}_\sigma & \text{if } \sigma \in R_\mathcal{Z} \\ \zeta^{\alpha_{\varphi(\sigma)}+1} \cdot \mathcal{B}_\sigma & \text{if } \sigma \in L_\mathcal{Z}, \end{cases}$$

and the isomorphism is recursive in $0^{(\alpha_{\varphi(\sigma)})}$. (Recall φ is a recursive function such that $\sigma \in L^\mathcal{Z} \iff \psi(x) \in 0^{(\alpha_{\varphi(\sigma)})}$.)

By the recursion theorem there is an index c such that $\{c\} = \{\mathcal{E}(c)\}$, where $\{e\}$ is the eth Turing function. Since $\mathcal{E}$ always returns indices for T-sequences, c is the index of a T-sequence $\bar{\mathcal{C}}$. Let $\mathcal{C} = \bar{\mathcal{C}}_\emptyset$. ♢

We claim that $\mathcal{C}$ is the desired $\mathcal{Z}$-linear ordering.

Lemma 3.9. *$\mathcal{C}$ satisfies conditions (C1)-(C3)*

Proof. For each n let $m_n = \varphi(X \upharpoonright n)$. We start by observing that for every n, there is an isomorphism

$$f_n : \bar{\mathcal{C}}_{X\upharpoonright n} \to \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^n + \zeta^{\alpha_{m_n}+3} \cdot \bar{\mathcal{C}}_{X\upharpoonright n+1} + \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^{n*},$$

which is recursive in $0^{(\alpha_{m_n})}$. This, by induction on n, implies that for each n there is an isomorphism $g_n \leq_T 0^{(\alpha_{m_n})}$,

$$g_n : \mathcal{C} \to \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^n + \zeta^{\alpha_{m_0}+3+\cdots+\alpha_{m_n}+3} \cdot \bar{\mathcal{C}}_{X\upharpoonright n+1} + \zeta^{\alpha\cdot\omega} \cdot \mathcal{Z} \cdot \omega^{n*}.$$

(We have used that $\mathbf{1}+\omega+\cdots+\omega^n$ is recursively isomorphic to ω^n, and $\zeta^{\alpha_i+3}\cdot\zeta^{\alpha\cdot\omega} \cong \zeta^{\alpha_i+3+\alpha\cdot\omega} \cong \zeta^{\alpha\cdot\omega}$.) Condition (C3) follows. Moreover, we can construct the maps g_n such that if $n_1 > n_0$, then g_{n_0} and g_{n_1} coincide on the initial segment of the form $\zeta^{\alpha\cdot\omega}\cdot\mathcal{Z}\cdot\omega^{n_0}$, and on the final segment of the form $\zeta^{\alpha}\cdot\mathcal{Z}\cdot\omega^{n_0*}$. Therefore, putting all these isomorphisms together, we get an isomorphism $g \leq_T 0^{(\alpha)}$,

$$g\colon \mathcal{C}\to\zeta^{\alpha\cdot\omega}\cdot\mathcal{Z}\cdot\omega^{\omega}+\mathcal{D}+\zeta^{\alpha\cdot\omega}\cdot\mathcal{Z}\cdot\omega^{\omega*},$$

for some possibly empty linear ordering $\mathcal{D}$. Also observe that for every $b\in\zeta^{\alpha}\cdot\mathcal{Z}\cdot\omega^{\omega}$, $p(g^{-1}(b))\in L_{\mathcal{Z}}$ and for every $b\in\zeta^{\alpha}\cdot\mathcal{Z}\cdot\omega^{\omega*}$, $p(g^{-1}(b))\in R_{\mathcal{Z}}$. Therefore, if $b\in\mathcal{D}$, $p(g^{-1}(b))$ is either the last element of $L_{\mathcal{Z}}$ of the first element of $R_{\mathcal{Z}}$. But $L_{\mathcal{Z}}$ has no last element and $R_{\mathcal{Z}}$ has no first element (because otherwise $\langle L_{\mathcal{Z}},R_{\mathcal{Z}}\rangle$ would be recursive), so $\mathcal{D}$ has to be empty. Conditions (C1) and (C2′) follow. Condition (C2) easily follows from (C2′). □

3.2. **Pairs of computable structures.** In this subsection we explain how the results in [AK00, Chapter 18] and [Ash91, §4] imply Lemma 3.5.

We start by defining the back-and-forth relations and the notion of α-friendliness. See [AK00, Sections 15.1 and 15.2] for more information on these concepts.

Definition 3.10. Let K be a class of structures for a fixed language. For each ordinal α, we define the *standard back-and-forth relation* $\leq_\alpha$ on pairs $(A,\bar a)$, where $A\in K$ and $\bar a$ is a tuple in A. Let $\bar a$ in A and $\bar b$ in B be tuples of the same length. Then,

(1) $(A,\bar a)\leq_1(B,\bar b)$ if and only if all Σ_1 formulas true of $\bar b$ in B are true of $\bar a$ in A.
(2) For $\alpha>1$, $(A,\bar a)\leq_n(B,\bar b)$ if and only if for each $\bar d$ in B, and each $\beta<\alpha$, there exists a $\bar c$ in A with $|\bar c|=|\bar d|$ such that $(B,\bar b,\bar d)\leq_\beta(A,\bar a,\bar c)$.

This definition can be extended to tuples of different length, but we are only interested in pairs of tuples of the same length. We may write $A\leq_n B$ instead of $(A,\emptyset)\leq_n(B,\emptyset)$.

A pair of structures $\{A_0,A_1\}$ is *α-friendly* if the structures A_i are computable, and for $\beta<\alpha$, the standard back-and-forth relations $\leq_\beta$ on pairs $(A_i,\bar a)$ with $\bar a\in A_i\in\{A_0,A_1\}$, are r.e. uniformly in β. That is, we can recursively enumerate all the triples $\langle\langle i,\bar a\rangle,\langle j,\bar b\rangle,\beta\rangle$ with $\beta<\alpha$, $\bar a\in\mathcal{A}_i$ and $\bar b\in\mathcal{A}_j$ such that $(A_i,\bar a)\leq_\beta(A_j,\bar b)$.

One observation that might give the reader some intuition about the back-and-forth relation is that $(A,\bar a)\leq_n(B,\bar b)$ if and only if all the Π_n infinitary formulas true of $\bar a$ in A are true of $\bar b$ in B [AK00, Proposition 15.1].

