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EXTENSIONS OF EMBEDDINGS BELOW COMPUTABLY ENUMERABLE DEGREES

ROD DOWNEY, NOAM GREENBERG, ANDREW LEWIS, AND ANTONIO MONTALBÁN

ABSTRACT. Toward establishing the decidability of the two quantifier theory of the Δ^0_2 Turing degrees with join, we study extensions of embeddings of upper-semi-lattices into the initial segments of Turing degrees determined by computably enumerable sets, in particular the degree of the halting set $\mathbf{0}'$. We obtain a good deal of sufficient and necessary conditions.

CONTENTS

1. INTRODUCTION

Since the introduction of the structure of the Turing degees $\mathcal{D}$ by Kleene and Post [KP54], one of the main interests of computability theory has been to understand its order-theoretic and algebraic properties; this pursuit was extended to many other degree structures as well. Particular attention was paid to countable classes of Turing degrees, with the ordering inherited from $\mathcal{D}$. These are usually classes which consist of the degrees of sets which are definable in arithmetic by formulas of some fixed complexity. For example, classes which were investigated extensively were the classes of computably enumerable degrees, of arithmetic degrees, and of hyper-arithmetic degrees.

In this paper we concentrate on another important such collection, which is also a principal initial segment of this structure: the upper-semi-lattice of the degrees computable from the greatest c.e. degree $\mathbf{0}'$, that we denote by $\mathcal{D}_{(\leq \mathbf{0}')}$. The sets that are computable in $\mathbf{0}'$ are the Δ^0_2-definable sets. They form a very natural class of sets, one of the reasons being that they are exactly the sets that can be computably approximated (see [Soa87, Limit Lemma III.3.3]).

Key words and phrases. Turing degrees, decidability, Extensions of embeddings.

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Extensions of embeddings. Much is known about the upper-semi-lattice $\mathcal{D}_{(\leq \mathbf{0}')}$, but we are far from having a clear understanding of what the structure really looks like. We do know it is a complicated structure; for instance we know that its theory is undecidable [Eps79, Ler83]. Moreover, its theory is one-to-one equivalent to true first order arithmetic (Shore [Sho81]). On the other hand, if we look only at existential sentences, we can decide which such sentences hold in $\mathcal{D}_{(\leq \mathbf{0}')}$: every existential sentence that does not obviously contradict the axioms of upper-semi-lattices holds in $\mathcal{D}_{(\leq \mathbf{0}')}$ (this follows from results in [KP54]). In other words, the one-quantifier theory of $\mathcal{D}_{(\leq \mathbf{0}')}$ is decidable.

In order to understand where the complexity of a certain structure lies, one natural question to ask is what fragments of its theory are decidable. It has always been the case that answers to this question, by exposing either decidability procedures or coding methods, have given us a good deal of information about the algebraic properties of the structure. In Figure 1 we show the results know so far for $\mathcal{D}_{(\leq \mathbf{0}')}$. We note that when dealing with fragments which are determined by few quantifier alterations, it makes sense to enrich the structure by functions and relations which are definable, but not by quantifier-free formulas. The work in this paper is oriented towards addressing the one question-mark left in the table: whether the two-quantifier theory of $\mathcal{D}_{(\leq \mathbf{0}')}$, expanded by adding the join (least upper bound) operation, is decidable or not.

These investigations have also been done for the whole structure $\mathcal{D}$ of the Turing degrees, for the structure $\mathcal{R}$ of the computably enumerable degrees, and for many other structures. We refer the reader to [Sho06] for a recent survey of known results.

Decidability results of $\exists$-theories and $\forall\exists$-theories are closely related to embeddability results. Given a finite relational language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{A}$, the $\exists \operatorname{Th}_{\mathcal{L}}(\mathcal{A})$ is decidable if and only if the the set of finite $\mathcal{L}$-structures $\mathcal{P}$ which embed into $\mathcal{A}$ is computable; the $\forall\exists \operatorname{Th}_{\mathcal{L}}(\mathcal{A})$ is decidable if and only if given a finite tuple of $\mathcal{L}$-structures $(\mathcal{P}, \mathcal{Q}_1, ..., \mathcal{Q}_m)$ with $\mathcal{P} \subseteq \mathcal{Q}_i$ for all $i \leqslant m$, it is decidable whether every embedding $\mathcal{P} \hookrightarrow \mathcal{A}$ has an extension $\mathcal{Q}_i \hookrightarrow \mathcal{A}$ for some i. The extensions-of-embeddings problem for $\mathcal{A}$ is the restriction of this latter problem to the case $m = 1$. Hence, in terms of computational complexity, the extensions-of-embedding problem for $\mathcal{A}$ lies between the $\exists$-theory and the $\forall\exists$-theory of $\mathcal{A}$.

For a recent survey of embeddability results in the Turing degrees see [Mon].

Definition 1.1. An *upper-semi-latice (usl)* is a partial ordering in which every pair of elements has a least upper bound. We denote the least upper bound of $\mathbf{a}$ and $\mathbf{b}$ by $\mathbf{a} \vee \mathbf{b}$. All the usls we consider will have a top element $\mathbf{1}$ and a bottom element $\mathbf{0}$. A *usl embedding* has to preserve not only the ordering, non-ordering, and join operation, but also the top and bottom elements. When we write $\mathcal{P} \subseteq \mathcal{Q}$ we mean that the the top and bottom elements of $\mathcal{P}$ and $\mathcal{Q}$ coincide, that is, the identity on $\mathcal{P}$ is a usl embedding into $\mathcal{Q}$.

Let $\mathbb{E}$ be the set of pairs of usls $(\mathcal{P}, \mathcal{Q})$, such that $\mathcal{P} \subseteq \mathcal{Q}$ and such that every usl embedding of $\mathcal{P}$ into $\langle \mathcal{D}_{(\leq \mathbf{0}')}, \leq, \vee, \mathbf{0}, \mathbf{0}' \rangle$, can be extended to an embedding of $\mathcal{Q}$ into $\langle \mathcal{D}_{(\leq \mathbf{0}')}, \leq, \vee, \mathbf{0}, \mathbf{0}' \rangle$. Thus $\mathbb{E}$ is the *extensions-of-embeddings problem* for $\mathcal{D}_{(\leq \mathbf{0}')}$.

In order to find a procedure for deciding $\forall\exists \operatorname{Th}(\mathbf{D}_{(\leq \mathbf{0}')}, \leq, \vee)$ we definitely have to start by solving the extensions-of-embeddings problem for this structure, that is, by showing that $\mathbb{E}$ is a computable set.

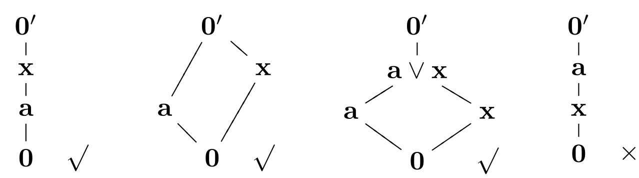
	$\exists$	$\forall\exists$	$\exists\forall\exists$
$(\mathcal{D}_{(\leq \mathbf{0}')}, \leq_T)$	decidable	decidable	undecidable
$(\mathcal{D}_{(\leq \mathbf{0}')}, \leq_T, \vee)$	decidable	**?**	undecidable
$(\mathcal{D}_{(\leq \mathbf{0}')}, \leq_T, \vee, \wedge)$	decidable	undecidable	undecidable

FIGURE 1. The decidability of $\exists\,\mathrm{Th}(\mathcal{D}_{(\leq \mathbf{0}')}, \leq_T, \vee)$ follows from the work of Kleene and Post [KP54]. The decidability of $\exists\,\mathrm{Th}(\mathcal{D}_{(\leq \mathbf{0}')}, \leq_T, \vee, \wedge)$, where $\wedge$ is the partial binary operation that is the greatest lower bound operation, follows from the Lachlan and Lebeuf lattice embedding theorem [LL76]. The undecidability of $\exists\forall\exists\,\mathrm{Th}(\mathcal{D}_{(\leq \mathbf{0}')}, \leq_T)$ is due to Schmerl and Lerman [Ler83]. The decidability of $\forall\exists\,\mathrm{Th}(\mathcal{D}_{(\leq \mathbf{0}')}, \leq_T)$ is due to Lerman and Shore [LS88]. The undecidability of $\forall\exists\,\mathrm{Th}(\mathcal{D}_{(\leq \mathbf{0}')}, \leq_T, \vee, \wedge)$, where $\wedge$ is any total binary operation that is the greatest lower bound operation when this exists, is due to R. Miller, Nies and Shore [MNS04].

Conversely, in some occasions, deciding the extensions-of-embeddings problem has been sufficient to show the decidability of $\forall\exists$-theories. This was the case with the decidability of $\forall\exists\,\mathrm{Th}(\mathcal{D}, \leq, \vee)$ by Jockusch and Slaman [JS93] and the decidability of $\forall\exists\,\mathrm{Th}(\mathcal{D}_{(\leq \mathbf{0}')}, \leq_T)$ by Lerman and Shore [LS88]. The extension-of-embedding problem for $(\mathcal{R}, \leq_T)$, proved decidable by Slaman and Soare [SS01], was the first one whose decision procedure was not trivial. This result did not produce a decision procedure for $\forall\exists\,\mathrm{Th}(\mathcal{R}, \leq)$. We expect a similar behavior for $\mathcal{D}_{(\leq \mathbf{0}')}$, in the sense that solving the extension of embeddings problem for $\mathcal{D}_{(\leq \mathbf{0}')}$ will not be enough to decide its $\forall\exists$-theory. We will give evidence for this suspicion in Subsection 1 below.

We have not yet found a decision procedure for the extension-of-embedding problem for $\mathcal{D}_{(\leq \mathbf{0}')}$. However, we have found a good deal of necessary and sufficient conditions that we expect will eventually lead to a solution of the problem. Many of the theorems we proved for this purpose are interesting in their own right, and provide a better understanding of the structure $\mathcal{D}_{(\leq \mathbf{0}')}$.

Known results. Let us start analyzing whether $(\mathcal{P}, \mathcal{Q}) \in \mathbb{E}$ for the simplest cases. Suppose $\mathcal{P} = \{\mathbf{0} < \mathbf{a} < \mathbf{1}\}$ is a 3-element chain, and that $\mathcal{Q} \supset \mathcal{P}$ is generated from $\mathcal{P}$ by adding a single element $\mathbf{x}$; we write $\mathcal{Q} = \mathcal{P}[\mathbf{x}]$. We identify $\mathcal{P}$ with its image under an embedding of $\mathcal{P}$ into $\mathcal{D}_{(\leq \mathbf{0}')}$, so $\mathbf{1} = \mathbf{0}'$. There are four different possibilities for $\mathcal{Q}$.



First, if $\mathbf{a} < \mathbf{x} < \mathbf{1}$, since $\mathbf{0}'$ is c.e. over $\mathbf{a}$, by the downwards density of the c.e. degrees (Sacks Splitting Theorem [Sac63]), we know that $\mathbf{0}'$ is not minimal over $\mathbf{a}$. Therefore, there is a degree $\mathbf{x}$ as wanted, and every embedding of $\mathcal{P}$ into $\mathcal{D}_{(\leq \mathbf{0}')}$ can be extended to an embedding of $\mathcal{Q}$ into $\mathcal{D}_{(\leqslant \mathbf{0}')}$.

Second, suppose that $\mathbf{a}$ and $\mathbf{x}$ are incomparable and $\mathbf{x} \vee \mathbf{a} = \mathbf{1}$. In this case, the extension is possible by the following theorem.

Theorem 1.2 (Robinson [Rob72], Posner and Robinson [PR81])**.** *For every degree* $\mathbf{a} < \mathbf{0}'$*, there exists* $\mathbf{x} < \mathbf{0}'$ *such that* $\mathbf{a} \vee \mathbf{x} = \mathbf{0}'$*.*

Third, comes the case where $\mathbf{a}$ and $\mathbf{x}$ are incomparable but $\mathbf{x} \vee \mathbf{a} < \mathbf{1}$. Since $\mathbf{0}'$ is c.e. over $\mathbf{a}$, there is a 1-$\mathbf{a}$-generic degree $\mathbf{x}$ below $\mathbf{0}'$ (see [Soa87, Ex. VI 3.9]). Therefore, $\mathbf{x}$ is incomparable to $\mathbf{a}$ and $\mathbf{x} \vee \mathbf{a}$ does not compute $\mathbf{0}'$ (as it does not compute any $\mathbf{a}$-c.e. set). So the extension is possible.

The last case is $\mathbf{0} < \mathbf{x} < \mathbf{a}$. In this case the extension will not be possible when $\mathbf{a}$ is a minimal degree, and we know there are minimal degrees below $\mathbf{0}'$ (Sacks [Sac61]). The analysis for this last case can be extended to a much more general setting:

Definition 1.3. If $\mathcal{P} \subseteq \mathcal{Q}$ are usls, we say that $\mathcal{Q}$ is an *end extension* of $\mathcal{P}$, if no element of $\mathcal{Q} \setminus \mathcal{P}$ lies below an element of $\mathcal{P}$, with the obvious exception of $\mathbf{1}$.

Lemma 1.4. *If* $(\mathcal{P}, \mathcal{Q}) \in \mathbb{E}$*, then* $\mathcal{Q}$ *is an end extension of* $\mathcal{P}$*.*

Proof. Suppose $\mathcal{Q}$ is not an end extension of $\mathcal{P}$. An embedding of $\mathcal{P}$, where $\mathcal{P} \setminus \mathbf{1}$ is an initial segment below $\mathbf{0}'$, would not have an extension to $\mathcal{Q}$. The existence of such an embedding of $\mathcal{P}$ was proved by Lerman and Shore [LS88]. So, $(\mathcal{P}, \mathcal{Q}) \notin \mathbb{E}$ □

There is another necessary condition for extension-of-embeddings of a different nature that follows from known results. Here is the key theorem.

Theorem 1.5 (Cooper [Coo89], Slaman, Steel [SS89])**.** *There are c.e. degrees* $\mathbf{0} < \mathbf{b} < \mathbf{c}$ *such that for no* $\mathbf{x} < \mathbf{c}$ *do we have* $\mathbf{b} \vee \mathbf{x} = \mathbf{c}$*.*

We say that $\mathbf{c}$ fails the join property, witnessed by $\mathbf{b}$. Using Jockusch and Shore's pseudo-jump inversion theorem [JS83], we obtain a c.e. degree $\mathbf{a}$, relative to which $\mathbf{0}'$ fails the join property: there is some $\mathbf{b}$, strictly between $\mathbf{a}$ and $\mathbf{0}'$, such that there is no $\mathbf{x}$ strictly between $\mathbf{a}$ and $\mathbf{0}'$ such that $\mathbf{b} \vee \mathbf{x} = \mathbf{0}'$.

Therefore, if $\mathcal{P} = \{\mathbf{0} < \mathbf{a} < \mathbf{b} < \mathbf{1}\}$ and $\mathcal{Q} = \mathcal{P}[\mathbf{x}]$ where $\mathbf{a} < \mathbf{x} < \mathbf{1}$ and $\mathbf{b} \vee \mathbf{x} = \mathbf{1}$, then $(\mathcal{P}, \mathcal{Q}) \notin \mathbb{E}$.

We will see that if $(\mathcal{P}, \mathcal{Q}) \in \mathbb{E}$, then the configuration we just described cannot appear inside $(\mathcal{P}, \mathcal{Q})$ in a sense we will specify later. We will also extend Theorem 1.5 and get other necessary conditions to have the extensions-of-embeddings property.

The $\forall\exists$-theory. When Jockusch and Slaman [JS93] proved the decidability of $\forall\exists\,\mathrm{Th}(\mathcal{D}, \leq, \vee)$, they proved that a pair or usls $\mathcal{P}$ and $\mathcal{Q}$ (with a bottom element, but without a top element) has the extensions-of-embedding property if and only if $\mathcal{Q}$ is an end extension of $\mathcal{P}$. The fact that this condition is necessary follows from the fact that any usl can be embedded as an initial segment of the Turing degrees. It then follows that given $(\mathcal{P}, \mathcal{Q}_1, ..., \mathcal{Q}_k)$ with $\mathcal{P} \subseteq \mathcal{Q}_i$, we have that every embedding of $\mathcal{P}$ into $(\mathcal{D}, \leq, \vee)$ has an extension to some $\mathcal{Q}_i$ if and only if some $\mathcal{Q}_i$ is an end extension of $\mathcal{P}$. Hence, $\forall\exists\,\mathrm{Th}(\mathcal{D}, \leq, \vee)$ is decidable. A very similar behavior occurred with Lerman and Shore's proof of the decidability of $\forall\exists\,\mathrm{Th}(\mathbf{D}_{(\leq \mathbf{0}')}, \leq_T)$. The following example shows that proving the decidability of $\forall\exists\,\mathrm{Th}(\mathbf{D}_{(\leq \mathbf{0}')}, \leq_T, \vee)$ will require more work than just solving the extensions-of-embedding problem.