Given this definition we can state the main theorem on pairs of computable structures that we will be using.

Theorem 3.11. *(Essentially* [AK00, 18.9]*) For each n, let $\mathcal{A}_n$ and $\mathcal{B}_n$ be structures such that $\mathcal{B}_n\leq_{\alpha_n}\mathcal{A}_n$ and $\{\mathcal{A}_n,\mathcal{B}_n\}$ is α_n-friendly, uniformly in n. Let S be a $\Pi^0_{(\alpha_n)}$ set. In other words, let S be such that that is there exists a computable function f such that $n\in S\iff f(n)\notin 0^{(\alpha_n)}$. Then, there is a uniformly computable sequence $\{\mathcal{C}_n\}_{n\in\mathcal{N}}$ such that*

$$\mathcal{C}_n\cong\begin{cases}\mathcal{A}_n & \text{if } n\in S\\ \mathcal{B}_n & \text{otherwise.}\end{cases}$$

Moreover, the isomorphisms above are recursive in $0^{(\alpha_n)}$ and an index for $\{\mathcal{C}_n\}_{n\in\mathcal{N}}$ can be obtained uniformly from indices for S, $\{\alpha_n : n\in\mathcal{N}\}$, $\{\mathcal{A}_n : n\in\mathcal{N}\}$, $\{\mathcal{B}_n : n\in\mathcal{N}\}$ and the back-and-forth relations.

Proof. The first part of the theorem (before the "Moreover") is exactly [AK00, Theorem 18.9]. The rest follows from the proof of [AK00, Theorem 18.9]. In the proof of [AK00, Theorem 18.9], for each n, a complicated apparatus, that outputs a computable structure $\mathcal{C}_n$ as desired, is constructed. It is constructed uniformly in n, and indices for $\mathcal{A}_n$, $\mathcal{B}_n$ and the back-and-forth relations between them. This apparatus is what they call an α_n-system (defined in [AK00, Chapter 14]), which is a $\Delta^0_{\alpha_n}$-priority construction. From the fact that the construction of each of these apparatuses is uniform in $\mathcal{A}_n$, $\mathcal{B}_n$ and the back-and-forth relations between them, we get that we can even get the sequence $\{\mathcal{C}_n\}$ uniformly from indices for S, $\{\alpha_n : n \in \mathcal{N}\}$, $\{\mathcal{A}_n : n \in \mathcal{N}\}$, $\{\mathcal{B}_n : n \in \mathcal{N}\}$ and the back-and-forth relations.

The isomorphism between $\mathcal{C}_n$ and either $\mathcal{A}_n$ or $\mathcal{B}_n$ is $\Delta^0_{\alpha_n}$ because it can be computed from a run of the α_n-system. See the proof of [AK00, 18.6]. □

The only structures we will be dealing with are linear orderings. The following two lemmas give us a way of computing the back-and-forth relations on linear orderings without having to refer to the definition given above.

Lemma 3.12. [AK00, 15.7] *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are linear orderings. Let $\bar{a} = \langle a_0, ..., a_{n-1}\rangle$ and $\bar{b} = \langle b_0, ..., b_{n-1}\rangle$ be increasing tuples form $\mathcal{A}$, $\mathcal{B}$ respectively. For each $i \leq n$ let $\mathcal{A}_i$ be the interval $(a_{i-1}, a_i)_{\mathcal{A}}$ (of course, $\mathcal{A}_0 = \mathcal{A}_{(<a_0)}$ and $\mathcal{A}_n = \mathcal{A}_{(>a_{n-1})}$). Define $\mathcal{B}_i$ analogously. Then $(\mathcal{A}, \bar{a}) \leq_\beta (\mathcal{B}, \bar{b})$ if and only if for all $0 \leq i \leq n$, $\mathcal{A}_i \leq_\beta \mathcal{B}_i$.*

Lemma 3.13. [AK00, 15.8] *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are linear orderings. Then $\mathcal{A} \leq_1 \mathcal{B}$ if and only if $\mathcal{A}$ is infinite or at least as large as $\mathcal{B}$. For $\beta > 1$, $\mathcal{A} \leq_\beta \mathcal{B}$ if and only if, for any $1 \leq \gamma < \beta$ and any finite partition of $\mathcal{B}$ into intervals $\mathcal{B}_1, ..., \mathcal{B}_k$, with end points in $\mathcal{B}$, there is a corresponding partition of A into intervals $\mathcal{A}_1, ..., \mathcal{A}_k$, such that for all $i < n$, $\mathcal{B}_i \leq_\gamma \mathcal{A}_i$.*

Now we use the results in [Ash91, §4] to prove that we can apply Theorem 3.11 to get Lemma 3.5.

Notation 3.14. Let $\xi_\beta = \sum_{\gamma<\beta} \zeta^\gamma \cdot \omega$ and $\nu_\beta = \xi_\beta + \xi^*_\beta$. Observe that $\zeta^\beta = \xi^*_\beta + \xi_\beta$, that every final segment of ζ^β with first element in ζ^β has order type ξ_β, and that every segment of ζ^β with both endpoints in ζ^β has order type

$$\xi_\gamma + \zeta^\gamma \cdot n + \xi^*_\gamma = \nu_\gamma \cdot (n+1),$$

for some $\gamma < \beta$ and $n \in \mathcal{N}$.

Lemma 3.15. *(Essentially* [Ash91, Proposition 4.8]*) Let α, β and γ be ordinals.*

1. *$\nu_\beta \cdot n \leq_{2\gamma} \nu_\alpha \cdot m$ if and only if either $\langle \beta, n\rangle = \langle \alpha, m\rangle$, or $\alpha, \beta \geq \gamma$.*
2. *$\nu_\beta \cdot n \leq_{2\gamma+1} \nu_\alpha \cdot m$ if and only if either $\langle \beta, n\rangle = \langle \alpha, m\rangle$, $\alpha \geq \gamma$ & $\beta > \gamma$, or $\alpha = \beta = \gamma$ & $n \geq m$.*

Proof. The proof is technical, but not complicated. It is by induction on γ and makes heavy use of Lemma 3.13. We only prove part (1) to illustrate the ideas.