Example 1.6 (Montalbán [Mon]). Let $\mathcal{P} = \{\mathbf{0} < \mathbf{a} < \mathbf{b} < \mathbf{1}\}$; let $\mathcal{Q}_1$ be the one element extension $\mathcal{P}[\mathbf{x}]$, where $\mathbf{0} < \mathbf{x} < \mathbf{a}$; let $\mathcal{Q}_2$ be the one element extension $\mathcal{P}[\mathbf{x}]$, where $\mathbf{a} < \mathbf{x} < \mathbf{1}$ and $\mathbf{x} \vee \mathbf{b} = \mathbf{1}$. See Figure 2. By our observations above, we see that there is an embedding of $\mathcal{P}$ which has no extensions to $\mathcal{Q}_1$, namely the one where $\mathbf{a}$ is a minimal degree. Also, there is an embedding of $\mathcal{P}$ which has no extension to $\mathcal{Q}_2$, namely the one we obtained by inverting the pseudo-jump operator given by the Cooper or the Slaman-Steel constructions, making $\mathbf{0}'$ fail the join property relative to $\mathbf{a}$, as witnessed by $\mathbf{b}$. However, every embedding of $\mathcal{P}$ into $\mathcal{D}_{(\leq \mathbf{0}')}$ can be extended to either $\mathcal{Q}_1$ or $\mathcal{Q}_2$: On the one hand, if $\mathbf{a}$ is not minimal, then we can extend to $\mathcal{Q}_1$; on the other hand, if $\mathbf{a}$ is minimal, then $\mathbf{a}$ is low$_2$, and hence $\mathbf{0}'$ is high over $\mathbf{a}$, and hence we can get $\mathbf{x}$ by Posner's [Pos77] join theorem. Posner's theorem is the generalization of Theorem 1.2 to any high degree in place of $\mathbf{0}'$.

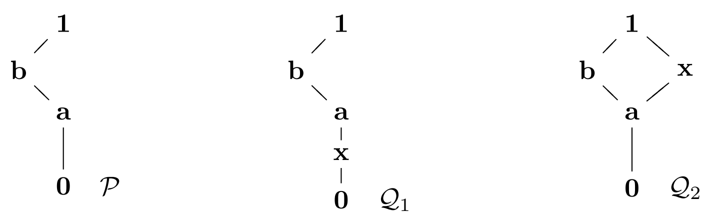


FIGURE 2. The $\forall\exists\,\mathrm{Th}(\mathcal{D}_{(\leq \mathbf{0}')}, \leq, \vee)$ is not immediately computable from the extensions-of-embeddings problem.

We show in this paper how Posner's Theorem and Theorem 1.2 can be extended to any non-low$_2$ degree, furthermore, to any non-generalized-low$_2$ degree.

Theorem 1.7. *Let $\mathbf{c}$ be a non-GL_2 degree. Then for every non-zero degree $\mathbf{a} < \mathbf{c}$ there is some $\mathbf{x} < \mathbf{c}$ such that $\mathbf{a} \vee \mathbf{x} = \mathbf{c}$.*

We will prove this theorem in Section 7, using ideas from Slaman and Steel's [SS89] uniform proof of the join theorem for $\mathbf{0}'$. We believe that this theorem, and maybe other theorems regarding non-low$_2$ degrees, will be important to solve the decidability of $\forall\exists\,\mathrm{Th}(\mathbf{D}_{(\leq \mathbf{0}')}, \leq, \vee)$, as illustrated in the following example.

Example 1.8. Let $\mathcal{P} = \{\mathbf{0} = \mathbf{a}_0 < \mathbf{b}_1 < \mathbf{a}_1 < \mathbf{b}_2 < \ldots < \mathbf{a}_n = \mathbf{1}\}$; for each $i = 1, \ldots, n$, let $\mathcal{Q}_i$ be the one element extension $\mathcal{P}[\mathbf{x}]$, where $\mathbf{a}_{i-1} < \mathbf{x} < \mathbf{a}_i$ and $\mathbf{b}_i \vee \mathbf{x} = \mathbf{a}_i$. See Figure 3, which illustrates the case $n = 3$. Then, as in the previous example, we can show that for each i, there is an embedding of $\mathcal{P}$ which has no extension to $\mathcal{Q}_i$. However, for every embedding of $\mathcal{P}$ there is some i such that the embedding extends to $\mathcal{Q}_i$. The reason is that for some i we have to have that $\mathbf{a}_i$ is non-low$_2$ over $\mathbf{a}_{i-1}$, because otherwise $\mathbf{0}' = \mathbf{a}_n$ would be low$_2$ over $\mathbf{0} = \mathbf{a}_0$. Then we apply Theorem 1.7 relative to $\mathbf{a}_{i-1}$.

Extensions below a c.e. set. In computability theory, most of the proofs are relativizable, and one would expect that the solution of the extension of embedding problem (if decidable) will also be relativizable. So, it makes sense to study the following extension-of-embeddings set, as it is possible it might end up being equal to $\mathbb{E}$.

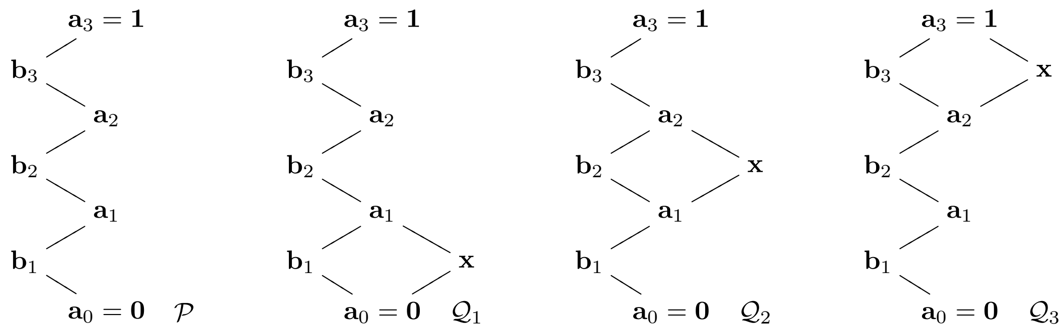


FIGURE 3. Example 1.8

Definition 1.9. Let $\mathbb{E}^{jump} = \{(\mathcal{P}, \mathcal{Q})$ finite usls: every embedding $h : \mathcal{P} \hookrightarrow \mathcal{D}$ with $h(\mathbf{1}) \equiv_T h(\mathbf{0})'$, has an extension to $\mathcal{Q} \hookrightarrow \mathcal{D}$ $\}$.

The method of pseudo-jump inversion used in Example 1.6 suggests that it will also be useful to study the extension-of-embeddings problem below any c.e. degree: essentially, given any c.e. degree $\mathbf{c}$, there is some degree $\mathbf{a} < \mathbf{0}'$ such that relative to $\mathbf{a}$, $\mathbf{0}'$ behaves like $\mathbf{c}$.

Definition 1.10. Let $\mathbb{E}^{c.e.} = \{(\mathcal{P}, \mathcal{Q})$ finite usls: every embedding $h : \mathcal{P} \hookrightarrow \mathcal{D}$ where $h(\mathbf{1})$ is c.e. in $h(\mathbf{0})$, has an extension to $\mathcal{Q} \hookrightarrow \mathcal{D}$ $\}$.

It is not hard to see that

$$(\mathcal{P}, \mathcal{Q}) \in \mathbb{E}^{c.e.} \implies (\mathcal{P}, \mathcal{Q}) \in \mathbb{E}^{jump} \implies (\mathcal{P}, \mathcal{Q}) \in \mathbb{E}.$$

The first of these implications cannot be reversed: take $\mathcal{P} = \{\mathbf{0} < \mathbf{a} < \mathbf{1}\}$ and $\mathcal{Q} = \mathcal{P}[\mathbf{x}]$ where $\mathbf{a} \vee \mathbf{x} = \mathbf{1}$. By Theorem 1.5, $(\mathcal{P}, \mathcal{Q}) \notin \mathbb{E}^{c.e.}$, but, using Therorem 1.2 we get $(\mathcal{P}, \mathcal{Q}) \in \mathbb{E}^{jump}$.

However, there is a restatement of the implication above that might be reversible. Given a usl $\mathcal{P}$, let $\mathcal{P}_*$ be $\mathcal{P} \cup \{\mathbf{0}_*\}$ were $\mathbf{0}_*$ is strictly below all the elements of $\mathcal{P}$. If $\mathbb{E}^{c.e.}, \mathbb{E}^{jump}$ and $\mathbb{E}$ are decidable, and proofs are relativizable, the pseudo-jump inversion technique would lead us to expect the following equivalence:

$$(\mathcal{P}, \mathcal{Q}) \in \mathbb{E}^{c.e.} \iff (\mathcal{P}_*, \mathcal{Q}_*) \in \mathbb{E}^{jump} \iff (\mathcal{P}_*, \mathcal{Q}_*) \in \mathbb{E}.$$

We thus believe that understanding $\mathbb{E}^{c.e.}$ is key to understand $\mathbb{E}$. The rest of the paper is dedicated to the study of $\mathbb{E}^{c.e.}$.

Necessary conditions. We have already shown that if $(\mathcal{P}, \mathcal{Q}) \in \mathbb{E}$, then $\mathcal{Q}$ is an end extension of $\mathcal{P}$. Since $\mathbb{E}^{c.e.} \subseteq \mathbb{E}$, the same holds for $\mathbb{E}^{c.e.}$. So, from now on, we will always assume that $\mathcal{Q}$ is an end extension of $\mathcal{P}$.

The other negative extension-of-embeddings result we have mentioned is Theorem 1.5. We would like to get a result saying that if $(\mathcal{P}, \mathcal{Q})$ contains a configuration similar to the one of that theorem, then $(\mathcal{P}, \mathcal{Q}) \notin \mathbb{E}^{c.e.}$.

The first extension of Theorem 1.5 that we obtain is the following one.

Theorem 1.11. *There are c.e. degrees* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *such that* $\mathbf{0} < \mathbf{a} < \mathbf{b} < \mathbf{c}$, *and for every* $\mathbf{x} \leq \mathbf{c}$, *if* $\mathbf{b} \not\leq \mathbf{x}$, *then* $\mathbf{b} \not\leq \mathbf{x} \vee \mathbf{a}$. *That is, in* $\mathcal{D}_{(\leqslant \mathbf{c})}$, *no degree non-trivially joins* $\mathbf{b}$ *above* $\mathbf{a}$.

This theorem follows from the following two lemmas, that use a notion similar to contiguity. Recall that a degree $\mathbf{a}$ is *strongly contiguous* if any two sets $A, B \in \mathbf{a}$ are weak truth-table equivalent. This notion was defined by Downey [Dow87], based on work by Ladner and Sasso [LS75]. We will not use contiguous degrees, but rather, the similar notion of a *contiguous pair*, which is a pair as in the following lemma, which we will prove in Subsection 2.1.

Lemma 1.12. *There exist c.e. sets B and C such that $\emptyset <_T B <_T C$, and such that for every set X such that $B \leq_T X \leq_T C$, we have $B \leqslant_{wtt} X$.*

Then, we will use the global-anti-cupping theorem for the weak truth-table degrees.

Lemma 1.13 (Downey [Dow87])**.** *For every noncomputable c.e. set B, there exists a noncomputable c.e. set $A <_T B$ such that for every set X,*

$$X \oplus A \geq_{wtt} B \implies X \geq_T B.$$

To prove Theorem 1.11, let B and C be the sets guaranteed by Lemma 1.12, and let A be the anti-cupping witness for B given by Lemma 1.13, and let $\mathbf{a} = \deg_T(A)$, $\mathbf{b} = \deg_T(B)$, and $\mathbf{c} = \deg_T(C)$.

Now, suppose that $\mathcal{P} \subseteq \mathcal{Q}$ are two usls, and suppose that there are $\mathbf{a} < \mathbf{b}$ in $\mathcal{P}$ and some $\mathbf{x} \in \mathcal{Q} \setminus \mathcal{P}$ which in $\mathcal{Q}$ non-trivially joins $\mathbf{a}$ above $\mathbf{b}$; that is, in $\mathcal{Q}$, $\mathbf{x} \ngeqslant \mathbf{b}$, but $\mathbf{a} \vee \mathbf{x} \geqslant \mathbf{b}$. We would like to use Theorem 1.11 to deduce that $(\mathcal{P}, \mathcal{Q}) \notin \mathbb{E}^{c.e.}$. In order to do this, we would like to find an embedding of $\mathcal{P}$ into $\mathcal{D}_{(\leqslant \mathbf{c})}$ for some c.e. degree $\mathbf{c}$, where the images of $\mathbf{a}$ and $\mathbf{b}$ satisfy the non-join property in $\mathcal{D}_{(\leqslant \mathbf{c})}$ as in Theorem 1.11; this would preclude an extension of this embedding to $\mathcal{Q}$. However, this plan is impossible if already in $\mathbb{P}$ there is some $\mathbf{y}$ which non-trivially joins $\mathbf{a}$ above $\mathbf{b}$.

Example 1.14. Suppose $\mathcal{P} = \{\mathbf{0} < \mathbf{a}_0, \mathbf{a}_1 < \mathbf{b} < \mathbf{1}\}$ where $\mathbf{a}_0 \vee \mathbf{a}_1 = \mathbf{b}$, and that $\mathcal{Q} = \mathcal{P}[\mathbf{x}]$ where $\mathbf{a}_0 \vee \mathbf{x} > \mathbf{b}$ and $\mathbf{a}_1 \vee \mathbf{x} > \mathbf{b}$. As we said above, we will not be able to get an embedding of $\mathcal{P}$ where for no $\mathbf{x} \ngeqslant \mathbf{b}$ we have that $\mathbf{a}_0 \vee \mathbf{x} > \mathbf{b}$ because $\mathbf{a}_1$ already has this property. However, by merging the proofs of the next lemma and Lemma 1.12 we can get an embedding of $\mathcal{P}$ such that for no $\mathbf{x} \ngeqslant \mathbf{b}$ we simultaneously have that $\mathbf{a}_0 \vee \mathbf{x} > \mathbf{b}$ and $\mathbf{a}_1 \vee \mathbf{x} > \mathbf{b}$, and so nonetheless we get $(\mathcal{P}, \mathcal{Q}) \notin \mathbb{E}^{c.e.}$. We will merge these proofs and other proofs in Theorem 1.17.

Lemma 1.15. *There exist Turing incomparable, disjoint c.e. sets A_0 and A_1 such that for <u>every</u> set $X \subseteq \omega$,*

$$X \vee A_0 \geq_{wtt} B \;\;\&\;\; X \vee A_1 \geq_{wtt} B \implies X \geq_T B,$$

where $B = A_0 \cup A_1$.

(We recall that $\deg_T(B) = \deg_T(A_0) \vee \deg_T(A_1)$.)

We will prove this lemma in Subsection 2.2. Of course, instead of having B split into two sets A_0 and A_1, we could split B into as many sets as we want.

We will exploit Example 1.14 and show that if a similar configuration occurs inside a pair $(\mathcal{P}, \mathcal{Q})$, then $(\mathcal{P}, \mathcal{Q}) \notin \mathbb{E}^{c.e.}$. Let us describe this in more detail.

Definition 1.16. Let $\mathcal{P}$ be an usl and $\mathcal{Q}$ be an extension of $\mathcal{P}$. We say that $(\mathcal{P}, \mathcal{Q})$ satisfies the *anti-cupping condition*, and write $(\mathcal{P}, \mathcal{Q}) \models \mathsf{ACC}$, if for every $\mathbf{x} \in \mathcal{Q} \setminus \mathcal{P}$

and for every $\mathbf{b} \in \mathcal{P}$ such that $\mathbf{b} \not\leq \mathbf{x}$, there exists $\mathbf{c} \in \mathcal{P}$ ($\mathbf{c}$ might be $\mathbf{0}$) such that $\mathbf{c} \not\geq \mathbf{b}$ and for every $\mathbf{a} < \mathbf{b}$ in $\mathbb{P}$ we have that

$$\mathbf{x} \vee \mathbf{a} \geq \mathbf{b} \implies \mathbf{c} \vee \mathbf{a} \geq \mathbf{b}.$$

In Section 3 we will prove the following theorem that says that this is a necessary condition.

Theorem 1.17. *Let $\mathcal{P}$ be a usl, and let $\mathcal{Q} = \mathcal{P}[\mathbf{x}]$ be an extension of $\mathcal{P}$ generated by a single element. If $(\mathcal{P}, \mathcal{Q}) \in \mathbb{E}^{c.e.}$, then $(\mathcal{P}, \mathcal{Q}) \models \mathsf{ACC}$.*

However, we do know that this condition is not sufficient. The reason is the following surprising theorem.

Theorem 1.18. *There exist c.e. sets A, B, C, D and E such that A, B, D and E are all Turing reducible to C and pairwise incomparable, and such that any Δ^0_2 set X which is computable in C and joins A above B also joins D above E.*

Theorem 1.18 implies that there is a pair $(\mathcal{P}, \mathcal{Q})$ such that $\mathcal{Q} = \mathcal{P}[\mathbf{x}]$ and such that $(\mathcal{P}, \mathcal{Q}) \models \mathsf{ACC}$ (and $\mathcal{Q}$ is an end-extension of $\mathcal{P}$) but $(\mathcal{P}, \mathcal{Q}) \notin \mathbb{E}^{c.e.}$. Let A, B, C, D, E be the sets given by Theorem 1.18, and let $\mathbf{a} = \deg_T(A)$, $\mathbf{b} = \deg_T(B)$, etc. Let $\mathcal{P}$ be the sub-usl of $\mathcal{D}_{(\leq \mathbf{c})}$ generated by the degrees $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$ and $\mathbf{e}$. Let $\mathcal{Q} = \mathcal{P}[\mathbf{x}]$ be such that $\mathbf{x} > \mathbf{b}$, and of course $\mathbf{x} < \mathbf{c}$, but no further inequalities hold between $\mathbf{x}$ and the elements of $\mathcal{P}$. Then certainly $\mathbf{x} \vee \mathbf{a} > \mathbf{b}$, but $\mathbf{x} \vee \mathbf{d} \not\geq \mathbf{e}$; so there is no extension of the identity embedding of $\mathcal{P}$ to an embedding of $\mathcal{Q}$ into $\mathcal{D}_{(\leq \mathbf{c})}$. $(\mathcal{P}, \mathcal{Q}) \models \mathsf{ACC}$ holds vacuously, because the only non-trivial join involving $\mathbf{x}$ is caused by $\mathbf{x} > \mathbf{b}$.

Theorem 1.18 says, in a sense, that if $X \leq_T C$ joins A above B, then there is a certain amount of information encoded in X, and this information is enough to join D above E. The natural question that follows is whether this information can compute some non-zero degree.

Question 1.19. Is there a c.e. set C, incomparable sets $A, B <_T C$ and a non-computable set E such that for any Δ^0_2 set X which is computable in C, if X joins A above B, then X computes E?

Sufficient Conditions. There are some cases where we know we always have the extensions-of-embeddings property. We start by looking at free extensions.

Definition 1.20. We say that $\mathcal{Q}$ is a *free extension* of $\mathcal{P}$ if $\mathcal{Q} = \mathcal{P}[F]$ for some finite set F, and given given $p_0 \vee \bigvee A_0$ and $p_1 \vee \bigvee A_1$ with $p_0, p_1 \in \mathcal{P}$, and $A_0, A_1 \subseteq F$, we have that

$$p_0 \vee \bigvee A_0 \leq p_1 \vee \bigvee A_1 \iff p_0 \leq p_1 \ \& \ A_0 \subseteq A_1$$

Lemma 1.21. *Every free extension belongs to $\mathbb{E}^{c.e.}$.*

To prove this lemma it is enough to consider F with one element. Because if $F = \{\mathbf{x}_1, ..., \mathbf{x}_k\}$, then $\mathcal{P}[F] = \mathcal{P}[\mathbf{x}_1][\mathbf{x}_2]....[\mathbf{x}_k]$, where each of these 1-generator extensions is free.

In the case where $\mathcal{P}$ has one element, that we discussed above, we used a 1-**a**-generic set to get the free extension and the fact that there are 1-generic degrees below any c.e. set. When $\mathcal{P}$ has more elements, we would like to get a set G, computable from $\mathbf{1}$, that is 1-generic relative to all the elements of $\mathcal{P} \setminus \{\mathbf{1}\}$. This set G is easily obtainable if $\mathcal{P} \setminus \{\mathbf{1}\}$ has a maximal element. However, we prove that we can get such a set G even if this is not the case.

Theorem 1.22. *Let C be a c.e. set and let $\{A_i : i \in \omega\}$ a uniformly C-computable list of sets. Then, there exists a set $G \leq_T C$ such that G is 1-generic relative to A_i, for every $i \in \omega$ such that $A_i <_T C$.*

Lemma 1.21 follows from Theorem 1.22 using two basic properties of 1-generic sets: 1-generic sets do not compute c.e. degrees; and if $A <_T B$ and G is 1-B-generic, then $A \oplus G \not\geq_T B$ [Joc80].

Let us now go back to the extension of Theorems 1.5 and 1.11. Suppose we have $\mathbf{0} < \mathbf{a} < \mathbf{d} < \mathbf{c}$ where $\mathbf{c}$ is c.e., and we want to get $\mathbf{x} \leq \mathbf{c}$ such that

$$\mathbf{d} \not\leq \mathbf{x} \quad \& \quad \mathbf{d} \leq \mathbf{x} \vee \mathbf{a}.$$

Theorem 1.11 tells us that we might find $\mathbf{a}, \mathbf{d}, \mathbf{c}$ such that no such $\mathbf{x}$ exists. Suppose now that we know there is some $\mathbf{b} < \mathbf{d}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{d}$. In this case we can find $\mathbf{x}$ as above just by letting $\mathbf{x} = \mathbf{b}$. However, suppose we do not want to cheat, and we want to get such an $\mathbf{x}$ that is not above $\mathbf{b}$. Can we still find $\mathbf{a}, \mathbf{b}, \mathbf{d}, \mathbf{c}$ so that there is no such $\mathbf{x}$? In other words, is it possible that $\mathbf{b}$ is the least degree below $\mathbf{c}$ such that $\mathbf{a} \vee \mathbf{b} \geq \mathbf{d}$? The answer is no.

Theorem 1.23 (No-least-join theorem)**.** *Let $\mathbf{c}$ be a noncomputable c.e. degree. Let $\mathbf{a}, \mathbf{b} < \mathbf{c}$ such that $\mathbf{a} \not\leq \mathbf{b}$ and $\mathbf{b} > \mathbf{0}$. Then there is a degree $\mathbf{x} \leq \mathbf{c}$ such that $\mathbf{a} \vee \mathbf{x} \geq \mathbf{b}$ but $\mathbf{x} \not\geq \mathbf{b}$.*

Figure 4 reflects the situation of the theorem in the particular case when $\mathbf{a} \vee \mathbf{x} < \mathbf{c}$ and $\mathbf{a}$ and $\mathbf{b}$ are incomparable.)

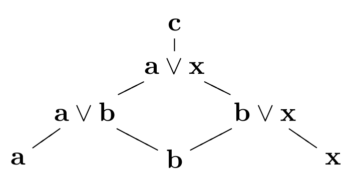


FIGURE 4. No least join

The restricted difference Filter. The questions we have raised in this paper indicate that an important object that we need to understand better is the following.

Definition 1.24. Given Turing degrees $\mathbf{b}$ and $\mathbf{a}$, we define the *difference filter* as

$$\mathbf{a} \rightarrow \mathbf{b} = \{\mathbf{x} \in \mathbf{D} : \mathbf{x} \vee \mathbf{a} \geq \mathbf{b}\}.$$

For the work in this paper, a more interesting notion is the *restricted difference filter*

$$\mathbf{a} \rightarrow_{\mathbf{c}} \mathbf{b} = \{\mathbf{x} \leq \mathbf{c} : \mathbf{x} \vee \mathbf{a} \geq \mathbf{b}\},$$

where $\mathbf{c}$ is a c.e. degree.

We call this set a *filter* just because it is closed upwards.

The following are known observations about the difference Filter. Let $\mathbf{a} \not\geq \mathbf{b}$. Then $\mathbf{a} \to \mathbf{b}$ is never an upper cone. Moreover, it always contains 1-generic degrees and minimal degrees. To see this let $\mathbf{d} = \mathbf{b} \vee \mathbf{0}'$. It follows from Slaman and Steel's proof of the join theorem for $\mathbf{0}'$ [SS89] that there are 1-generic degrees $\mathbf{x}$ such that $\mathbf{x} \vee \mathbf{a} = \mathbf{d} \geq \mathbf{b}$. Using the minimal complementation theorem [Lew05], we get a minimal degree $\mathbf{x}$ with $\mathbf{x} \vee \mathbf{a} = \mathbf{d} \geq \mathbf{b}$.

Jockusch and Slaman [JS93] proved the following result (stated in a different way):

$$\mathbf{a} \to \mathbf{b} \subseteq \mathbf{d} \to \mathbf{e} \iff \text{ either } \mathbf{e} \geq \mathbf{a} \;\&\; \mathbf{d} \geq \mathbf{e} \vee \mathbf{b}, \text{ or } \mathbf{e} \geq \mathbf{d}$$

The behavior of the restricted difference filter is rather different. Theorem 1.11 states that there are c.e. degrees $\mathbf{0} < \mathbf{a} < \mathbf{b} < \mathbf{c}$ such that $\mathbf{a} \to_{\mathbf{c}} \mathbf{b}$ is the cone of degrees above $\mathbf{b}$, of course, restricted to $\mathcal{D}_{(\leq \mathbf{c})}$. On the other hand, the no-least-join theorem 1.23 states that if $\mathbf{0} < \mathbf{a}, \mathbf{b} < \mathbf{c}$, $\mathbf{c}$ is c.e. and $\mathbf{a}|\mathbf{b}$, then $\mathbf{a} \to_{\mathbf{c}} \mathbf{b}$ is never an upper cone.

Jockusch's and Slaman's condition does not hold anymore for the restricted filter: Theorem 1.18 provided c.e. degrees $\mathbf{a}, \mathbf{b}, \mathbf{d}, \mathbf{e}$, all incomparable and below a c.e. degree $\mathbf{c}$, such that $\mathbf{a} \to_{\mathbf{c}} \mathbf{b} \subseteq \mathbf{d} \to_{\mathbf{c}} \mathbf{e}$.

Background and Notation. Our notation is standard and mostly follows [Soa87].

Many of our constructions will be organized on a tree of strategies. We assume the reader is familiar with this type of construction. See [Soa87, Chapter XIV] for background on tree constructions. Let us very quickly refresh the reader about the notation on this type of construction: Each node α in the tree of strategies is assigned a requirement $\mathbf{R}_\alpha$. These requirement have certain possible *outcomes*, and for each of these outcomes o, $\alpha^\frown o$ is another node in the tree of strategies. The idea is that each node in the tree of strategies codes the outcomes of the requirements of higher priority. The set of outcomes is linearly ordered, and this induces a lexicographic partial ordering $<_L$ on the whole tree, where nodes comparable under $\subseteq$ are incomparable under $<_L$. Nodes to the left have higher priority than nodes to the right. Also, nodes that are initial segments of other nodes have higher priority. At each stage s we will pick a node α_s in the tree to be our current *approximation to the true path*. At this stage we will act for each requirement $\mathbf{R}_\alpha$ for $\alpha \subseteq \alpha_s$. If $\alpha \subseteq \alpha_s$, we say that α is *accessible* at s, and that s is an *α-stage*. The *true path* of a construction is the leftmost path visited infinitely often. At the end, we will only worry about satisfying requirements $\mathbf{R}_\alpha$ for α in the true path.

A set G is *1-A-generic* if for every c.e. set of strings W_e, there exists $\sigma \subset G$ such that either $\sigma \in W_e^A$, or no extension of σ is in W_e^A. If G is 1-A-generic, then G does not compute any A-c.e. set and if $D <_T A$, then $D \oplus G \not\geq_T A$.

A degree $\mathbf{c}$ is GL_2 if $(\mathbf{c} \vee \mathbf{0}')' \geq \mathbf{c}''$. If f_1 is computable in $\mathbf{c} \vee \mathbf{0}'$, and $\mathbf{c}$ is not GL_2, then there exists a $\mathbf{c}$-computable function f_2 that is not dominated by f_1.

2. Almost contiguity and the global anti-cupping property

In this section we prove Lemmas 1.12 and 1.15 getting Example 1.14 as a corollary. The ideas used in the proofs will be used in later sections.

2.1. **An almost contiguous pair.** As we said in the introduction, a degree $\mathbf{a}$ is *strongly contiguous* if any two sets $A, B \in \mathbf{a}$ are *wtt*-equivalent. Downey [Dow87] used these degrees to transfer properties about the structure of *wtt*-degrees to the structure of Turing degrees. We use the same idea here. However, we do not need to use strongly contiguous degrees. We just need the following lemma.

Lemma 2.1 (Lemma 1.12)**.** *There exist c.e. sets B, A, such that $0 <_T B <_T A$ such that for every set X with $B \leq_T X \leq_T A$, we have $B \leqslant_{wtt} X$.*

Proof. We have three types of requirements:

$$\begin{aligned} \mathbf{P}_e^{B,A}: &\quad \Psi_e^B \neq A, \\ \mathbf{P}_e^{\emptyset,B}: &\quad \Psi_e \neq B, \\ \mathbf{M}_e: &\quad \Phi_e(A) = X \text{ total } \wedge \Gamma_e(X) = B \Longrightarrow B \leqslant_{\mathrm{wtt}} X. \end{aligned}$$

The requirements will be arranged on a tree of strategies as usual. To each node σ in the tree of strategies we assign a requirement $\mathbf{R}_\sigma$. If $|\sigma| = 3e$ then we let $\mathbf{R}_\sigma = \mathbf{M}_e$, if $|\sigma| = 3e+1$ let $\mathbf{R}_\sigma = \mathbf{P}_e^{B,A}$ and if $|\sigma| = 3e+2$ let $\mathbf{R}_\sigma = \mathbf{P}_e^{\emptyset,B}$.

We also need to make sure that $B \leq_T A$. To this end, every time we enumerate a number x into B, we also enumerate it into A.

2.1.1. *Requirements* $\mathbf{P}^{B,A}$. We use Friedberg-Muchnick's strategy [Fri57, Muc56]. These requirements have two possible outcomes `sat` for satisfied, and `wait` for waiting. We order the outcomes by `sat` < `wait`. They will choose a follower x and wait until $\Phi_e^B(x) \downarrow [s] = 0$. If this ever happens, they will enumerate x into A and preserve the computation by initializing weaker priority requirements that wanted to enumerate followers into B. Furthermore, every time a requirement $\mathbf{P}^{B,A}$ acts, by either appointing a follower or enumerating a follower, all the $\mathbf{P}$ requirements of weaker priority are initialized and we move on to the next stage $s+1$.

2.1.2. *Requirement* $\mathbf{P}_e^{\emptyset,B}$. They work in the same way as $\mathbf{P}^{B,A}$.

2.1.3. *Requirements* $\mathbf{M}_e$. The possible outcomes of these requirements are ∞ < `fty`. Requirement $\mathbf{M}_e$ will be monitoring the length of agreement between $\Gamma_e(\Phi_e(A))$ and B. When this length of agreement grows large enough, we take the infinite outcome; we call these stages *expansionary stages*. More precisely: suppose $\mathbf{R}_\sigma = \mathbf{M}_e$; the first σ-stage is a *σ-expansionary stage*, and every time this length of agreement goes beyond the largest number ever seen in the previous σ-expansionary stage, we have another σ-expansionary stage. At σ-expansionary stages we take the ∞ outcome and at the non-expansionary stages we take the finite outcome. There is no action performed by these requirements other than deciding what is the next node in the tree of strategies to be visited. This decision might have the effect of cancelling a whole bunch of followers, and therefore preserving A and B.

At every σ-stage we initialize all the requirements $\mathbf{R}_\tau$ for $\sigma <_L \tau$. We appoint at most one follower at every stage. We enumerate at most one element at every stage.

2.1.4. *Verifications.* The tree of strategies is finitely branching. So there is a leftmost path that we call the *true path*.

Suppose first that σ is on the true path and $\mathbf{R}_\sigma = \mathbf{P}_e^{B,A}$. Let s be a stage after which we never go to the left of σ and such none of the positive requirements $\mathbf{R}_\tau$ with $\tau \subsetneq \sigma$ act ever again. The next time $\mathbf{P}_e^{B,A}$ is active, we will choose a follower x for it, if we have not yet. This follower will never be cancelled and hence, if $\Phi_e^B(x) \downarrow = 0$, the follower will eventually go into A. When this happens, all followers of weaker priority will be cancelled. Since none of the followers of stronger priority will enter B, we have that nobody below the use of this computation will enter B.

If σ is in the true path and $\mathbf{R}_\sigma = \mathbf{P}_e^{\emptyset,B}$, a similar argument applies.

Suppose now that σ is in the true path and $\mathbf{R}_\sigma = \mathbf{M}_e$. Also, suppose that $\Phi_e(A)$ total , $X = \Phi_e(A)$ and $\Gamma_e(X) = B$. So, there will be infinitely many σ-expansionary true stages and hence $\sigma^\frown\infty$ will be in the true path. We need to show that $B \leqslant_{\rm wtt} X$. Fix n; we want to decide whether $n \in B$. Wait for the first σ-expansionary stage s where the length of agreement is larger than n. Let $v(n)$ be the X use for computing this length of agreement at that stage and let $u(n)$ be the A use for computing this length of agreement. Using use $v(n)$ we want to X-compute $B(n)$. Let $t > s$ be the least σ-expansionary stage where X is correct up to $v(n)$. We claim that $n \in B$ if and only if $n \in B_t$. If X did not change below $v(n)$ between stages s and t, then $B(n)[t] = \Gamma_e^X(n)[t] = \Gamma_e^X(n)$ did not change either and will not ever change. If it did it is because some number got enumerated into A below $u(n)$, and maybe also into B. If this number is below n, then the follower n would have been cancelled and hence never enumerated into B. On the other hand, when follower n was appointed, all the weaker priority followers were cancelled, and at the σ-expansionary stage s, all the followers to the right of the true path were cancelled again. The next follower appointed after s is greater than s, and s is greater than $u(n)$. So, nobody below $u(n)$ is enumerated into A between stages s and t without cancelling n. □

2.2. Global anti-cupping property.

Lemma 2.2 (Lemma 1.15)**.** *There exist incomparable disjoint c.e. sets A_0, A_1 such that for* every *set X,*

$$X \vee A_0 \geq_{wtt} B \ \ \& \ \ X \vee A_1 \geq_{wtt} B \implies X \geq_T B,$$

where $B = A_0 \cup A_1$.