Suppose that $\alpha, \beta \geq \gamma$; we want to show that $\nu_\beta \cdot n \leq_{2\gamma} \nu_\alpha \cdot m$. Let $\delta < \gamma$. Let $\mathcal{A}_1, ..., \mathcal{A}_k$ be a partition of $\nu_\alpha \cdot m$ into intervals with endpoints in $\nu_\alpha \cdot m$. Then, there exists ordinals $\alpha_i : i = 1, ..., k$ and numbers $m_1, ..., m_k$ such that for each i, $\mathcal{A}_i = \nu_{\alpha_1} \cdot m_1$. Necessarily $\max \alpha_i = \alpha$ and $\sum_{i:\alpha_i=\alpha} m_i = m$. For i with $\alpha_i < \delta$, let $\beta_i = \alpha_i$ and $n_i = m_i$. For the first i such that $\alpha_i = \alpha$, let $\beta_i = \beta$ and $n_i = n$. For all the other i, let $\beta_i = \delta$ and $n_i = 1$. Note that for every $i = 0, ..., k$,

$\nu_{\alpha_i} \cdot m_i \leq_{2\delta+1} \nu_{\beta_i} \cdot n_i$ and that $\sum_{i\leq k} \nu_i \cdot n_i \cong \nu_\beta \cdot n$. For $i = 1, .., k$, let $\mathcal{B}_i = \nu_{\beta_i} \cdot n_i$. By Lemma 3.13, this shows that $\nu_\beta \cdot n \leq_{2\gamma} \nu_\alpha \cdot m$.

Now assume that $\nu_\beta \cdot n \leq_{2\gamma} \nu_\alpha \cdot m$; we want to show that if $\langle \beta, n\rangle \neq \langle \alpha, m\rangle$, then $\alpha, \beta \geq \gamma$. If either α or β is equal to δ and $\delta + 1 < \gamma$, then, by inductive hypothesis, $\nu_\beta \cdot n \not\leq_{2\delta+2} \nu_\alpha \cdot m$, so $\nu_\beta \cdot n \not\leq_{2\gamma} \nu_\alpha \cdot m$. If $\gamma = \alpha = \beta + 1$, then, again by inductive hypothesis, $\nu_\beta \cdot n \not\leq_{2\beta+1} \nu_\alpha \cdot m$, so $\nu_\beta \cdot n \not\leq_{2\gamma} \nu_\alpha \cdot m$. If $\gamma = \beta = \alpha + 1$, then $\nu_\alpha \cdot m \not\leq_{2\alpha+1} \nu_\beta \cdot n$, so $\nu_\beta \cdot n \not\leq_{2\gamma} \nu_\alpha \cdot m$. □

Lemma 3.16. *For any linear orderings $\mathcal{B}$ and $\mathcal{D}$ and an ordinal α, $\zeta^{\alpha+1} \cdot \mathcal{B}$ and $\zeta^{\alpha+1} \cdot \mathcal{D}$ are α-back-and-forth equivalent and α-friendly.*

Proof. The proof is by induction on α. Let $\beta < \alpha$. Let $\langle z_0, d_0\rangle, ..., \langle z_{k-1}, d_{k-1}\rangle \in \zeta^\alpha \cdot (\zeta \cdot \mathcal{D})$ be any ordered tuple. Let $c_0, ..., c_{k-1} \in \zeta \cdot \mathcal{B}$ be such that $d_i < d_j \iff c_i < c_j$. Observe that for each i, the interval $[\langle z_i, d_i\rangle, \langle z_{i+1}, d_{i+1}\rangle]_{\zeta^{\alpha+1}\cdot\mathcal{D}}$ is β-back and forth equivalent to $[\langle z_i, c_i\rangle, \langle z_{i+1}, c_{i+1}\rangle]_{\zeta^{\alpha+1}\cdot\mathcal{C}}$.

They are α-friendly because any closed interval interval of $\zeta^{\alpha+1} \cdot \mathcal{B}$ is either isomorphic to $\nu_\beta \cdot m$ for some $\beta \leq \alpha$, or of the form $\xi_{\alpha+1} + \xi^*_{\alpha+1} \cong \nu_{\alpha+1}$, or of the form $\xi_{\alpha+1} + \zeta^{\alpha+1} \cdot \mathcal{C} + \xi^*_{\alpha+1}$ for some $\mathcal{C}$, which is α-back and forth equivalent to $\xi_{\alpha+1} + \zeta^{\alpha+1} + \xi^*_{\alpha+1} \cong \nu_{\alpha+1} \cdot 2$. In all these cases we know how to compute the β-back-and-forth relations recursively. □

Lemma 3.5 now follows from Theorem 3.11 and the lemma above. This finishes the proof of Theorem 3.1.

Remark 3.17. The result of Lemma 3.16 is not sharp. Possibly, one could prove that α, $\zeta^{\alpha+1} \cdot \mathcal{B}$ and $\zeta^{\alpha+1} \cdot \mathcal{D}$ are β-back-and-forth equivalent for some $\beta > \alpha$. Therefore, Lemma 3.5 is not sharp either. But this is not relevant for our results.

4. The Game Statements

In this section we prove all the implications in Theorem 1.12 that have to do with game statements. All these statements were introduced in Subsection 1.2. We work in RCA_0.

Σ^1_1-$\mathsf{AC}_0 \implies$ $\mathsf{DG\text{-}AC}$: Let $\{T_n : n \in \mathcal{N}\}$ be a sequence of determined trees. If for some n, *II* has a winning strategy for $G(T_n)$, then then *I* has the following winning strategy in $\sum_n T_n$: Start by playing n, and then use *II*'s winning strategy in $G(T_n)$. (Recall that *I* is the second player in $G(T_n)$.) Suppose now that for every n, *I* has a winning strategy in $G(T_n)$. We will show that then, *II* has a winning strategy in $\sum_n T_n$. Using Σ^1_1-AC_0, let $\langle s_n : n \in \mathcal{N}\rangle$ be such that s_n is a winning strategy for *I* in $G(T_n)$. Now, if *I* starts playing n, *II* continues following s_n in $G(T_n)$ and wins.