Proof. We have three types of requirements:

$$\begin{aligned}
\mathbf{P}_e^{A_0,A_1}: &\quad \Psi_e^{A_0} \neq A_1,\\
\mathbf{P}_e^{A_1,A_0}: &\quad \Psi_e^{A_1} \neq A_0,\\
\mathbf{N}_e: &\quad \hat{\Phi}_e(A_0 \oplus X) = \hat{\Phi}_e(A_1 \oplus X) = B \Longrightarrow X \geq_T B,
\end{aligned}$$

where we use $\hat{\Phi}_e$ to denote a *wtt*-Turing functional with use φ_e. Notice that φ_e might not be total, in which case $\hat{\Phi}_e$ will not be a *wtt*-turing functional. But it will otherwise.

The requirements are ordered by priorities in some way and put in a tree of strategies.

The positive requirements $\mathbf{P}_e^{A_0,A_1}$ and $\mathbf{P}_e^{A_1,A_0}$ work exactly as $\mathbf{P}^{B,A}$ in the previous construction.

We also have that every time a requirement $\mathbf{P}^{B,A}$ acts, by either appointing a follower or enumerating a follower, all the positive requirements of weaker priority are initialized and the stage is stopped. Furthermore, every requirement to the right of the current approximation to the true path is also initialized.

2.2.1. *Requirements* $\mathbf{N}_e$. All these requirements do is to impose a restraint on the $\mathbf{P}$ requirements of weaker priority. Actually, rather than imposing restraint, these requirements will decide part of the true path and hence play a roll initializing requirements to the right of the true path.

Requirement $\mathbf{N}_e$ guesses whether the use function φ_e is total or not. So, they have two outcomes $\infty <$ `fty`; the former guessing total and the latter non-total. Given σ in the tree of strategies with $\mathbf{R}_\sigma = \mathbf{N}_e$, we define a set of *$\sigma$-expansionary stages* as we did for requirement $\mathbf{M}_e$ in the previous construction. The first σ-stage is a σ-expansionary stage. A σ-stage s is σ-expansionary if $\varphi_e[s]$ converges at the largest number ever seen in the previous σ-expansionary stage (and it also converges on all the numbers below it). At σ-expansionary stages we output ∞, and we output `fty` otherwise.

Again, at every σ stage we initialize all the requirements $\mathbf{R}_\tau$ for $\sigma <_L \tau$ and either we appoint at most one follower at every stage or enumerate at most one element.

We now show how this is enough to satisfy $\mathbf{N}_e$. Let σ be in the true path such that $\mathbf{R}_\sigma = \mathbf{N}_e$. Suppose that φ_e is total and $\hat{\Phi}_e(A_0 \oplus X) = \hat{\Phi}_e(A_1 \oplus X) = B$ for some real X. We claim that $X \geq_T B$. Wait until a stage after which σ is never injured, and after which all the requirements of higher priority had acted already. Pick $n \in \omega$; we want to X-compute $B(n)$. Wait until we have (inductively) decided $B \upharpoonright n$. Wait a bit longer until a $\sigma^\frown\infty$-stage s at which n is appointed as a follower or we know n will never be appointed as a follower. So, n has been appointed by $\tau \supseteq \sigma^\frown\infty$ and suppose that n is a follower directed to enter A_1 and not A_0. After the following $\sigma^\frown\infty$-stage s_1, we will never enumerate anything else in A_0 smaller than $\varphi_e(n)$. Notice that we know that nothing below n will ever get enumerated in A_0 because $B \upharpoonright n$ has reached it final value already and $A_0 \subseteq B$. Wait until the first $\sigma^\frown\infty$-stage $s_2 \geq s_1$ such that $\hat{\Phi}_e^{A_0 \oplus X} = B \upharpoonright n+1[s_2]$. We claim that if n has not been enumerated by then, it will not be enumerated ever. This holds because if n is enumerated in B, then, since we will never enumerate anything else in A_0 smaller than $\varphi_e(n)$, we will have $\hat{\Phi}_e^{A_0 \oplus X} \neq B \upharpoonright n+1$. Therefore, $B(n) = \hat{\Phi}_e^{A_0[s_2] \oplus X}(n)$. $\square$

3. The generalized anti-cupping condition

This section is dedicated to proving Theorem 1.17.

3.1. Upper-semi-lattice embeddings into the c.e. degrees.

We start by describing the standard method of embedding a usl in the c.e. degrees. We include the proof here because we will use a modification of it later. Also, we describe it in a way that is compatible with this modification.

Lemma 3.1 (Friedberg-Muchnik [Fri57,Muc56]). *Every finite usl can be embedded into the c.e. degrees.*

Proof. Let $\mathcal{P}$ be a finite usl. We build c.e. sets $\mathcal{A}^{\mathbf{a}}$ for each $\mathbf{a} \in \mathcal{P}$ such that the map $\mathbf{a} \mapsto A^{\mathbf{a}}$ is a usl-embedding.

Let $\mathcal{P}$ be a usl. A subset F of $\mathcal{P}$ is said to be a *filter* if it is closed upwards and whenever $\mathbf{a} \vee \mathbf{b} \in F$, at least one of $\mathbf{a}$ or $\mathbf{b}$ is in F.

When $\mathbf{a} \leq \mathbf{b}$, we need to get $A^{\mathbf{a}} \leq_T A^{\mathbf{b}}$. For this purpose, we impose the condition that whenever we enumerate some x into $A^{\mathbf{a}}$ we also enumerate it in $A^{\mathbf{b}}$. When $\mathbf{a} \vee \mathbf{b} \geq \mathbf{c}$ we want to have that $A^{\mathbf{a}} \oplus A^{\mathbf{b}} \geq_T A^{\mathbf{c}}$. For this purpose, we impose the condition that whenever we enumerate some x into $A^{\mathbf{c}}$, we also enumerate it into either $A^{\mathbf{a}}$ of $A^{\mathbf{b}}$. We summarize these two conditions into one:

(F1) Whenever some number x is enumerated into some set, there is a filter F such that x is enumerated in every set $A^{\mathbf{a}}$ for $\mathbf{a} \in F$.

It is not hard to see that if this condition is satisfied by our construction, then we will get that $A^{\mathbf{a}} \leq_T A^{\mathbf{b}}$ whenever $\mathbf{a} \leq \mathbf{b}$, and that $A^{\mathbf{a}} \oplus A^{\mathbf{b}} \geq_T A^{\mathbf{c}}$ whenever $\mathbf{a} \vee \mathbf{b} \geq \mathbf{c}$. We will only use filters of the form

$$F_{\mathbf{d}} = \{\mathbf{z} \in \mathcal{P} : \mathbf{z} \not\leq \mathbf{d}\}.$$

The next thing we need is that whenever $\mathbf{d} \not\geq \mathbf{e}$, $A^{\mathbf{d}} \not\geq_T A^{\mathbf{e}}$. Note that since $\mathcal{P}$ is an upper-semi-latice, $\mathbf{e} \vee \mathbf{d} \in \mathcal{P}$ and it is enough to get $A^{\mathbf{d}} \not\geq_T A^{\mathbf{e} \vee \mathbf{d}}$. Hence, it is enough to satisfy the following requirements. For each $\mathbf{d}, \mathbf{e} \in \mathcal{P}$ with $\mathbf{d} < \mathbf{e}$, and each $e \in \omega$, we have a requirement

$$\mathbf{P}_e^{\mathbf{d},\mathbf{e}}: \qquad \Psi_e^D \neq E.$$

These requirements are assigned priorities in some way as usual, and put into a tree of strategies.

3.1.1. *Requirements* $\mathbf{P}^{\mathbf{d},\mathbf{e}}$. These requirements work in a very similar way to the requirements in §2.1.1, except that they will enumerate their followers into a filter rather than into a single set. They have two outcomes $\mathtt{sat} < \mathtt{wait}$. Let σ be a node in the tree of strategies and $\mathbf{R}_\sigma = \mathbf{P}^{\mathbf{d},\mathbf{e}}$. The first time we are in a σ-stage since $\mathbf{R}_\sigma$ has been initialized, we appoint a follower x greater than every number seen before. At every following σ-stage s, we check whether $\Phi_e^{A^{\mathbf{d}}}(x) \downarrow [s] = 0$. If not, we output $\mathtt{wait}$, declare s to be a $\sigma^\frown\mathtt{wait}$-stage and go to the module for $\mathbf{R}_{\sigma^\frown\mathtt{wait}}$. If yes, we enumerate x into $A^{\mathbf{z}}$ for every $\mathbf{z} \in F_{\mathbf{d}}$, and in particular into $A^{\mathbf{e}}$ and stop the stage. At every following σ-stage, so long as $\mathbf{R}_\sigma$ is not initialized, we output $\mathtt{sat}$ and move directly to the module of $\mathbf{R}_{\sigma^\frown\mathtt{sat}}$.

Every time a requirement $\mathbf{P}_e^{\mathbf{d},\mathbf{e}}$ acts, by either appointing or enumerating a follower, all the $\mathbf{P}$ requirements of weaker priority are initialized.

Let us now verify that each $\mathbf{P}_e^{\mathbf{d},\mathbf{e}}$ is satisfied. Let σ be the node in the true path of the construction such that $\mathbf{R}_\sigma = \mathbf{P}_e^{\mathbf{d},\mathbf{e}}$. Since σ is in the true path, there is a first σ-stage s, after which $\mathbf{R}_\sigma$ is never initialized again. At s, $\mathbf{R}_\sigma$ will enumerate a follower x. If $\Phi_e^{A^{\mathbf{d}}}(x) \downarrow [t] = 0$ is ever true at a σ-stage t, then we will have $A^{\mathbf{e}}(x) = 1$. Since $\mathbf{R}_\sigma$ is never initialized again, nobody of higher priority than $\mathbf{R}_\sigma$ enumerates anything ever again. Since all the requirements of weaker priority than $\mathbf{R}_\sigma$ are initialized, there will be no new followers below the use of this computation. Therefore, the computation $\Phi_e^{A^{\mathbf{d}}}(x)[t] = 0$ is preserved for ever. □

3.2. **A negative condition.** In this subsection we put together the constructions of the previous three subsections and prove Theorem 1.17. So, let us consider a pair of usls $\mathcal{P} \subset \mathcal{Q}$ such that $(\mathcal{P}, \mathcal{Q}) \not\models \mathsf{ACC}$. There has to be some $\mathbf{x} \in \mathcal{Q} \setminus \mathcal{P}$ and a $\mathbf{b} \in \mathcal{P}$, $\mathbf{b} \not\leq \mathbf{x}$ such if $\mathbf{a}_1, ..., \mathbf{a}_k$ are the $\mathcal{P}$-minimal elements below $\mathbf{b}$ such that $\mathbf{a}_i \vee \mathbf{x} \geq \mathbf{b}$, then for every $\mathbf{c}$ with $\mathbf{c} \not\geq \mathbf{b}$, there exists some i such that $\mathbf{a}_i \vee \mathbf{c} \not\geq \mathbf{b}$.

We want to construct an embedding of $\mathcal{P}$ which has no extensions to $\mathcal{P}[\mathbf{x}]$. We build c.e. sets $\mathcal{A}^{\mathbf{a}}$ for each $\mathbf{a} \in \mathcal{P}$ such that the map $\mathbf{a} \mapsto A^{\mathbf{a}}$ is an usl-embedding. We use A to denote $A^{\mathbf{1}}$ and B to denote $A^{\mathbf{b}}$. Moreover, we build this embedding so that there is no $X \leq_T A$ such that $X \not\geq_T B$ and for every $i = 1, ..., k$, $X \oplus A^{\mathbf{a}_i} \geq_T B$. For this purpose, we build B and A such that for every Y, if $B \leq_T Y \leq_T A$, we have $B \leq_{wtt} Y$ and such that for every X, if for every $i = 1, ..., k$, $X \oplus A^{\mathbf{a}_i} \geq_{wtt} B$, then $X \geq_T B$.

For each $\mathbf{d}, \mathbf{e} \in \mathcal{P}$ with $\mathbf{d} < \mathbf{e}$, we have requirements:

$$\begin{aligned} \mathbf{P}_e^{\mathbf{d},\mathbf{e}} : &\quad \Psi_e^{A^{\mathbf{d}}} \neq A^{\mathbf{e}}, \\ \mathbf{M}_e : &\quad \Phi_e(A) \text{ total } \wedge \Gamma_e(\Phi_e(A)) = B \Longrightarrow B \leqslant_{\text{wtt}} \Phi_e(A), \\ \mathbf{N}_e : &\quad \forall i = 1, ..., k \ \hat{\Phi}_e(A^{\mathbf{a}_i} \oplus X) = B \Longrightarrow X \geq_T B. \end{aligned}$$

Also, if $\mathbf{d}, \mathbf{e} \in \mathcal{P}$ with $\mathbf{d} \leq \mathbf{e}$, we need to get that $A^{\mathbf{d}} \leq_T A^{\mathbf{e}}$. We will obtain this as a feature of our construction using a condition similar to (F1).

The requirements are ordered by priorities in some way and assigned nodes in a tree of strategies as usual.

3.2.1. *Requirement* $\mathbf{M}_e$. This requirement works exactly like the $\mathbf{M}$ requirements in the construction of Section 2.1. It monitors the length of agreement between $\Gamma_e(\Phi_e(A))$ and B. When this length of agreement goes beyond the largest number ever seen in the previous σ-expansionary stage, we have another σ-expansionary stage and we output ∞.

At every σ stage we initialize all the requirements $\mathbf{R}_\tau$ for $\sigma <_L \tau$. We appoint at most one follower at every stage. We enumerate at most one element at every stage.

The verification works exactly as in 2.1.4.

3.2.2. *Requirement* $\mathbf{P}_e^{\mathbf{d},\mathbf{e}}$. We relax condition (F1) a bit so we do not injure lower priority $\mathbf{N}_i$ requirements.

(F2) Whenever some number is enumerated into some set, there are two filters $F_0 \subseteq F_1$ and numbers $x_0 \leq x_1$ such that x_0 is enumerated in every set $A^{\mathbf{a}}$ for $\mathbf{a} \in F_0$ and x_1 is enumerated in every set $A^{\mathbf{a}}$ for $\mathbf{a} \in F_1 \setminus F_0$. Also, $\mathbf{b} \in F_0$ and for some i, $\mathbf{a}_i \notin \mathcal{F}_0$.

We call x_1 a *follower* and x_0 an *agitator*.

Notice that this type of permitting is still enough for our purposes: Suppose $\mathbf{a} < \mathbf{c}$ and suppose we enumerate a number x into $A^{\mathbf{a}}$. Then x is either x_0 or x_1 of condition (F2). If $x = x_0$ and $\mathbf{a} \in F_0$, then $\mathbf{c} \in F_0$, and x_0 is enumerated into $A^{\mathbf{c}}$. If $x = x_1$, and $\mathbf{a} \in F_1 \setminus F_0$, then $\mathbf{c} \in F_1$ and either x_0 or x_1 is enumerated into $A^{\mathbf{c}}$. In either case, something less than or equal to x is enumerated $A^{\mathbf{c}}$, and hence $A^{\mathbf{a}} \leq_T A^{\mathbf{c}}$. Suppose now that $\mathbf{f} \leq \mathbf{a} \vee \mathbf{c}$ and that x is enumerated into $\mathcal{A}^{\mathbf{f}}$, where x is either x_0 or x_1 of condition (F2). If $x = x_0$ and $\mathbf{f} \in F_0$, then either $\mathbf{a} \in F_0$ or $\mathbf{c} \in F_0$, and x_0 is enumerated into either $A^{\mathbf{a}}$ or $A^{\mathbf{c}}$. If $x = x_1$ and $\mathbf{f} \in F_1 \setminus F_0$, then either $\mathbf{a} \in F_1$ or $\mathbf{c} \in F_1$, and either x_0 or x_1 is enumerated into either $A^{\mathbf{a}}$ or $A^{\mathbf{c}}$. In any case, something less than or equal to x is enumerated into either $A^{\mathbf{a}}$ or $A^{\mathbf{c}}$, and hence $A^{\mathbf{f}} \leq_T A^{\mathbf{a}} \vee A^{\mathbf{c}}$.

Now we show how we can always find filters as needed for either (F1) or (F2), maybe with $F_0 = F_1$.

Lemma 3.2. *For every* $\mathbf{d} < \mathbf{e}$*, one of the following holds.*

1. *There exists a filter* F *such that* $\mathbf{d}, \mathbf{b} \notin F$*,* $\mathbf{e} \in F$*;*
2. *There exists a filter* F *such that* $\mathbf{d} \notin F$*,* $\mathbf{e}, \mathbf{b} \in F$ *but for some* i*,* $\mathbf{a}_i \notin F$*;*
3. *There exists filters* F_0 *and* F_1 *as in condition (F2) such that* $\mathbf{d} \notin F_0, F_1$*,* $\mathbf{e} \in F_1$*, and* F_0 *omits some* $\mathbf{a}_i$ *but contains* $\mathbf{b}$*.*

Proof. First, if $\mathbf{b} \vee \mathbf{d} \not\geq \mathbf{e}$, then we can let $F = \{\mathbf{z} : \mathbf{z} \not\leq \mathbf{b} \vee \mathbf{d}\}$. Second, if for some $\mathbf{a}_i$, $\mathbf{a}_i \vee \mathbf{d} \not\geq \mathbf{e}$, we can let $F = \{\mathbf{z} : \mathbf{z} \not\leq \mathbf{a}_i \vee \mathbf{d}\}$. Otherwise we go for the third option. Let F_1 be $\{\mathbf{z} : \mathbf{z} \not\leq \mathbf{d}\}$. Note that since $\mathbf{b} \vee \mathbf{d} \geq \mathbf{e} \in F_1$ and $\mathbf{d} \notin F_1$, we have $\mathbf{b} \in F_1$. Also, since $\mathbf{b} \vee \mathbf{d} \geq \mathbf{e}$, we cannot have $\mathbf{d} \geq \mathbf{b}$, because otherwise $\mathbf{d} = \mathbf{b} \vee \mathbf{d} \geq \mathbf{e} > \mathbf{d}$. Since $(\mathcal{P}, \mathcal{P}[\mathbf{x}]) \not\models \mathsf{ACC}$, there is an $\mathbf{a}_{i_0}$ such that $\mathbf{d} \vee \mathbf{a}_{i_0} \not\geq \mathbf{b}$. Let $F_0 = \{\mathbf{z} : \mathbf{z} \not\leq \mathbf{d} \vee \mathbf{a}_{i_0}\}$. □

Now we describe how $\mathbf{P}_e^{\mathbf{d},\mathbf{e}}$ acts. If (3) of the lemma above applies, let F_0 and F_1 be as given there. If either (1) or (2) apply, let $F_0 = F_1 = F$. Suppose $\mathbf{R}_\sigma = \mathbf{P}_e^{\mathbf{d},\mathbf{e}}$. The first time we are at a σ-stage after initialization we pick a large agitator x that is going to go into F_0 and we stop the stage. The second time we are at a σ-stage we pick a large follower y for $F_1 \setminus F_0$ and stop the stage. At every subsequent σ-stage we check whether $\Phi_e^B(y) \downarrow [s] = 0$. If not, we output `wait` and pass control to the module for $\mathcal{R}_{\sigma^\frown \mathtt{wait}}$. If yes, we enumerate y into $F_1 \setminus F_0$ and x into F_0 and we stop the stage. Recall that $\mathbf{d}$ does not belong to F_1, and hence nothing is enumerated in $A^{\mathbf{d}}$. At any following σ-stage we output `sat`, unless the requirement is initialized, in which case we have to start over. Every time a $\mathbf{P}$ requirement acts, by either appointing or enumerating an agitator or a follower, all the $\mathbf{P}$ requirements of weaker priority are initialized.

3.2.3. *Requirement* $\mathbf{N}_e$. These requirements work like the $\mathbf{N}_e$ requirements in § 2.2.1. The only difference is that now the $\mathbf{P}$ requirements are respecting condition (F2).

$\mathbf{N}_e$ is guessing whether the use function φ_e is total or not. So, they have two outcomes $\infty <$ `fty`; the former guessing total and the latter non-total. Given σ in the tree of strategies with $\mathbf{R}_\sigma = \mathbf{N}_e$, we define a set of *$\sigma$-expansionary stages* as before. The first σ-stage is a σ-expansionary stage. A σ-stage s is σ-expansionary if $\varphi_e[s]$ converges at the largest number ever seen in the previous σ-expansionary stage (and it also converges on all the numbers below this one). At σ-expansionary stages we output ∞, and we output `fty` otherwise. Again, at every σ-stage we initialize all the requirements $\mathbf{R}_\tau$ for $\sigma <_L \tau$ and either we appoint at most one agitator or follower at every stage or enumerate at most one element.

We now show how this is enough to satisfy $\mathbf{N}_e$. Let σ be in the true path such that $\mathbf{R}_\sigma = \mathbf{N}_e$. Suppose that φ_e is total $\forall i = 1, ..., k$ $\hat{\Phi}_e(A^{\mathbf{a}_i} \oplus X) = B$ for some real X. We claim that $X \geq_T B$. Pick $n \in \omega$; we want to X-compute $B(n)$. Wait until a stage after which σ is never injured, and after which all the requirements of higher priority had acted already. Also wait until we have decided $B \upharpoonright n$. Wait even a bit longer until a $\sigma^\frown\infty$-stage s at which n is either appointed as an agitator for B or we know n will never be appointed to potentially enter B. (Note that if n is appointed as a follower, then something smaller than or equal n would be appointed as a agitator, so we only need to worry about the case when n is an agitator.) So, n has been appointed by $\tau \supseteq \sigma^\frown\infty$ and suppose $\mathbf{a}_{i_0}$ is not in the filter enumerating n. After the following $\sigma^\frown\infty$-stage s_1, we will never enumerate anything else in $A^{\mathbf{a}_{i_0}}$ smaller than $\varphi_e(n)$. Notice that nothing below n will be enumerated into $A^{\mathbf{a}_{i_0}}$ because $B \upharpoonright n$ has already reached its final state and $A^{\mathbf{a}_{i_0}} \subseteq B$. Wait until the first $\sigma^\frown\infty$-stage $s_2 \geq s_1$ such that $\hat{\Phi}_e^{A^{\mathbf{a}_{i_0}} \oplus X} = B \upharpoonright n+1[s_2]$. We claim that if n has not been enumerated by then, it will not be enumerated ever. This holds because, if n is enumerated, then, since we will never enumerate anything else in $A^{\mathbf{a}_{i_0}}$ smaller than $\varphi_e(n)$, $\hat{\Phi}_e^{A^{\mathbf{a}_{i_0}} \oplus X} \neq B \upharpoonright n+1$. Therefore, $B(n) = \hat{\Phi}_e^{A^{\mathbf{a}_{i_0}}[s_2] \oplus X}(n)$.

4. Joins that imply other joins

In this section we prove the following theorem.

Theorem 4.1 (Theorem 1.18)**.** *There exist c.e. sets A, B, C, D and E such that A, B, D and E are all Turing reducible to C and pairwise incomparable, and such that any Δ^0_2 set X which is computable in C and joins A above B also joins D above E.*

The proof is also compatible with the usl embedding proof of Section 3. So, one could extend this theorem and get a more general necessary condition on pairs $(\mathcal{P}, \mathcal{Q})$ to have the extensions-of-embeddings property.

We have to satisfy two types of requirements. For every e let Y_e and Z_e be distinct elements of $\{A, B, D, E\}$ and let

$$\mathbf{P}_e^{Z_e,Y_e}: \qquad Y_e \neq \Psi_e(Z_e).$$

$$\mathbf{M}_i : (\Psi_i^C = X \text{ and } \Psi_i^{A,X} = B) \ \Rightarrow X \oplus D \geq_T E,$$

We shall ensure that A, B, D and E are all Turing reducible to C simply by putting $C = A \oplus B \oplus D \oplus E$. At any stage we let σ_i be the longest finite binary string which is an initial segment of $\Psi_i(C)$. The only role of C here is that, by fixing long initial segments of A, B, D and E, we can fix long initial segments of σ_i.

4.1. **Requirement $\mathbf{P}_e^{Z,Y}$.** These requirements work in a similar way to the requirements $\mathbf{P}_e^{\mathbf{d},\mathbf{e}}$ in the previous construction. Suppose $\mathbf{P}_e^{Z,Y} = \mathbf{R}_\sigma$. At the first σ-stage (after initialization) we pick a large number m that we will use as an *agitator*, to make sure we do not injure lower priority $\mathbf{M}$ requirements, and then we stop the stage. The next time we are at a σ-stage we pick an even larger follower n for Y and stop the stage. At every subsequent σ-stage we check whether $\Phi_e^Z(n)\downarrow[s] = 0$. If not, we output `wait` and pass control to the module for $\mathcal{R}_{\sigma^\frown \mathtt{wait}}$. If yes, we enumerate n into Y and m into D, unless $Z = D$, in which case we enumerate m into B. At any following σ-stage we output `sat`, unless the requirement is initialized, in which case we have to start over. Every time a $\mathbf{P}$ requirement acts, by either appointing or enumerating a follower or an agitator, all the $\mathbf{P}$ requirements of weaker priority are initialized. The verification that this strategy works is exactly as in previous constructions.

4.2. **Requirement $\mathbf{M}_i$.** This requirement works in a similar way to the requirements $\mathbf{M}$ and $\mathbf{N}$ of the previous construction. It will be monitoring the length of agreement between Ψ_i^{A,σ_i} and B, and will outcome ∞ at expansionary stages and `fty` at non-expansionary stages. That is all these requirements do.

Let us verify that they work. Suppose $\Psi_i^C = X$ and $\Psi_i^{A,X} = B$, and that $\mathbf{M}_i = \mathbf{R}_\sigma$, with σ in the true path. So, σ has ∞ outcome in the true path. Let s_0 be a stage after which we never go to the left of σ and such that by this stage all the requirements of higher priority that ever act, have already done so. Now, we want to compute $E(x)$ using X and D. Let $s_1 > s_0$ be the first $\sigma^\frown\infty$-stage after which either x has been appointed as a follower for E, or we know it will never be appointed as one. Let $s_2 > s_1$ be a $\sigma^\frown\infty$-stage such that we have already decided $E \restriction x$, D is correct up to x and X is correct up to its use computing $\Psi_i^{A,X} = B\restriction x[s_2]$. We claim that $x \in E$ if and only if $x \in E_{s_2}$. Suppose x gets enumerated into E at some later stage s_3, as the follower n of some requirement $\mathbf{P}_e^{Y,E}$. Then, the agitator

$m < n$ for that requirement enters either D or B. It cannot enter D, because we are assuming $D \upharpoonright x$ was already correct at stage s_2. So, m enters $B \upharpoonright x$. This is going to lead us to a contradiction. Since at stage s_2, X is correct up to the use of $\Psi_i^{A,X} = B \upharpoonright x[s_2]$, we have that A needs to change at some time after s_2. Let a be the least number that enters $C = A \oplus B \oplus D \oplus E$ after stage s_2 as a follower. Since $a \leq x$ is less than the use of $\Psi_i^{A,X} = B \upharpoonright x[s_2]$, a was appointed as a follower by some requirement $\mathbf{P}_e^{Y,Z}$, before stage s_2. Since it was not cancelled at s_2, we have that $\mathbf{P}_e^{Y,Z} = \mathbf{R}_\tau$ for some $\tau \supseteq \sigma^\frown\infty$. The same requirement $\mathbf{R}_\tau$ had previously chosen an agitator b to enter either D or B. Since D is correct up to x at stage s_2, we have that b had to enter B. Since $\tau \supseteq \sigma^\frown\infty$, we have that a has been picked at a $\sigma^\frown\infty$-stage following the $\sigma^\frown\infty$-stage at which b was picked. Therefore, a is larger than the use of $\Psi_i^{A,X} = B(b)$ at that stage, and also larger than the C use used to compute the part of X necessary in that computation. Since a was not cancelled by stage s_2, nobody below it has been enumerated anywhere. So, at s_2, both A and X have been preserved below a, and hence $\Psi_i^{A,X} = B(b)$ has been preserved. Moreover, A is going to be preserved for ever below a since a is the least number we enumerate in C after s_2. Also, X is correct up to a because we chose s_2 so that X was correct up to the use of $\Psi_i^{A,X} = B \upharpoonright x[s_2]$. Therefore, when b is enumerated into B we loose the agreement $\Psi_i^{A,X} = B(b)$ for ever.

5. Multi-generic

This section is dedicated to prove the following theorem.

Theorem 5.1 (Theorem 1.22)**.** *Let C be a c.e. set and let $\{A_i : i \in \omega\}$ a uniformly C-computable list of sets. Then, there exists a set $G \leq_T C$ such that G is 1-generic relative to A_i, for every i with $A_i <_T C$.*

We have to satisfy the following genericity requirements.

$\mathbf{R}_{e,i}$**:** There is some $\sigma \subset G$ such that either $\sigma \in W_e^{A_i}$ or no extension of σ is in $W_e^{A_i}$.

If such a σ exists, then, in the former case we say that $\mathbf{R}_{e,i}$ has been satisfied by *forcing inside* $W_e^{A_i}$, and in the latter case we say we have *forced outside* $W_e^{A_i}$.

During the construction we use a computable enumeration of C, and use this enumeration of C to obtain uniform computable approximations $\langle A_i[s] : s \in \omega\rangle$ of the sets A_i. The requirements are associated with levels of the tree of strategies as usual. The requirements will work roughly as follows. Suppose α is in the tree of strategies and $\mathbf{R}_{e,i} = \mathbf{R}_\alpha$. Requirement $\mathbf{R}_\alpha$ monitors $W_e^{A_i}$, looking for possible $\sigma \in W_e^{A_i}$ so that we can make σ an initial segment of G and satisfy the requirement by forcing inside $W_e^{A_i}$. But not every $\sigma \in W_e^{A_i}$ will be *eligible* to be an initial segment of G. First, we have to respect the work done by higher priority requirements, and hence these strings σ will have to extend a string that we will call $\mathbf{r}(\alpha)$ given to $\mathbf{R}_\alpha$ as its input by the requirement of immediately higher priority. Second, since we want to get $G \leq_T C$, we will only consider stings σ that are permitted by C in a sense we will specify later. Third, even if we see that $\sigma \in W_e^{A_i}$ at a stage s, our approximation A_i might change later, and σ might be removed from this set. So, the plan for $\mathbf{R}_\alpha$ is to collect, into a set $P(\alpha)$, pairs (τ, σ), where $\sigma \in W_e^\tau$, τ is a potential initial segment of A_i, and σ is eligible according to certain conditions we will specify later. Then, if we see that one of these τ is actually an initial segment of A_i, we will try to make σ an initial segment of G.

As we mentioned above, requirement $\mathbf{R}_\alpha$ will be given an input string $\mathbf{r}(\alpha)$ that $\mathbf{R}_\alpha$ must keep as an initial segment of G. The possible outcomes of $\mathbf{R}_\alpha$ are ∞ or a pair $(\tau, \sigma) \in (2^{<\omega})^2$. The outcome ∞ means that $\mathbf{R}_\alpha$ will be satisfied by forcing outside of $W_e^{A_i}$. In this case, the next requirement $\mathbf{R}_{\alpha^\frown\infty}$ will receive input $\mathbf{r}(\alpha)$ too. An outcome $(\tau, \sigma) \in (2^{<\omega})^2$ codes that $\sigma \in W_e^\tau$, that $\tau \in A_i$, and that σ is an initial segment of G. Therefore, it is coding that $\mathbf{R}_\alpha$ is being satisfied by forcing inside $W_e^{A_i}$. In this case, the next requirement $\mathbf{R}_{\alpha^\frown(\tau,\sigma)}$ will receive input σ.

Let us now describe the construction in more detail.

The tree of strategies consists of those finite strings α such that:

- For all $i < |\alpha|$, either $\alpha(i) = \infty$ or $\alpha(i) \in (2^{<\omega})^2$;
- If $i < j < |\alpha|$, $\alpha(i) = (\tau_1, \sigma_1)$ and $\alpha(j) = (\tau_2, \sigma_2)$, then $\sigma_1 \subseteq \sigma_2$.
- If $i < |\alpha|$, $\alpha(i) = (\tau, \sigma)$ and $\alpha(i)$ works for requirement $R_{e,j}$, then $\sigma \in W_e^\tau$.

The *restraint* $\mathbf{r}(\alpha)$ imposed on a node α is the union of all strings σ where for some τ and some $i < |\alpha|$, $\alpha(i) = (\tau, \sigma)$.

Each node α keeps track of a finite set of possible outcomes $P(\alpha) \subset (2^{<\omega})^2$. This is the set of outcomes which have been permitted by C.

5.1. **Construction.** At each stage s we will define an approximation α_s to the true path, where α_s is a string in the tree of strategies of length s. We will use $\mathbf{r}(\alpha_s)$ as our stage-s-approximation to G.

Stage 0. Only the root $\langle\rangle$ is accessible, so $\alpha_0 = \langle\rangle$. We do not do anything else.

Stage $s > 0$. We start by defining α_s. Suppose that we have already declared that α of length less than s is accessible at stage s. Suppose that α works for $\mathbf{R}_{e,i}$.

If s is the first α-stage greater than $|\alpha|$, then we let $\alpha^\frown\infty$ be accessible.

Otherwise, there are two possible choices:

- If there is some $(\tau, \sigma) \in P(\alpha)$ such that $\tau \subset A_i[s]$, then we let $\alpha^\frown(\tau, \sigma)$ be accessible.
- If there is no such (τ, σ) , we let $\alpha^\frown\infty$ be accessible.