$\mathsf{DG\text{-}AC} \implies$ $\mathsf{DG\text{-}CA}$: Let $\{T_n : n \in \mathcal{N}\}$ be a sequence of determined trees. Now, for each n, consider the game $\bar{G}_n = G_n + G^*_n$. Where $G_n = G(T_n)$ and G^*_n is in Definition 1.8. So $\bar{G}_n$ is the game in which player *I* starts by choosing whether to play G_n or G^*_n. Since G_n is determined for each n, *I* has a winning strategy for $\bar{G}_n$ in which he starts by choosing whichever of G_n, or G^*_n has a winning strategy for *II*. Therefore, by $\mathsf{DG\text{-}AC}$, *II* has a winning strategy s for $\sum_n \bar{G}_n$. Let X be the set of n such that if *I* starts playing n in $\sum_n \bar{G}_n$, then *II*, following s, starts playing $\bar{G}_n$ by choosing G^*_n. So, X is the set of n such that *II* has a winning strategy in G^*_n, or equivalently, such that *I* has a winning strategy in G_n.

Δ^1_1-$\mathsf{CA}_0 \implies \mathsf{DG\text{-}CA}$: Let $\{T_n : n \in \mathcal{N}\}$ be a sequence of determined trees. Since there exists a winning strategy for I in $G(T_n)$ if and only if there is no winning strategy for $I\!I$ in $G(T_n)$. By Δ^1_1 we can define a set X such that $n \in X$ if and only if there exists a winning strategy for I. $\mathsf{DG\text{-}CA}$ follows.

$\mathsf{DG\text{-}CA} \implies \Delta^1_1$-$\mathsf{CA}_0$: Let φ and ψ be Σ^1_1 formulas such that $\forall n(\psi(n) \iff \neg\varphi(n))$. We want to show that there exists a set X such that $\forall n(n \in X \iff \psi(n))$ By [Sim99, Theorem V.1.7′], there exist a sequences of trees $\{T_n : n \in \mathcal{N}\}$ and $\{S_n : n \in \mathcal{N}\}$ such that for every n,

$$S_n \text{ has a path} \iff \psi(n) \iff \neg\varphi(n) \iff T_n \text{ is well-founded.}$$

Consider the game, $\bar{G}_n$, in which players I and $I\!I$ alternatively play numbers $a_0, a_1, \ldots$. Player I, when playing a_{2i}, has to make sure that $\langle a_0, a_2, \ldots, a_{2i}\rangle \in S_n$ because otherwise he looses. Player $I\!I$, when playing a_{2i+1}, has to make sure that $\langle a_1, a_3, \ldots, a_{2i+1}\rangle \in T_n$; otherwise he looses.

Suppose that T_n is well-founded and S_n is not. Let X be a path through S_n. Notice that if I plays $X(i)$ in his ith move, he will surely win. Analogously, if S_n is well-founded and T_n is not, $I\!I$ has a winning strategy in the game $\bar{G}_n$. So, we have that for each n, $\bar{G}_n$ is determined. By $\mathsf{DG\text{-}CA}$, there exists a set X such that

$$n \in X \iff I \text{ has a winning strategy in } \bar{G}_n \iff S_n \text{ has a path} \iff \psi(n).$$

Δ^1_1-$\mathsf{CA}_0 \implies$ weak-Σ^1_1-AC_0 is not hard to prove. See for example [Sim99].

weak-Σ^1_1-$\mathsf{AC}_0 \implies \mathsf{CDG\text{-}CA}$: Let T be a well founded tree. We first show that d is the unique winning function of $G(T)$ if and only if

$$\forall \sigma \in T(d(\sigma) = \mathtt{W} \iff \exists n(\sigma^\frown n \in T \;\&\; d(\sigma^\frown n) = \mathtt{L})). \tag{4.1}$$

It is clear that if d is a winning function, then (4.1) holds. Suppose now that $d\colon T \to \{\mathtt{W}, \mathtt{L}\}$ satisfies (4.1). For each $\sigma \in T$ with $d(\sigma) = \mathtt{W}$ we have to define a winning strategy s_σ for I in $G(T_\sigma)$, and if $d(\sigma) = \mathtt{L}$ we have to define a winning strategy s_σ for $I\!I$ in $G(T_\sigma)$.

If $d(\sigma) = \mathtt{W}$ and $\tau \in T_{\sigma,I}$, we let $s_\sigma(\tau)$ be the least $n \in \mathcal{N}$ such that $\tau^\frown n \in T_\sigma$ and $d(\sigma^\frown\tau^\frown n) = \mathtt{L}$. We claim that s_σ is a winning strategy for I in $G(T_\sigma)$. Observe that one can easily prove by induction that if $\tau \in T_\sigma$ is a partial run of $G(T_\sigma)$ following s_σ, then, if $\tau \in T_{\sigma,I}$, $d(\sigma^\frown\tau) = \mathtt{W}$ and, if $\tau \in T_{\sigma,I\!I}$, $d(\sigma^\frown\tau) = \mathtt{L}$. Since for every end node τ of T_σ, $d(\sigma^\frown\tau) = \mathtt{L}$, we have that if I follows s_σ, he surely wins.

If $d(\sigma) = \mathtt{L}$ and $\tau \in T_{\sigma,I\!I}$, we let $s_\sigma(\tau)$ be the least $n \in \mathcal{N}$ such that $\tau^\frown n \in T_\sigma$ and $d(\sigma^\frown\tau^\frown n) = \mathtt{L}$. An argument like the one above shows that s_σ is a winning strategy for $I\!I$ in $G(T_\sigma)$.

Now, in order to prove $\mathsf{CDG\text{-}CA}$, consider a family of completely determined trees $\{T_n : n \in \mathcal{N}\}$. We want to show that there exists a set X such that $n \in X$ if and only if I wins $G(T_n)$. For each n, there is a unique function d_n such that (4.1) holds. So, by weak-Σ^1_1-AC_0, the sequence $\langle d_n : n \in \mathcal{N}\rangle$ exists. Let $X = \{n : d_n(\emptyset) = \mathtt{W}\}$.

$\mathsf{CDG\text{-}AC} \implies \mathsf{CDG\text{-}CA}$: If d is the winning function of $\sum_n T_n$ given by $\mathsf{CDG\text{-}AC}$, let $X = \{n : d(\langle n\rangle) = \mathtt{W}\}$.

CDG-CA $\Longrightarrow$ CDG-AC: Suppose we are given a sequence $\{T_n : n \in \mathcal{N}\}$ of completely determined trees. Consider the family $\{T_{n,\sigma} : n \in \mathcal{N}, \sigma \in T_n\}$, where $T_{n,\sigma}$ is the tree $\{\tau : \sigma^\frown\tau \in T_n\}$. By CDG-CA, there is a set X such that $\forall n \forall \sigma \in T_n(\langle n, \sigma\rangle \in X \iff I$ has a winning strategy in $G(T_{n,\sigma}))$. Let $\langle d_n : n \in \mathcal{N}\rangle$ be such that for all $n \in \mathcal{N}$ and $\sigma \in T_n$, $d_n(\sigma) = \mathtt{W} \iff \langle n, \sigma\rangle \in X$ and $d_n(\sigma) = \mathtt{L}$ otherwise. Note that each d_n is a winning function for $G(T_n)$.

CDG-CA $\Longrightarrow$ JI. Let α be an ordinal and suppose that for every $\beta < \alpha$, $0^{(\beta)}$ exists. By recursive transfinite induction, we construct a family of tree games $\{G_{\beta,n} : \beta < \alpha, n \in \mathcal{N}\}$, such that $n \in 0^{(\beta)} \iff I$ has a winning strategy in $G_{\beta,n}$. For $\beta = 0$ and any n, let $T_{0,n} = \emptyset$. In the game $G(\emptyset)$ player I starts loosing, and $I\!I$ always wins. If β is a limit ordinal, then $n \in 0^{(\beta)} \iff n = \langle m_n, \gamma_n\rangle$ & $\gamma_n < \beta$ & $m_n \in 0^{(\gamma_n)}$. So, let $T_{\beta,n} = T_{\gamma_n,m_n}$, if $n = \langle m_n, \gamma_n\rangle$ & $\gamma_n < \beta$, and let $T_{\beta,n} = \emptyset$ otherwise. If $\beta = \gamma + 1$, then there exists a recursive function f such that $\forall n(n \in 0^{(\beta)} \iff \exists s(f(n,s) \notin 0^{(\gamma)}))$. Let $T_{\beta,n} = \sum_{s\in\mathcal{N}} T_{\gamma,f(n,s)}$. Then, I has a winning strategy in $T_{\beta,n}$ if and only if, for some s, $I\!I$ has a winning strategy in $T_{\gamma,f(n,s)}$, which happens if and only if $n \in 0^{(\beta)}$. Moreover, we claim that, using our assumption that for every $\beta < \alpha$, $0^{(\beta)}$ exists, we can prove that each game $T_{\beta,n}$ is completely determined: By recursive transfinite induction we define $0^{(\beta)}$-recursive indices for winning functions $d_{\beta,n}$ for $T_{\beta,n}$. Of course, for $\beta = 0$, $d_{0,n}$ is the empty function. When β is a limit ordinal we just let $d_{\beta,n} = d_{\gamma_n,m_n}$, if $n = \langle m_n, \gamma_n\rangle$ & $\gamma_n < \beta$, and let $d_{\beta,n} = \emptyset$ otherwise. When $\beta = \gamma + 1$, we let $d_{\beta,n}(\emptyset) = \mathtt{W} \iff \exists s(f(n,s) \notin 0^{(\gamma)})$ and let $d_{\beta,n}(\langle s\rangle^\frown\sigma) = d_{\gamma,f(n,s)}(\sigma)$.

So, we have a family $\{G_{\beta,n} : \beta < \alpha, n \in \mathcal{N}\}$ of completely determined games such that I wins $G_{\beta,n}$ if and only if $n \in 0^{(\beta)}$. By CDG-CA, there exits a set X such that $\langle \beta, n\rangle \in X \iff I$ wins $G(T_{\beta,n})$. This X is $0^{(\alpha)}$.

5. JI does not imply CDG-CA

This section is dedicated to prove the following theorem.

Theorem 5.1. *There is an ω-model of JI which is not a model of CDG-CA. Therefore, RCA+JI does not imply CDG-CA, and hence does not imply weak-Σ^1_1-AC$_0$ either.*

We will define a sequence $\{\langle T_i^G, d_i^G, h_i^G\rangle : i \in \omega\}$ in a generic way. Then, we will let $\mathcal{M}_\infty$ be the least ω-model closed under hyperarithmetic reduction, which contains the sequence $\{T_i^G : i \in \omega\}$ and each of the functions d_i^G. We will prove that, in $\mathcal{M}_\infty$, each T_i^G is a well-founded tree and d_i^G is a winning function for it. Even though $\mathcal{M}_\infty$ contains all the functions d_i^G, we will prove that it does not contain the sequence $\{d_i^G : i \in \omega\}$. Moreover, we will prove that it does not contain the set $\{n : d_i^G(\emptyset) = \mathtt{W}\}$. This will imply that CDG-CA does not hold in $\mathcal{M}_\infty$. To show that JI holds in $\mathcal{M}_\infty$ we will show that for every $X \in \mathcal{M}_\infty$ and ordinal α, $X^{(\alpha)} \in \mathcal{M}_\infty$ if and only if $\alpha < \omega_1^{CK}$. This will easily imply JI.

The functions h_i^G are going to be a kind of rank functions on T_i^G that we will specify later. We use them to ensure that the trees T_i^G look well-founded in $\mathcal{M}_\infty$, and to prove properties about the forcing notion.

5.1. **Ranked games.** Given a tree T, a *game rank* for T is a pair of functions $d\colon T \to \{\mathtt{L}, \mathtt{W}\}$ and $h\colon T \to \omega_1$ such that

(1) If $\sigma \in T$ and $d(\sigma) = \mathtt{L}$, then for every immediate successor τ of σ in T, $d(\tau) = \mathtt{W}$ and $h(\sigma) = \sup\{h(\tau) + 1 : \tau \in T \ \&\ \tau^- = \sigma\}$.
(2) If $\sigma \in T$ and $d(\sigma) = \mathtt{W}$, then for some immediate successor τ of σ in T, $d(\tau) = \mathtt{L}$ and $h(\sigma) = \min\{h(\tau) : d(\tau) = \mathtt{L} \ \&\ \tau^- = \sigma\}$.

Observe, that d is a winning function for $G(T)$, even when T is not well founded. By this we mean that if $d(\sigma) = \mathtt{W}$, then player I has a strategy in $G(T_\sigma)$ that will lead him to win in finitely many steps. This is because when player I moves, he always has the option to move to a node labeled $\mathtt{L}$ without increasing the ordinal label. On the other hand, player $I\!I$ is always forced to play to a node labeled $\mathtt{W}$ and with a strictly smaller ordinal label.