We keep on defining this list of accessible nodes at s until we get to α of length s, which we call α_s. Before moving to the next stage we may also update $P(\alpha)$ for every α such that $\alpha^\frown\infty$ is accessible at s:

We add (τ, σ) to $P(\alpha)$ in case there is some $\alpha^\frown\infty$-stage $r \leqslant s$ such that:

- $\tau \subset A_i[r], A_i[s]$ and $|\tau| < r$;
- $\sigma \in W_e^\tau$;
- $\sigma \supset \mathbf{r}(\alpha)$;
- If $n \leqslant |\sigma|$ and $C \upharpoonright n\,[r] = C \upharpoonright n\,[s]$ then $\sigma \supseteq \mathbf{r}(\alpha_r) \upharpoonright n$.

The last condition says we need to have permission from C for σ to be eligible as a potential initial segment of G: If at some $\alpha^\frown\infty$-stage we see $\sigma \in W_e^\tau$ that is potentially eligible, and n is the least such that $\sigma(n) \neq \mathbf{r}(\alpha_t)(n)$, then we have to wait until something less than or equal to n gets enumerated in C to be able to enumerate $(\sigma, \tau) \in P(\alpha)$.

5.2. **Verifications.**

Lemma 5.2. *A true path exists: if α is accessible infinitely often, then either $\alpha^\frown(\infty)$ is accessible infinitely often (and for no (τ, σ) is $\alpha^\frown(\tau, \sigma)$ accessible infinitely often), or there are strings τ, σ such that from some stage, whenever α is accessible, so is $\alpha^\frown(\tau, \sigma)$.*

Proof. If for some $(\tau,\sigma) \in P(\alpha)$ we have $\tau \subset A_i$, then once the approximation to A_i settles on τ, at every α-stage will have that $\alpha^\frown(\tau,\sigma)$ is accessible. Otherwise, for every $(\tau,\sigma) \in P(\alpha)$, there will be a point after which $\alpha^\frown(\tau,\sigma)$ is never accessible. Therefore, for every given s, there is a later stage t that is not $\alpha^\frown(\tau,\sigma)$ accessible for any $(\tau,\sigma) \in P(\alpha)[s]$. So, there has to be some $\alpha^\frown\infty$-stage after s, for every given s. Hence $\alpha^\frown(\infty)$ is accessible infinitely often and for no (τ,σ) is $\alpha^\frown(\tau,\sigma)$ accessible infinitely often. □

We define

$$G = \bigcup\{\mathbf{r}(\alpha) : \alpha \in \text{true path}\}.$$

(To show that G is actually an infinite string one can use the fact that for every n there exists an e such that for every X, $W_e^X = 2^{>n}$, and we will always be able to force inside this set.)

Lemma 5.3. *Suppose that $\mathbf{R}_{e,i}$ fails. Then $A_i \geqslant_T C$.*

Proof. Let α on the true path work for $\mathbf{R}_{e,i}$. So $\alpha^\frown\infty$ is on the true path as well, because otherwise we would have forced inside $W_e^{A_i}$ and satisfy $\mathbf{R}_{e,i}$.

Suppose that at an $\alpha^\frown\infty$-stage r, there is some $\tau \subset A_i[r], A_i$ and some $\sigma \supseteq \mathbf{r}(\alpha_r) \upharpoonright n, \mathbf{r}(\alpha)$ in W_e^τ. Then we know that $C \upharpoonright n = C \upharpoonright n[r]$, because otherwise at some later stage we would add (τ,σ) to $P(\alpha)$.

On the other hand, there are infinitely many such stages (with arbitrarily large n) because $W_e^{A_i}$ is dense around G. We can then use A_i to find these stages and compute C. □

The main difficulty in proving that G is computable from C comes from the fact that the true path is not necessarily computable from C. This is the most interesting part of the proof. The global idea, is that to compute $G \upharpoonright n$ using C, we will wait until a stage s where we are sure that for every stage $t > s$, $\mathbf{r}(\alpha_s) \upharpoonright n = \mathbf{r}(\alpha_t) \upharpoonright n$, and hence $G \upharpoonright n = \mathbf{r}(\alpha_s) \upharpoonright n$.

Definition 5.4. We say that a stage s is *n-correct* if, for every $\beta \subset \alpha_s \upharpoonright n$ the following conditions hold, where $\gamma = \mathbf{r}(\alpha_s \upharpoonright n)$ and $|\beta| = \langle e,i\rangle$:

1. $C \upharpoonright |\gamma| = C_s \upharpoonright |\gamma|$.
2. If there exists $(\tau,\sigma) \in P(\beta)[s]$ with $\tau \subseteq A_i$, then $\forall t \geq s$, $\tau \subseteq A_i[t]$, and $\beta^\frown(\tau,\sigma) \subseteq \alpha_s$.
3. If there is no such (τ,σ), then for every $(\tau,\sigma) \in P(\beta)[s]$,
   (a) either $\sigma \supseteq \gamma$,
   (b) or $\forall t \geq s$, $\tau \not\subseteq A_i[t]$.

Since C computes the sets A_i, and the approximation to these sets is built using these C-computations, C can decide whether a stage s is n-correct or not. We now have to prove that such stages exists, and that if s is n-correct then $\mathbf{r}(\alpha_s \upharpoonright n)$ is an initial segment of G.

Lemma 5.5. *Suppose that s is n-correct. Then for every $t \geq s$, $\mathbf{r}(\alpha_t \upharpoonright n) = \mathbf{r}(\alpha_s \upharpoonright n)$.*

Proof. Let $\gamma = \mathbf{r}(\alpha_s \upharpoonright n)$. By induction on $i \leq n$ we prove the following statement:

Then for every $(\alpha_s \upharpoonright n - i)$-stage $t \geq s$, $\mathbf{r}(\alpha_t) \supseteq \gamma$.

Note that this is enough, since every t is an $\langle\rangle$-stage, and $\langle\rangle = \alpha_s \restriction n - n$. For the induction basis $i = 0$, it is clear that at every $(\alpha_s \restriction n)$-stage t we have $\alpha_s \restriction n \subseteq \alpha_t$, and hence $\mathbf{r}(\alpha_t) \supseteq \mathbf{r}(\alpha_s \restriction n)$. For the induction step, let $\beta = \alpha_s \restriction n - i$, and suppose we have proved the claim for $\alpha_s \restriction n - i + 1$. Let $t \geq s$ be a β-stage and let $\langle e, i\rangle = |\beta|$. Suppose first there is no $(\tau, \sigma) \in P(\beta)[t]$ with $\tau \subseteq \mathcal{A}_i[t]$, and that t is a $\beta^\frown\infty$-stage. Then, there was no such (τ, σ) at stage s either, because s is n-correct. So, $\beta^\frown\infty \subseteq \alpha_s$ and hence t is an $\alpha_s \restriction n - i + 1$-stage. But then, by the induction hypothesis, $\mathbf{r}(\alpha_t) \supseteq \mathbf{r}(\alpha_s \restriction n)$. Second, suppose that we have $(\tau, \sigma) \in P(\beta)[t]$ with $\tau \subseteq \mathcal{A}_i[t]$ and that t is a $\beta^\frown(\tau, \sigma)$-stage. If s was also a $\beta^\frown(\tau, \sigma)$-stage, then, by the induction hypothesis, $\mathbf{r}(\alpha_t) \supseteq \mathbf{r}(\alpha_s \restriction n)$. So, suppose $\beta^\frown\infty \subseteq \alpha_s$. If (τ, σ) was already in $P(\beta)[s]$, since $\tau \subseteq \mathcal{A}_i[t]$, we have to have that $\sigma \supseteq \gamma$ where $\gamma = \mathbf{r}(\alpha_s \restriction n)$, and hence $\mathbf{r}(\alpha_t) \supseteq \sigma \supseteq \gamma$. If not, we could have enumerated (τ, σ) into $P(\beta)$ at some $\beta^\frown\infty$-stage s_1 between s and t. As we mentioned above, at every $\beta^\frown\infty$-stage s_1 we have $\mathbf{r}(\alpha_{s_1}) \supseteq \gamma$. Since $C \restriction |\gamma| = C_s \restriction |\gamma|$, we could have only enumerated (τ, σ) into $P(\beta)[s_1]$ if $\sigma \supseteq \mathbf{r}(\alpha_{s_1}) \restriction |\gamma| = \gamma$. Therefore $\mathbf{r}(\alpha_t) \supseteq \sigma \supseteq \gamma$ as wanted. □

Lemma 5.6. *For every n there exists an s that is n-correct.*

Proof. Let α be the node of length n that is in the true path. Let $\gamma = \mathbf{r}(\alpha)$.

By induction on $i \leq n$ we prove the following statement:

> There exists a stage s_i such that for every $(\alpha \restriction n - i)$-stage $t \geq s_i$, $\mathbf{r}(\alpha_t) \supseteq \gamma$.

For $i = 0$, this is clearly true. For convenience, let s_0 be such that $C \restriction |\gamma| = C_{s_0} \restriction |\gamma|$. Let $\beta = \alpha \restriction n - i$ and suppose we have showed the claim for $i - 1$. Let $\langle e, i\rangle = |\beta|$. If $\beta^\frown(\tau, \sigma) \subseteq \alpha$, then, let $s_i > s_{i-1}$ be such that for every $t \geq s_i$, $\tau \subseteq A_i[t]$. Then, every β-stage $t \geq s_1$ is a $\beta^\frown(\tau, \sigma)$-stage, and hence by the induction hypothesis, $\mathbf{r}(\alpha_t) \supseteq \gamma$. Suppose now that $\beta^\frown\infty \subseteq \alpha$. Let $s_i > s_{i-1}$ be such that for every $(\tau, \sigma) \in P(\beta)[s_{i-1}]$

(1) either $\sigma \supseteq \gamma$,
(2) or $\forall t \geq s_i$, $\tau \not\subseteq A_i[t]$.

By the induction hypothesis we know that at every $\beta^\frown\infty$-stage t we have $\mathbf{r}(\alpha_t) \supseteq \gamma$. Therefore, since $C \restriction |\gamma| = C_t \restriction |\gamma|$, at every $\beta^\frown\infty$-stage $t \geq s_{i-1}$, every new pair (τ, σ) that is enumerated into $P(\beta)[t]$ has to have $\sigma \supseteq \gamma$. Hence, if $t \geq s_i$ is a $\beta^\frown(\tau, \sigma)$-stage for some $(\tau, \sigma) \in P(\beta)[t]$, then $\mathbf{r}(\alpha_t) \supseteq \sigma \supseteq \gamma$. This finishes our induction.

It is not hard to see from the construction above, that we also have that for $\beta = (\alpha \restriction n - i)$ we have that for every β-stage $t \geq s_i$ the conditions of Definition 5.4 hold. So, every α-stage s after s_n is n-correct. □

To compute G, all we need to do is use C to find n-correct stages s and then we know that $\mathbf{r}(\alpha_s \restriction n) \subseteq G$.

6. No least join

Theorem 6.1 (Theorem 1.23)**.** *Let A and B be Δ^0_2 sets; assume that B is non-computable and that B does not compute A. Suppose that C is a c.e. set which computes both A and B. Then there is some $X \leqslant_T C$ which does not compute B, but such that $B \leqslant_T A \oplus X$.*

6.1. **Discussion.** The heart of the argument is its reliance on non-uniformity. That is, we build two sets X_0 and X_1, one of which will satisfy the conditions required of the set X of the theorem. To explain why we are driven to this non-uniformity, we first describe a naïve, uniform plan, and show the problem with carrying it out.

So, suppose that we computably approximate a Δ^0_2 set X such that $B \not\leqslant_T X$ but $B \leqslant_T A \oplus X$. To ensure that $B \not\leqslant_T X$ we meet, for each e, the requirement

R_e**:** $\Psi_e(X) \neq B$,

where $\langle \Psi_e \rangle$ is an effective enumeration of all Turing functionals. Here is a plan to meet a single requirement R_e. Let $\langle A_s \rangle$ and $\langle B_s \rangle$ computable approximations for A and B, respectively.

Suppose that the work we do to meet $R_{e'}$ for $e' < e$ certifies that $\alpha \subset A$, $\beta \subset B$, and forces us to make $\xi \subset X$ for some finite binary strings α, β and ξ; for simplicity, assume that $|\alpha| = |\beta|$. Let $n = |\alpha|$.

Suppose that for a long time, we see that $A_s(n) = i$, and that $B_s\restriction_{n+1}$ has also been stable for a while. A *certification* of $A(n) = i$ will come in the form of a pair of strings σ and τ, possible initial segments of X, which both extend ξ, and which form a Ψ_e*-splitting*: $\Psi_e(\sigma) \perp \Psi_e(\tau)$. If we do not find such a splitting, then we argue that R_e is met without work; for if $\Psi_e(X) = B$, the assumption that B is not computable will ensure the existence of such a splitting.

Given such certification, we can make a computable promise, that if $B_s\restriction_{n+1} \subset B$, then $A(n) = i$. For then we have two possibilities: if $A(n) \neq i$, then a subsequent change in $A_s\restriction_{n+1}$ will force a change in C, which will allows us to ensure that either $\sigma \subset X$ or $\tau \subset X$, choosing so that we can make $\Psi_e(X) \perp B$ and meet the requirement R_e (while placing only finitely much restraint on weaker requirements). And if $A(n) = i$, then we record this fact, and move to try to compute $A(n+1)$ with oracle B in the same fashion. Since we assume that $A \not\leqslant_T B$, this process must stop after finitely many iterations.

Now here is the difficulty. To ensure that $B \leqslant_T A \oplus X$, we enumerate a Turing functional Γ and intend that $\Gamma(A, X) = B$. The challenge is to keep Γ a *consistent* functional. Suppose that we discover a Ψ_e-splitting σ and τ, and receive permission to direct X to extend, say σ. But in a past life, a different requirement, possibly weaker than R_e, has asked X to extend σ, and has enumerated an axiom into Γ which is compatible with σ and the new version of A, but which outputs a pre-historic, and incorrect, version of B. This would make σ ineligible for usage, and derail our strategy.

Indeed, we are not familiar with any direct way to prevent this occurrence. This is where the non-uniformity gives a way to overcome the problem. We now build X_0 and X_1. We ensure that for both $j < 2$, we have $X_j \leqslant_T C$ and $B \leqslant_T A \oplus X_j$; and we make sure that either $B \not\leqslant_T X_0$ or $B \not\leqslant_T X_1$. Using a Posner-style trick, it is sufficient to meet the following requirements:

R_e**:** either $\Psi_e(X_0) \neq B$ or $\Psi_e(X_1) \neq B$.

To meet R_e, we follow the naïve strategy, threatening to compute A from B, but this time we certify A-configurations using pairs of Ψ_e-splittings, one for each X_j. Whenever we get permission to use one such split for X_j and diagonalize, we direct the other X_{1-j} through a fresh string not used before. This will eventually allow us to argue that whenever we discover a pair of splittings, one of them will be eligible for us to use and would not destroy the consistency of the corresponding functional Γ_j.

6.2. **Preliminaries.** As mentioned above, we are given computable approximations $\langle A_s \rangle$ and $\langle B_s \rangle$ for A and B respectively.

6.2.1. *The role of C.* the fact that C is computably enumerable means that we may assume, by choosing appropriate computable approximations for A and B, that C can compute a modulus function for A and B. Specifically, we let, for $n < \omega$, $m(n)$ be the least stage s such that for all $t \geqslant s$, $A_t\restriction_{n+1}= A\restriction_{n+1}$ and $B_t\restriction_{n+1}= B\restriction_{n+1}$. We pick approximations $\langle A_s \rangle$ and $\langle B_s \rangle$ so that $m \leqslant_T C$. From now, we avoid all reference to C, and use the oracle m instead.

The function $m\colon \omega \to \omega$ has a left-c.e. approximation. For $s < \omega$, let m_s be the stage s approximation for m: $m_s(n)$ is the greatest $\bar{s} \leqslant s$ such that for all $t \in [\bar{s}, s]$, we have $A_t\restriction_{n+1}= A_s\restriction_{n+1}$ and $B_t\restriction_{n+1}= B_s\restriction_{n+1}$. The approximation being "left-c.e." means that if $s < t$, $\mu \subset m_s$ and $\mu \subset m_t$, then $\mu \not\subset m$, indeed $\mu \not\subset m_u$ for any $u \geqslant t$; for all s, m_s lies lexicographically to the left of m_{s+1}. We have $m_s\restriction_n\neq m_{s+1}\restriction_n$ if and only if either $A_s\restriction_n\neq A_{s+1}\restriction_n$ or $B_s\restriction_n\neq B_{s+1}\restriction_n$.

6.2.2. *Functionals.* Rather than defining X_0 and X_1 directly, we enumerate Turing functionals Ξ_0 and Ξ_1, and in the end, let, for $j < 2$, $X_j = \Xi_j(m)$. We mention now that these functionals need not be, strictly speaking, consistent; but we will show that $\Xi_0(m), \Xi_1(m) \in 2^\omega$. See the discussion in Section 6.2.7.

We also enumerate functionals Γ_0 and Γ_1, and will show that for both $j < 2$, $B = \Gamma_j(A, X_j)$; these functionals will be consistent.