But, if a tree T has a game rank function, it is not necessarily well-founded. For example, consider the tree $T = \{0^n : n \in \omega\} \cup \{0^n{}^\frown 1 : n \in \omega\}$. A game rank function for T is defined as follows. Let $d(0^n{}^\frown 1) = \mathtt{L}$, $h(0^n{}^\frown 1) = 0$, $d(0^n) = \mathtt{W}$, $h(0^n) = 0$. The following condition guaranties that T is well-founded.

Definition 5.2. We say that a game rank d, h on T is *uniform* if whenever $\sigma \in T$, $d(\sigma) = \mathtt{W}$ and τ is an immediate successor of σ we have that if $d(\tau) = \mathtt{L}$, $h(\tau) = h(\sigma)$, and if $d(\tau) = \mathtt{W}$, $h(\tau) < h(\sigma)$.

Note that not every well-founded tree has a uniform game rank.

5.2. **The forcing notion.** Let $\bar{\xi}$ be a recursive ordering of order type $\omega_1^{CK} \cdot (1+\eta)$ (i.e., a Harrison linear ordering [Har68]), and let $\xi = \bar{\xi} \cup \{\infty\}$, where ∞ is a new symbol grater than all the elements of $\bar{\xi}$. We let $\infty < \infty$. We intend the functions d_i^G, h_i^G mentioned above, to act like uniform game ranks on the trees T_i^G. They will not be actual game ranks because the image of h_i^G will not be an ordinal, but ξ. The advantage of using the Harrison ordering, instead of ω_1^{CK} as Steel does in [Ste78], is that the forcing notion is then computable.

Definition 5.3. We let $I\!P$ be the forcing notion which consist of conditions p of the form $\langle\langle T_i^p, d_i^p, h_i^p\rangle : i < n^p\rangle$, where

(1) $n^p \in \omega$ and each T_i^p is a finite subtree of $\omega^{<\omega}$;
(2) $d_i^p\colon T_i^p \to \{\mathtt{L}, \mathtt{W}\}$ and $h_i^p\colon T_i^p \to \xi$;
(3) If $\sigma \in T_i^p$, $d_i^p(\sigma^-) = \mathtt{L}$ then $d_i^p(\sigma) = \mathtt{W}$, and $h_i^p(\sigma) < h_i^p(\sigma^-)$;
(4) If $\sigma \in T_i^p$, $d_i^p(\sigma^-) = \mathtt{W}$ then, if $d_i^p(\sigma) = \mathtt{L}$, $h_i^p(\sigma) = h_i^p(\sigma^-)$, and if $d_i^p(\sigma) = \mathtt{W}$, $h_i^p(\sigma) < h_i^p(\sigma^-)$;
(5) $h_i^p(\emptyset) = \infty$.

We use T^p to denote $\{\langle i, \sigma\rangle : i < n^p, \sigma \in T_i^p\}$ and d^p and h^p to denote the partial functions defined by $d^p(\langle i, \sigma\rangle) = d_i^p(\sigma)$ and $h^p(\langle i, \sigma\rangle) = h_i^p(\sigma)$. Given $p, q \in I\!P$, we let $q \leq_{I\!P} p$ if, $n^q \geq n^p$, $T^q \supseteq T^p$, $d^q \supseteq d^p$ and $h^q \supseteq h^p$ as functions. Let G be a *hyperarithmetically generic* filter. That is, G is a filter and meets every hyperarithmetic dense subset of $I\!P$. We define $\langle\langle T_i^G, d_i^G, h_i^G\rangle : i \in \omega\rangle$ in the obvious way.

Given $F \subset_f \omega$, we let $G_F = \langle T_i^G : i \in \omega\rangle \oplus \bigoplus_{j \in F} d_j^G$, and let $\mathcal{M}_F$ be the set of all sets which are hyperarithmetic in G_F. (By $F \subset_f \omega$ we mean that F is a finite subset of ω.) Let $\mathcal{M}_\infty = \bigcup_{F \subset_f \omega} \mathcal{M}_F$.

Note that being hyperarithmetically generic over $\mathbb{P}$ is a Σ^1_1 condition: It can easily be written as a formula φ of the form $(\forall X \leq_H \emptyset)\psi$, where ψ is arithmetic. The Spector-Gandy Theorem [Spe60, Gan60] says that every such φ is equivalent to a Σ^1_1 formula. So, we can take G generic such that $\omega_1^G = \omega_1^{CK}$. This is because of the Gandy low basis theorem [Sac90, Corollary III.1.5] which says that every non-empty Σ^1_1 class has a hyperarithmetically low member. (A set $Y \subseteq \omega$ is *hyperarithmetically low* if $\omega_1^Y = \omega_1^{CK}$.) Fix such a G. Therefore, every set $X \in \mathcal{M}_\infty$ is computable from $G_F^{(\alpha)}$ for some $F \subset_f \omega$ and $\alpha < \omega_1^{CK}$, and hence is of the form $\{x : \psi(x, G_F)\}$ for some computable infinitary formula ψ. See [AK00, Chapter 7] for a definition of computable infinitary formulas.

We shall prove that $\mathcal{M}_\infty \models \mathsf{JI} \,\&\, \neg\mathsf{CDG\text{-}CA}$.

5.3. **The forcing relation.** Both Steel [Ste78] and Van Wesep [Van77] used a ramified language as a forcing language, when they worked with tagged trees forcing. Instead, we use computable infinitary formulas in the language of first-order arithmetic augmented with a unary relation symbol $\cdot \in T$ and binary relation symbols $d_i(\cdot) = \cdot$, for each $i \in \omega$. For $F \subset_f \omega$, we denote the set of formulas which do not mention d_i for $i \notin F$ by $\mathcal{L}^F$. We let $\mathcal{L}^\infty = \bigcup_{F \subset_f \omega} \mathcal{L}^F$. We associate to each formula of $\mathcal{L}^\infty$ a *rank* $\alpha < \omega_1^{CK}$ defined by transfinite induction as follows: if φ is an atomic formula of arithmetic, then $\mathrm{rk}(\varphi) = 0$, $\mathrm{rk}(x \in T) = 1$, $\mathrm{rk}(d(x) = \mathtt{L}) = \mathrm{rk}(d(x) = \mathtt{W}) = 2$, $\mathrm{rk}(\forall x \psi(x)) = \mathrm{rk}(\neg\psi) = \mathrm{rk}(\psi) + 1$ and $\mathrm{rk}(\bigwedge_{i\in\omega} \psi_i) = \sup\{\mathrm{rk}(\psi_i) + 1 : i \in \omega\}$. (The motivation for the base case in the definition of rk is just to prove Lemma 5.8.)