For axioms, we use the following notation: $\mu \mapsto \eta$ means that if μ is an initial segment of the oracle, then the output extends η. As Γ_0 and Γ_1 use two oracles, the axioms will be of the form $(\sigma, \eta) \mapsto \beta$, where σ is an initial segment of the first oracle, η is an initial segment of the second, and β is an initial segment of the output.

We thus think, set-theoretically, of Γ_0 and Γ_1 as binary relations, and denote, for $j < 2$, $\operatorname{dom}\Gamma_j$, to be the collection of pairs (σ, τ) such that there is some axiom $(\sigma, \eta) \mapsto \beta$ in Γ_j.

At stage s, we let $\Xi_{0,s}, \Xi_{1,s}, \Gamma_{0,s}$ and $\Gamma_{1,s}$ be the collection of axioms enumerated into the corresponding functionals by the *end* of stage s.

We note that as the intended oracle of Ξ_j is m, at every stage s, we only enumerate into Ξ_j axioms of the form $\mu \mapsto \eta$ where $\mu \subset m_s$.

6.2.3. *Procedures.* Recall that we try to meet the requirements

R_e**:** Either $\Psi_e(X_0) \neq B$, or $\Psi_e(X_1) \neq B$.

To work toward meeting a requirement R_e, we will, from time to time, appoint a *procedure* p. The procedure may, at some times, *call* another procedure to work on the next requirement; see Section 6.2.6. The procedure may also be cancelled at a later stage.

The following pieces of information are attached to a procedure.

(1) A number e_p; this is the index of the requirement toward meeting which the procedure p works.
(2) A (computable) function α_p. This functions records the attempt to compute A from B; setting $\sigma = \alpha_p(\beta)$ denotes that p has obtained certification that if $\beta \subset B$, then $\sigma \subset A$. We will ensure that for all $\beta \in \operatorname{dom}\alpha_p$, $|\alpha_p(\beta)| = |\beta|$.

(3) A finite string μ_p. The procedure p guesses that $\mu_p \subset m$. Since $\langle m_s \rangle$ is left-c.e., whenever the guess seems incorrect, it is verified to be incorrect, and the procedure is cancelled.
(4) The domain of α_p will in fact be a tree of binary strings, with extensions being one-bit extensions. The root of this tree will be a string denoted by ρ_p; we will ensure that $|\rho_p| = |\mu_p|$. If $\beta \in \operatorname{dom} \alpha_p$ is not ρ_p, then $\beta^- = \beta\restriction_{|\beta|-1}$, the immediate predecessor of β in $2^{<\omega}$, is also in $\operatorname{dom} \alpha_p$.
(5) Two binary strings $\eta_{p,0}$ and $\eta_{p,1}$. The procedure p declares that if it is never cancelled, then $\eta_{p,0} \subset X_0$, and $\eta_{p,1} \subset X_1$.
(6) Suppose that $\beta \in \operatorname{dom} \alpha_p$ and $\beta \neq \rho_p$. Then the reduction $\sigma = \alpha_p(\beta)$ must be certified. This certification comes in the following form: we define a number $k_p(\beta)$ and some $j_p(\beta) < 2$. For every binary string $\delta \supset \beta$ of length $k_p(\beta)$, we appoint a binary string $\zeta_p(\delta)$. The meaning of this is that if $\beta \subset B$ and $\alpha_p(\beta) \not\subset A$, and β is minimal with respect to this property, then we want $\zeta_p(\delta) \subset X_{j_p(\beta)}$, where δ is the unique extension of β of length $k_p(\beta)$ which is an initial segment of B. This will be useful for diagonalization, because we will require that $\Psi_{e_p}(\zeta_p(\delta)) \perp \delta$. The choice of $j_p(\beta)$ indicates that the strings $\zeta_p(\delta)$ are eligible to be initial segments of $X_{j_p(\beta)}$, and that such an appointment would not make the functional $\Gamma_{j_p(\beta)}$ inconsistent.

We remark that it seems that for definiteness, we should have used the notation $\zeta_{p,\beta}(\delta)$, rather than merely $\zeta_p(\delta)$. The point is that we shall ensure that if $\beta, \beta' \in \operatorname{dom} \alpha_p$ are not ρ_p, then $k_p(\beta) \neq k_p(\beta')$; indeed, $k_p(\beta)$ will be chosen large at the stage at which β is added to $\operatorname{dom} \alpha_p$, and at most one string is added to $\operatorname{dom} \alpha_p$ at each stage. Thus $|\delta| = k_p(\beta)$ determines β, and there is no overlap between $\delta \supset \beta$ of length $k_p(\beta)$ and $\delta' \supset \beta'$ of length $k_p(\beta')$, and so no ill-definedness for the expression $\zeta_p(\delta)$.

6.2.4. *Some notation for strings.* For binary strings σ and τ, we let $\sigma \subseteq \tau$ denote that σ is an initial segment of τ, and $\sigma \subset \tau$ denote that σ is a proper initial segment of τ. We let $\sigma \perp \tau$ denote that σ and τ are incomparable.

For a non-empty string σ, we let $\neg(\sigma)$ be the string of length $|\sigma|$ which agrees with σ on all but the last bit.

6.2.5. *Free extensions.* For any stage s, let $\#(s)$ be the largest number used or observed at stage s. We may assume that for all s, $\#(s) < \#(s+1)$. A number is called *large* at stage s if it is greater than $\#(s-1)$.

Whenever we choose a string $\eta_{p,j}$ or $\zeta_r(\delta)$ (as a potential initial segment of some X_j), we always choose a string whose last digit is 1.

Definition 6.2. Let $s < \omega$, and let η be any string. A *free extension* of η at stage s is any string of the form $\eta^\frown 0^k 1$, where $k > \#(s-1)$.

Free extensions are useful because no old axioms apply to them:

Lemma 6.3. *If η is a free extension of $\bar{\eta}$ at stage s, and a string ζ was chosen as a potential initial segment of some X_j (that is, as some $\eta_{p,j}$ or $\zeta_r(\delta)$) at a stage $t < s$, then $\zeta \subseteq \eta$ implies that $\zeta \subseteq \bar{\eta}$.*

Proof. Since $\bar{\eta} \subset \eta$, if $\zeta \subseteq \eta$ then $\zeta \not\perp \bar{\eta}$. Suppose that the lemma fails; then $\bar{\eta} \subset \zeta$. But $|\zeta| \leqslant \#(s-1)$, and the last bit of ζ is 1, so if $k > \#(s-1)$ we must have $\zeta \perp \bar{\eta}^\frown 0^k$. Then $\zeta \perp \eta$, contradicting our assumption. □

Hence, if η is a free extension of $\bar{\eta}$ at stage s, then for any σ, we have $\Gamma_{j,s-1}(\sigma,\eta) = \Gamma_{j,s-1}(\sigma,\bar{\eta})$.

6.2.6. *The combined tree of procedures; activity.* At the end of every stage, there will be finitely many procedures running. These will be nested: apart from p_0, the first procedure (which is never cancelled), every procedure is called by the previous procedure on the list. We write $p < q$ to denote that p appears before q on the list of procedures, and thus has stronger priority than q. We maintain this nested structure of procedures by following two rules:

- If a procedure p is cancelled at a stage s, then every procedure weaker than p is also cancelled at stage s.
- Only the weakest procedure running is allowed to call a new procedure.

Say that a procedure p called a procedure q. This calling will be done for the benefit of one particular string $\beta \in \operatorname{dom}\alpha_p$; the idea is that q guesses that β witnesses the success of the procedure p to meet its requirement R_{e_p}. We will call p the *mother* of q and β the *father* of q; and call q the *child* of p (and β). To sum up this information up, we write $\texttt{parents}(q) = (p,\beta)$.

The following will be useful notation. We let $\mathbb{P}_s$ be the collection of pairs (p,β) such that p is a procedure which is running at the end of stage s, and such that $\beta \in \operatorname{dom}\alpha_p$ at the end of stage s. [1]

On $\mathbb{P}_s$ we put a partial ordering, which we also denote by $<$. This is the transitive closure of the following two cases:

(1) If $\beta \subset \gamma$, then $(p,\beta) < (p,\gamma)$;
(2) If q is not the strongest procedure, then $\texttt{parents}(q) < (q,\rho_q)$.

The fact that the priority ordering linearly orders $\operatorname{dom}\mathbb{P}_s$, the collection of procedures which are running at the end of stage s, and that ancestry implies stronger priority, implies that $<$ is indeed a partial ordering on $\mathbb{P}_s$. The fact that $p < q$ iff p can be obtained from q by a sequence of ancestries, implies that the image of $<$ under the projection map $(p,\beta) \mapsto p$ is exactly the priority ordering on $\operatorname{dom}\mathbb{P}_s$.

In fact, the map $p \mapsto (p,\rho_p)$ is an order-preserving injection of $\operatorname{dom}\mathbb{P}_s$ into $\mathbb{P}_s$. So we sometimes identify p with (p,ρ_p).

We let $\mathbb{Q}_s$ be the collection of pairs (p,β) in $\mathbb{P}_s$ such that $\beta \subset B_s$ and $\alpha_p(\beta) \subset A_s$. Recall that β guesses that $\beta \subset B$, and β and p guess together that $\alpha_p(\beta) \subset A$. Thus $\mathbb{Q}_s$ is the collection of pairs $(p,\beta) \in \mathbb{P}_s$ which seem, at stage s, to be guessing correctly.

6.2.7. *The pseudo-consistency of* Ξ_j. Strictly speaking, the functionals Ξ_0 and Ξ_1 need not be consistent: it is possible, for example, that at some stage t, we enumerate an axiom $m_t\restriction_k \mapsto \zeta$ into Ξ_j ($j < 2$), and at a later stage s such that $m_s\restriction_k \neq m_t\restriction_k$, we enumerate an axiom $m_s\restriction_n \mapsto \zeta'$ into the same Ξ_j, with $n < k$, $m_s\restriction_n \subset m_t\restriction_k$, and $\zeta' \perp \zeta$.

The point is that the approximation $\langle m_s \rangle$ of m is left-c.e.; since m_s moved to the right of $m_t\restriction_k$, we know that $m_t\restriction_k$ is not an initial segment of m, and so at stage s we can consider the axiom $m_t\restriction_k \mapsto \zeta$ as if it's been discarded from Ξ_j. Formally, what we will actually want is that $\Xi_j(m) \in 2^\omega$.

[1] For any t we let $\mathbb{P}_{<t} = \bigcup_{s<t}\mathbb{P}_s$, $\mathbb{P}_{<\omega} = \bigcup_s \mathbb{P}_s$, etc.

Lemma 6.4. *Let $j < 2$. Suppose that for all s, $\Xi_{j,s}(m_s) \in 2^{<\omega}$. Then $\Xi_j(m) \in 2^{\leqslant\omega}$.*

Proof. Standard, for those familiar with working with left-c.e. oracles. Say for $i < 2$, both axioms $m\restriction_{n_i}, \mapsto \zeta_i$ are in Ξ_j. For sufficiently late s, for both $i < 2$, $m\restriction_{n_i} \subset m_s$, and the axiom $m\restriction_{n_i} \mapsto \zeta_i$ is in $\Xi_{j,s}$. Thus $\zeta_0, \zeta_1 \subseteq \Xi_{j,s}(m_s)$, and hence are comparable. □

We will need to prove that $\Xi_{j,s}(m_s) \in 2^{<\omega}$ during the verification (Section 6.4.5). However, we will need this consistency in order for the instructions of the construction to make sense. Circularity will be avoided by a step-by-step induction.

> To perform stage s of the construction,
> we assume that $\Xi_{j,s-1}(m_{s-1}) \in 2^{<\omega}$.

Lemma 6.5. *If $\Xi_{j,s-1}(m_{s-1}) \in 2^{<\omega}$, then $\Xi_{j,s-1}(m_s) \in 2^{<\omega}$.*

Proof. Since the approximation $\langle m_s\rangle$ is left-c.e., and since at all stages $t < s$, we only enumerate axioms of the form $m_t\restriction_n \mapsto \zeta$ into Ξ_j, we have

$$\Xi_{j,s-1}(m_s) \subseteq \Xi_{j,s-1}(m_{s-1}).$$ □

For brevity, we let, for $n < \omega$, $\xi_{j,t,n} = \Xi_{j,t}(m_t\restriction_n)$ and $\xi^*_{j,t,n} = \Xi_{j,t-1}(m_t\restriction_n)$. The proof of Lemma 6.5 shows:

Lemma 6.6. *If $\Xi_{j,s-1}(m_{s-1}) \in 2^{<\omega}$, then for all n, $\xi_{j,s-1,n} \in 2^{<\omega}$ and $\xi^*_{j,s,n} \in 2^{<\omega}$; indeed $\xi^*_{j,s,n} \subseteq \xi_{j,s-1,n}$.*

6.3. **Construction.**

At stage 0, we call the first procedure p_0, set $e_{p_0} = 0$, $\mu_{p_0} = \langle\rangle$, let $\rho_{p_0} = \langle\rangle$, and define $\alpha_p(\langle\rangle) = \langle\rangle$, and $\eta_{p_0,0} = \eta_{p_0,1} = \langle\rangle$.

Let $s > 0$. Stage s consists of three steps.

1. Cancelling procedures. We cancel every procedure p such that $\mu_p \not\subset m_s$.

2. Extending trees. Let $p \in \operatorname{dom}\mathbb{P}_{s-1}$ be a procedure which is still running, i.e., was not cancelled at step 1, and suppose that it is not the weakest such procedure; let q be the child of p. Let β be the longest initial segment of B_s in $\operatorname{dom}\alpha_p$ such that $\alpha_p(\beta) \subset A_s$.[2] Let $\beta^+ = B_s\restriction_{|\beta|+1} = \beta^\frown B_s(|\beta|)$, and let $\alpha^+ = A_s\restriction_{|\beta|+1} = \alpha_p(\beta)^\frown A_s(|\beta|)$. Let $j < 2$ be such that there are no $\sigma \supseteq \neg(\alpha^+)$ and $\zeta \subseteq \eta_{q,j}$ such that $(\sigma, \zeta) \in \operatorname{dom}\Gamma_{j,s-1}$.[3]

Let k be large. The procedure p requires attention if $(p, \beta^+) \notin \mathbb{P}_{s-1}$, and for every $\gamma \supset \beta^+$ of length k we can find a long string $\zeta_\gamma \supset \eta_{q,j}$ such that $\Psi_{e_p}(\zeta_\gamma) \perp \gamma$.

If there is a procedure which requires attention, let p be the strongest one. We then do the following:

(1) Set $k_p(\beta^+) = k$ and $j_p(\beta^+) = j$; for every $\gamma \supset \beta^+$ of length k, let $\zeta_p(\gamma) = \zeta_\gamma$.

[2] β is the longest string such that (p, β) is in both $\mathbb{P}_{s-1}$ and $\mathbb{Q}_s$, unless p is cancelled at stage s. Lemma 6.12 ensures the existence of β.

[3] Corollary 6.37 ensures the existence of such j. We do not need to assume the lemma holds for the construction to make sense; if there were no such j, p simply would not require attention.

(2) Define $\alpha_p(\beta^+) = \alpha^+$.
(3) Cancel all procedures weaker than p; end stage s.

3. Calling a new procedure. If no procedure requires attention, let p be the weakest procedure running. Define β and β^+ as in step 2.

We define an integer k as follows:

- If $(p, \beta^+) \notin \mathbb{P}_{s-1}$, let t be the least stage such that $(p, \beta) \in \mathbb{P}_t$, and for all $u \in [t, s]$, $(p, \beta) \in \mathbb{Q}_u$; let $k = \#(t) + 1$.[4]
- Otherwise, we let $k = k_p(\beta^+)$.

Let $\gamma = B_s\restriction_k$. We call a new procedure q, weaker than p. We set it up as follows:

(1) $e_q = e_p + 1$.
(2) $\mu_q = m_s\restriction_k$.
(3) $\texttt{parents}(q) = (p, \beta)$.
(4) $\rho_q = \gamma$ and $\alpha_q(\gamma) = A_s\restriction_k$.
(5) If $(p, \beta^+) \notin \mathbb{P}_{s-1}$, then for both $j < 2$, we let $\eta_{q,j}$ be some free extension of $\xi^*_{j,s,k}$.
(6) If $(p, \beta^+) \in \mathbb{P}_{s-1}$, then for $j = j_p(\beta^+)$ we let $\eta_{q,j}$ be some free extension of $\zeta_p(\gamma)$; for $j \neq j_p(\beta^+)$, we let $\eta_{q,j}$ be some free extension of $\xi^*_{j,s,|\beta|}$.

We also enumerate new axioms: for both $j < 2$, we enumerate the axiom $\mu_q \mapsto \eta_{q,j}$ into Ξ_j, and the axiom $(\alpha_q(\rho_q), \eta_{q,j}) \mapsto \rho_q$ into Γ_j.

6.4. **Verification.** We fix a stage s, and assume that for all $t < s$, for both $j < 2$, $\Xi_{j,t}(m_t) \in 2^{<\omega}$. Hence the construction up to and including stage s can be performed.