Definition 5.4. The forcing relation for formulas of $\mathcal{L}^\infty$ is defined as usual:

(1) $p \Vdash \psi \iff \psi$ when ψ is a quantifier free formula of arithmetic;
(2) $p \Vdash \langle i, \sigma\rangle \in T$ if either $|\sigma| < 2$, or $\sigma^{--} \in T_i^p$ and $h_i^p(\sigma^{--}) \geq 1$;
(3) $p \Vdash d_i(\sigma) = \mathtt{L}$ if one of the following holds:
  - $\sigma \in T_i^p$ and $d_i^p(\sigma) = \mathtt{L}$,
  - $\sigma^- \in T_i^p$, $d_i^p(\sigma^-) = \mathtt{W}$ and $h_i^p(\sigma^-) = 0$,
  - $\sigma^{--} \in T_i^p$, $d_i^p(\sigma^{--}) = \mathtt{L}$ and $h_i^p(\sigma^{--}) = 1$;
(4) $p \Vdash d_i(\sigma) = \mathtt{W}$ if one of the following holds:
  - $\sigma \in T_i^p$ and $d_i^p(\sigma) = \mathtt{W}$,
  - $\sigma^- \in T_i^p$, $d_i^p(\sigma^-) = \mathtt{L}$ and $h_i^p(\sigma^-) > 0$;
(5) $p \Vdash \forall x \psi(x)$ if for all n, $p \Vdash \psi(n)$;
(6) $p \Vdash \bigwedge_{i\in\omega} \psi_i$ if for every i, $p \Vdash \psi_i$;
(7) $p \Vdash \neg\psi$ if for every $q \leq_{\mathbb{P}} p$, $q \nVdash \psi$.

It can be proved by induction on the formulas that $p \Vdash \psi$ if and only if whenever G is a hyperarithmetically generic filter, $p \in G$ and $\mathcal{M}_\infty$ is the model defined from G, we have that $\mathcal{M}_\infty \models \psi$. This property is what motivated the definition of $p \Vdash d_i(\sigma) = \mathtt{L}$ and $p \Vdash d_i(\sigma) = \mathtt{W}$.

Observe that for a formula ψ of rank α, $0^{(\alpha)}$ can decide whether $p \Vdash \psi$ uniformly in ψ, p and α. This can be easily proved by transfinite induction. (Actually, less than $0^{(\alpha)}$ is required.)

We are now ready to prove that $\mathcal{M}_\infty \models \mathsf{JI}$.

Lemma 5.5. *Let $\alpha \in \mathcal{M}_\infty$ be a linear ordering and $X \in \mathcal{M}_\infty$ be an $H(\emptyset, \alpha)$-set. Then α and is a well ordering and $\alpha < \omega_1^{CK}$.*

(See definition of $H(\emptyset, \alpha)$-set in Subsection 1.5.)

Proof. Since $X \in \mathcal{M}_\infty$, there exist F and $\beta < \omega_1^{CK}$ such that $X \leq_T G_F^{(\beta)}$. Suppose toward a contradiction that α is not a well ordering. Then, there is a decreasing sequence $a_0 > a_1 > a_2 > ...$ of elements of α, and we have that for every k, $X^{[a_k]} \geq_T (X^{[a_{k+1}]})'$. Then, by [Sac90, Lemma III.3.3], we have that for every recursive ordinal γ, $0^{(\gamma)} \leq_T X \leq_T G_F^{(\beta)}$ uniformly. So, there is a computable infinitary $\Sigma^0_{\beta+1}$ formula φ such that for every $\delta < \omega_1^{CK}$, $\{n : \varphi(G, \delta, n)\} = 0^{(\delta)}$. Let γ be the rank of the formula φ. We will get a contradiction by proving that $0^{(\gamma)} \geq_T 0^{(\gamma+1)}$. Let $p \Vdash \{n : \varphi(G, \gamma+1, n)\} = 0^{(\gamma+1)}$. Now, given n, recursively in $0^{(\gamma)}$ find $q \leq_{I\!P} p$ which decides $\varphi(G, \gamma+1, n)$. Then $n \in 0^{(\gamma+1)}$ if and only if $q \Vdash \varphi(G, \gamma+1, n)$.

If we had $\alpha \geq \omega_1^{CK}$, we would also have that for every recursive ordinal γ, $0^{(\gamma)} \leq_T X \leq_T G_F^{(\beta)}$ uniformly, and we would get a contradiction the same way. □

It follows from the lemma above that for $\alpha, X \in \mathcal{M}_\infty$, $X^{(\alpha)} \in \mathcal{M}_\infty$ if and only if $\alpha < \omega_1^{CK}$.

Lemma 5.6. *$\mathcal{M}_\infty$ satisfies* JI*.*

Proof. Let X and α be such that, in $\mathcal{M}_\infty$, α is an ordinal and $\forall \beta < \alpha$, $X^{(\beta)} \in \mathcal{M}_\infty$. In particular, we have that for all $\beta < \alpha$, $0^{(\beta)} \in \mathcal{M}_\infty$, and hence, by the previous lemma, $\alpha < \omega_1^{CK}$. It then follows that $X^{(\alpha)} \in \mathcal{M}_\infty$. □

5.4. **Retaggings.** The goal of this subsection is to prove that CDG-CA does not hold in $\mathcal{M}_\infty$. We need to show that in $\mathcal{M}_\infty$ all the trees T_i^G are well-founded and completely determined by d_i^G, but that the set $\{n : d_n^G(\emptyset) = \mathtt{W}\}$ is not in $\mathcal{M}_\infty$.

The next definition and lemma are key when forcing with tagged trees.

Definition 5.7. Let $p, p^* \in I\!P$, $F \subset_f \omega$ and $\alpha \in \omega_1^{CK}$. We say that p^* is an *α-F-absolute retagging* of p, and we write $\mathrm{Ret}(\alpha, F; p, p^*)$, if

(1) $n^p = n^{p^*}$, $T^p = T^{p^*}$ and for $i \in F$, $d_i^p = d_i^{p^*}$;
(2) for all $i < n^p$ and $\sigma \in T_i^p$, if $h_i^p(\sigma) < \alpha$, then $h_i^{p^*}(\sigma) = h_i^p(\sigma)$ and $d_i^{p^*}(\sigma) = d_i^p(\sigma)$; and
(3) if $h_i^p(\sigma) \geq \alpha$, then $h_i^{p^*}(\sigma) \geq \alpha$.

Lemma 5.8. *Let ψ be a formula in $\mathcal{L}^F$ of rank less than or equal to α and let $p, p^* \in I\!P$ be α-F-*absolute retaggings*. Then, $p^* \Vdash \psi$ if and only if $p \Vdash \psi$.*

Proof. The proof is by transfinite induction on α. All the cases are trivial except for $\psi = \neg\varphi$. Suppose that $p^* \Vdash \neg\varphi$; we want to show that $p \Vdash \neg\varphi$. Consider $q \leq_{I\!P} p$; we need to show that $q \nVdash \varphi$. Let $\beta < \alpha$ be the rank of φ. We claim that that there is a $q^* \leq_{I\!P} p^*$ which is a β-F-absolute retagging of q. From the claim we would get what we want because, since $p^* \Vdash \neg\varphi$, we have that $q^* \nVdash \varphi$, and hence $q \nVdash \varphi$.

Let us now prove the claim. Note that we can assume that $T^q \smallsetminus T^p$ has only one element $\langle j, \sigma\rangle$; we can then prove our claim for a general q by induction on $|T^q \smallsetminus T^p|$. Let $T^{q^*} = T^q = T^{p^*} \cup \{\langle j, \sigma\rangle\}$ and for $\tau \in T^{p^*}$, let $h_i^{q^*}(\tau) = h_i^{p^*}(\tau)$ and $d_i^{q^*}(\tau) = d_i^{p^*}(\tau)$. Now, if $\sigma = \emptyset$, let $d_j^{q^*}(\sigma) = d_j^q(\sigma)$ and $h_j^{q^*}(\sigma) = h_j^q(\sigma) = \infty$. Suppose now that $\sigma \neq \emptyset$ and let $\tau = \sigma^-$. There are two cases. The first case is $h_j^q(\sigma) < \alpha$, where we need to define $h_j^{q^*}(\sigma) = h_j^q(\sigma)$ and $d_j^{q^*}(\sigma) = d_j^q(\sigma)$. We need to verify that this definition is consistent. To do this we have to look at all the possible values of $h_j^{p^*}(\tau)$ and $d_j^{p^*}(\tau)$. All the possibilities are easy to analyze. The second case is $h_j^q(\sigma) > \alpha$. In this case we only need to worry to define $h_j^{q^*}(\sigma) \geq \beta$ and $d_j^{q^*}(\sigma)$ to be consistent with $h_j^{p^*}(\tau)$ and $d_j^{p^*}(\tau)$, which is not hard to do. □

Lemma 5.9. *The trees T_i^G, $i \in \omega$, have no infinite paths in $\mathcal{M}_\infty$.*

Proof. Suppose, toward a contradiction, that $X \in \mathcal{M}_\infty$ is a path through T_i^G. The sequence $\{h_i^G(X\restriction n) : n \in \omega\}$ is a descending sequence in ξ, and therefore, for every n, $h_i^G(X\restriction n) > \omega_1^{CK}$. There are some $F \subset_f \omega$ and formula $\varphi \in \mathcal{L}^F$ such that $(\forall n, m)(X(n) = m \iff \varphi(n, m))$. Let $\psi(k)$ be the formula that says that $\{\langle n, m\rangle : \varphi(n, m, G_F)\}$ is a path through T_i^G and that $\varphi(0, k)$ (i.e., the path starts with $\langle k\rangle$). Let α be the rank of $\psi(k)$ and let $p \in G$ force $\psi(k)$ for some $k \in \omega$. It is not hard to prove that there exists q such that $\mathrm{Ret}(\alpha, F; p, q)$ and $h_i^q(\langle k\rangle) < \omega_1^{CK}$, using the fact that $h_i^p(\langle k\rangle) > \omega_1^{CK} > \alpha$. Then, by the previous lemma, we have that $q \Vdash \psi$, which is impossible because, since $h_i^q(\langle k\rangle) < \omega_1^{CK}$, there cannot be any path though T_i starting with $\langle k\rangle$ in any model defined from a generic extension of q. □

Corollary 5.10. *In $\mathcal{M}_\infty$, $\{T_i^G : i \in \omega\}$ is a sequence of completely determined well-founded trees.*

Proof. From the definition of $\mathbb{P}$ and the fact that G is generic, we get that for every i, d_i^G satisfies (4.1) and hence is a winning function for $G(T_i^G)$. □

Lemma 5.11. *In $\mathcal{M}_\infty$, there is no set X such that $n \in X$ if and only if I has a wining strategy in the game determined by T_n.*

Proof. If such a set X existed in $\mathcal{M}_\infty$, there would be a formula $\varphi(n) \in \mathcal{L}^F$, for some $F \subset_f \omega$, such that $(\forall n)\varphi(n) \iff d_n(\emptyset) = \mathtt{W}$. Let $\alpha = \mathrm{rk}((\forall n)\varphi(n) \iff d_n(\emptyset) = \mathtt{W})$, and let $p \in G$ be such that $p \Vdash (\forall n)\varphi(n) \iff d_n(\emptyset) = \mathtt{W}$ and for some $i \in \omega \setminus F$, $d_i^p(\emptyset) = \mathtt{L}$ and $p \Vdash \neg\varphi(i)$. Such p has to exists by the genericity of G. We will get a contradiction by proving that there exists q such that $\mathrm{Ret}(\alpha, F; p, q)$ and $d_i^q(\emptyset) = \mathtt{W}$. Let q be such that $T^q = T^p$, $h^q = h^p$, and except at $\langle i, \emptyset\rangle$, $d^q = d^p$. Let $d_i^q(\emptyset) = \mathtt{W}$. Since $h_i^q(\emptyset) = \infty$, we have that $\mathrm{Ret}(\alpha, F; p, q)$. To show that $q \in \mathbb{P}$, observe that for all immediate successors σ of $\emptyset$ in T^q, $d_i^q(\sigma) = \mathtt{W}$, so condition 5.3(4) is satisfied just because ∞ is grater than any other element of ξ. □

Theorem 5.1 now follows.

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