6.4.1. *Basic facts.*

Lemma 6.7. *p_0 is never cancelled.*

Proof. We always have $\langle\rangle \subset m_s$. □

Lemma 6.8. *For all $(p, \beta) \in \mathbb{P}_s$, if $\beta \neq \rho_p$, then $k_p(\beta) > |\beta|$.*

Proof. $k_p(\beta)$ is chosen to be large, at a stage $t \leqslant s$ such that $(p, \beta) \in \mathbb{P}_{t-1}$. Hence $k_p(\beta) > \#(t-1) \geqslant |\beta|$. □

Lemma 6.9. *Suppose that $q \neq p_0$ and $q \in \operatorname{dom} \mathbb{P}_s$. Then the length of ρ_q is greater than the length of the father of q.*

Proof. Let $(p, \beta) = \texttt{parents}(q)$; let $t \leqslant s$ be the stage at which q is called; let $k = |\rho_q|$. If $(p, \beta^+) \in \mathbb{P}_{t-1}$, then $k = k_p(\beta^+) > |\beta^+| > |\beta|$ by Lemma 6.8. Otherwise, $k > \#(u)$ where $u \leqslant t$ is a stage such that $(p, \beta) \in \mathbb{P}_u$; so $\#(u) \geqslant |\beta|$. □

Corollary 6.10. *If $(p, \beta), (q, \gamma) \in \mathbb{P}_s$ and $(p, \beta) < (q, \gamma)$, then $\beta \subset \gamma$.*

Proof. We show that if β is the father of q then $\beta \subset \rho_q$; the rest is immediate. So let $q \in \operatorname{dom} \mathbb{P}_s$, $q \neq p_0$, and let $(p, \beta) = \texttt{parents}(q)$. Let $t \leqslant s$ be the stage at which q is called. At stage t, we define $\rho_q \subset B_t$. We also have $\beta \subset B_t$. Since $|\beta| < |\rho_q|$ (Lemma 6.9), we get $\beta \subset \rho_q$. □

Lemma 6.11. *If $p, q \in \operatorname{dom} \mathbb{P}_s$ and $p < q$, then $\mu_p \subset \mu_q$.*

[4] If $t = s$, then instead of $\#(s)$, we take the greatest number seen so far in the construction.

Proof. Let $t \leqslant s$ be the stage at which q is called. Since $p \in \operatorname{dom} \mathbb{P}_t$, we have $\mu_p \subset m_t$; and we have $\mu_q \subset m_t$. We have $|\mu_p| = |\rho_p|$ and $|\mu_q| = |\rho_q|$; by Corollary 6.10, $\rho_p \subset \rho_q$, so $|\rho_p| < |\rho_q|$. ☐

6.4.2. *Comparability of* α*'s, and* $\mathbb{Q}_s$.

Lemma 6.12. *If* $p \in \operatorname{dom} \mathbb{P}_s$*, then* $(p, \rho_p) \in \mathbb{Q}_s$.

Proof. For $p = p_0$, we always have $\langle\rangle \subset B_s$ and $\langle\rangle \subset A_s$.

Suppose that $p \neq p_0$. Let $t \leqslant s$ be the stage at which p was called; let $k = |\rho_p|$. At stage t, we set $\rho_p = B_s\restriction_k$ and $\alpha_p(\rho_p) = A_s\restriction_k$; and we set $\mu_p = m_s\restriction_k$. Suppose that the lemma fails; let u be the least stage in $(t, s]$ such that $\rho_p \not\subset B_u$ or $\alpha_p(\rho_p) \not\subset A_u$. Then either $B_{u-1}\restriction_k \neq B_u\restriction_k$ or $A_{u-1}\restriction_k \neq A_u\restriction_k$. In either case, we get $m_{u-1}\restriction_k \neq m_u\restriction_k$. Hence $\mu_p \not\subset m_u$; so p is cancelled at step 1 of stage u, contradicting $p \in \operatorname{dom} \mathbb{P}_s$. ☐

Lemma 6.13. *If* $(p, \beta), (q, \gamma) \in \mathbb{P}_s$ *and* $(p, \beta) < (q, \gamma)$*, then* $\alpha_p(\beta) \subset \alpha_p(\gamma)$.

Proof. We prove two cases:

(1) If $\texttt{parents}(q) = (p, \beta)$, then $\alpha_p(\beta) \subset \alpha_q(\rho_q)$;
(2) If β^+ is an immediate successor of β in $\operatorname{dom} \alpha_p$, then $\alpha_p(\beta) \subset \alpha_p(\beta^+)$.

For (1), let $t \leqslant s$ be the stage at which q was called. We have $(p, \beta) \in \mathbb{Q}_t$, so $\alpha_p(\beta) \subset A_t$. We also define $\alpha_q(\rho_q) = A_t\restriction_{|\rho_q|}$. Since $|\alpha_p(\beta)| = |\beta|$ and $|\rho_q| > |\beta|$ (Lemma 6.9), we get the desired result.

For (2), let $t \leqslant s$ be the stage at which β^+ was added to $\operatorname{dom} \alpha_p$. Again, we have $\alpha_p(\beta) \subset A_t$ and define $\alpha_p(\beta^+) \subset A_t$; and we have $|\alpha_p(\beta)| = |\beta|$, whereas $|\alpha_p(\beta^+)| = |\beta^+| = |\beta| + 1$. ☐

Corollary 6.14. *If* $(q, \gamma) \in \mathbb{Q}_s$*,* $(p, \beta) \in \mathbb{P}_s$*, and* $(p, \beta) < (q, \gamma)$*, then* $(p, \beta) \in \mathbb{Q}_s$.

Proof. By Corollary 6.10, $\beta \subset \gamma$. By Lemma 6.13, $\alpha_p(\beta) \subset \alpha_q(\gamma)$. Since $\gamma \subset B_s$ and $\alpha_q(\gamma) \subset A_s$, we get $\beta \subset B_s$ and $\alpha_p(\beta) \subset A_s$. ☐

Thus, $\mathbb{Q}_s$ is a linearly ordered initial segment of $\mathbb{P}_s$.

Lemma 6.15. *Suppose that* $q \in \operatorname{dom} \mathbb{P}_s$*,* $q \neq p_0$*. Let* $(p, \beta) = \texttt{parents}(q)$*. Then* β *is the longest string such that* $(p, \beta) \in \mathbb{Q}_s$.

Proof. By Corollary 6.14, because $\texttt{parents}(q) < (q, \rho_q)$ and $(q, \rho_q) \in \mathbb{Q}_s$ (Lemma 6.12), we get that $\texttt{parents}(q) \in \mathbb{Q}_s$.

Maximality is proved by induction on the stages since the stage $t \leqslant s$ at which q was called. This maximality holds by design at stage t. Suppose that $s > t$, and that maximality holds at stage $s - 1$. If the corollary fails at stage s, then $(p, \beta^+) \in \mathbb{Q}_s$, where $\beta^+ = B_s\restriction_{|\beta|+1}$.

If $(p, \beta^+) \notin \mathbb{P}_{s-1}$, then q would be cancelled at step 2 of stage s, which we assume is not the case. Hence $(p, \beta^+) \in \mathbb{P}_{s-1} \setminus \mathbb{Q}_{s-1}$. The fact that $(p, \beta^+) \in \mathbb{Q}_s \setminus \mathbb{Q}_{s-1}$ implies that $m_s\restriction_{|\beta^+|} \neq m_{s-1}\restriction_{|\beta^+|}$. Since $|\beta| < |\rho_q|$ (Lemma 6.9) and $|\beta^+| = |\beta| + 1$, we get $m_s\restriction_{|\rho_q|} \neq m_{s-1}\restriction_{|\rho_q|}$, which would imply that q gets cancelled at step 1 of stage s. This is a contradiction. ☐

6.4.3. *Comparability of η's.*

Lemma 6.16. *Let $t < s$. If $p \in \operatorname{dom}\mathbb{P}_t$, then for both $j < 2$, $\eta_{p,j} \subseteq \xi_{j,t,|\rho_p|}$.*

Proof. For $p = p_0$ this is immediate, so assume that $p \neq p_0$. Let $u \leqslant t$ be the stage at which p is called. At stage u, we enumerate the axiom $\mu_p \mapsto \eta_{p,j}$ into Ξ_j. Since p is not cancelled at stage t, we have $\mu_p \subset m_t$, so since $|\mu_p| = |\rho_p|$, $\eta_{p,j} \subseteq \Xi_{j,t}(m_t\restriction_{|\rho_p|}) = \xi_{j,t,|\rho_p|}$. □

Lemma 6.17. *If $p \in \operatorname{dom}\mathbb{P}_{s-1}, \operatorname{dom}\mathbb{P}_s$, then for both $j < 2$, $\eta_{p,j} \subseteq \xi^*_{j,s,|\rho_p|}$.*

Proof. The assumption implies that $m_s\restriction_{|\rho_p|} = m_{s-1}\restriction_{|\rho_p|} = \mu_p$. Hence $\xi^*_{j,s,|\rho_p|} = \xi_{j,s-1,|\rho_p|}$. So the conclusion follows from Lemma 6.16. □

Lemma 6.18. *Let $p, q \in \operatorname{dom}\mathbb{P}_s$, and suppose that $p < q$. Then for both $j < 2$, $\eta_{p,j} \subset \eta_{q,j}$.*

Proof. We prove the lemma by induction on s. Assuming the lemma holds at all stages $t < s$, we need to prove that if q is called at stage s, and p is the mother of q, then for both $j < 2$ we have $\eta_{p,j} \subset \eta_{q,j}$.

Let β be the father of q. There are two cases.

The first case is when $\eta_{q,j}$ is chosen to be a free extension of $\xi^*_{j,s,|\beta|}$ or of $\xi^*_{j,s,|\rho_q|}$. We note that $(p, \beta) \in \mathbb{P}_{s-1}$, so $p \in \operatorname{dom}\mathbb{P}_{s-1}, \operatorname{dom}\mathbb{P}_s$, so by Lemma 6.17, $\eta_{p,j} \subseteq \xi^*_{j,s,|\rho_p|}$. Of course $\xi^*_{j,s,|\rho_p|} \subseteq \xi^*_{j,s,|\beta|}, \xi^*_{j,s,|\rho_q|}$.

The second case is when $\eta_{q,j}$ is chosen to be a free extension of $\zeta_p(\rho_q)$. Let $\beta^+ = \rho_q\restriction_{|\beta|+1} = B_s\restriction_{|\beta|+1}$; in this case we have $(p, \beta^+) \in \mathbb{P}_{s-1}$. Let $t < s$ be the stage at which β^+ was added to $\operatorname{dom}\alpha_p$, and let $\bar{q}$ be p's child at stage t. By induction, we have $\eta_{p,j} \subset \eta_{\bar{q},j}$. Chosen at stage t, $\zeta_p(\rho_q)$ is required to extend $\eta_{\bar{q},j}$, so $\zeta_p(\rho_q)$ extends $\eta_{p,j}$. Again we get the desired result. □

6.4.4. *What happens when $q \in \operatorname{dom}\mathbb{P}_{<s} \setminus \operatorname{dom}\mathbb{P}_s$ but $\mu_q \subset m_s$.* Recall that we are assuming that for all $t < s$, $\Xi_{j,t}(m_t) \in 2^{<\omega}$. To complete the induction at stage s, we need an extra piece of information.

Lemma 6.19. *For all $p \in \operatorname{dom}\mathbb{P}_s$, $\xi_{j,s,|\rho_p|} = \eta_{p,j}$.*

We assume that the lemma holds at every $t < s$.

In this section, we investigate a particular scenario. Suppose that $t < s$, $q \in \operatorname{dom}\mathbb{P}_t$ and $q \notin \operatorname{dom}\mathbb{P}_s$, but $\mu_q \subset m_s$.

Let r be the weakest procedure in $\operatorname{dom}\mathbb{P}_t$ which is in $\operatorname{dom}\mathbb{P}_s$ (of course r exists, because $p_0 \in \operatorname{dom}\mathbb{P}_t, \operatorname{dom}\mathbb{P}_s$). Since $q \notin \operatorname{dom}\mathbb{P}_s$, we have $r < q$. Let $\bar{q}$ be r's child at stage t, and let β be $\bar{q}$'s father. So $\bar{q} \leqslant q$, and $\bar{q}$ is cancelled at some stage $u \in (t, s]$.

By Lemma 6.11, $\mu_{\bar{q}} \subseteq \mu_q$. Hence $\mu_{\bar{q}} \subset m_s$; since $\mu_{\bar{q}} \subset m_t$, we must have $\mu_{\bar{q}} \subset m_u$. It follows that $\bar{q}$ was cancelled not at step 1 of stage u, but because at stage u, some string is added to $\operatorname{dom}\alpha_r$.

By Lemma 6.15, β is the maximal string such that $(r, \beta) \in \mathbb{Q}_t$. Because $|\beta| < |\rho_q|$ (Corollary 6.10, as $(p, \beta) < (q, \rho_q)$), we get $m_t\restriction_{|\beta|+1} = m_u\restriction_{|\beta|+1} = m_s\restriction_{|\beta|+1}$. It follows that β is the longest string such that $(p, \beta) \in \mathbb{P}_{u-1}, \mathbb{Q}_u$. Hence, at stage u, the string $\beta^+ = B_u\restriction_{|\beta|+1}$ is added to $\operatorname{dom}\alpha_r$. By the definition of $\alpha_r(\beta^+)$, we have $(r, \beta^+) \in \mathbb{Q}_u$. Since $m_u\restriction_{|\beta|+1} = m_s\restriction_{|\beta|+1}$, we get $(r, \beta^+) \in \mathbb{Q}_s$.

Claim 6.20. If a procedure p is called at stage s, then $|\rho_p| > |\rho_q|$

Proof. This is proved by induction on $s > u$. Note that at stage u, no new procedure is called. Now suppose that p is called at stage s. Let $\bar{p}$ be the child of r at stage s; so $\bar{p} \leqslant p$. If $\bar{p} < p$, then, since $\bar{p}$ was called after stage u but before stage s, we have, by induction, $|\rho_{\bar{p}}| > |\rho_q|$. By Corollary 6.10, we have $\rho_{\bar{p}} \subset \rho_p$, so we're done in this case.

Suppose then that r is p's mother; let δ be p's father. δ is the longest string such that $(r, \delta) \in \mathbb{Q}_s$. Since $(r, \beta^+) \in \mathbb{Q}_s$, we have $\beta^+ \subseteq \delta$.

Let $\delta^+ = B_s \upharpoonright_{|\delta|+1}$, and let $k = |\rho_p|$. If $(r, \delta^+) \in \mathbb{P}_{s-1}$, then $k = k_r(\delta^+)$. Since δ^+ is added to $\operatorname{dom} \alpha_r$ after stage u, and $k_r(\delta^+)$ is chosen large, we have $k_r(\delta^+) > \#(u) > \#(t) \geqslant |\rho_q|$ as required.

Suppose that $(r, \delta^+) \notin \mathbb{P}_{s-1}$. We have $k = \#(v) + 1$ for some stage v such that $(r, \delta) \in \mathbb{P}_v$. Again, since $\delta \supseteq \beta^+$, δ is added to $\operatorname{dom} \alpha_r$ not before stage u; so $v \geqslant u$. Again we get $k > \#(u) > \#(t) \geqslant |\rho_q|$. □

If follows that if p is called at stage s, then $\rho_q \subset \rho_p$ (as they are both extended by B_s), $\alpha_q(\rho_q) \subset \alpha_p(\rho_p)$ (they are both extended by A_s), and $\mu_q \subset \mu_p$ (they are both extended by m_s).

Claim 6.21. For both $j < 2$, $\eta_{q,j} \subseteq \xi^*_{j,s,|\rho_q|}$.

Proof. Let $j < 2$. We first note that $\xi^*_{j,s,|\rho_q|} = \xi_{j,t,|\rho_q|}$. If $\mu \subset m_t \upharpoonright_{|\rho_q|}$ and $\mu \mapsto \zeta$ is in $\Xi_{j,t}$, then $\mu \subset m_s$, and so $\zeta \subseteq \xi^*_{j,s,|\rho_q|}$; so $\xi_{j,t,|\rho_q|} \subseteq \xi^*_{j,s,|\rho_q|}$.

On the other hand, suppose that $\mu_p \mapsto \eta_{p,j}$ is an axiom which is enumerated into Ξ_j at a stage $v \in (t, s)$. Claim 6.20, applied at stage v, implies that $|\mu_p| > |\rho_q|$, so this new axiom does not apply to the computation $\Xi_{j,s-1}(m_s \upharpoonright_{|\rho_q|})$. Hence we get the desired equality.

Now by Lemma 6.16, $\eta_{q,j} \subseteq \xi_{j,t,|\rho_q|}$, which proves the lemma. □

Claim 6.22. If $p \in \operatorname{dom} \mathbb{P}_s$ and $(r, \gamma) = \texttt{parents}(p)$, then $(r, \gamma) \notin \mathbb{P}_t$.

Proof. Because $(r, \beta^+) \in \mathbb{Q}_s$, and so (Lemma 6.15), $\beta^+ \subseteq \gamma$. Since β^+ was added to $\operatorname{dom} \alpha_r$ at stage $u > t$, γ must have been added at a stage no earlier than u. □

Claim 6.23. If $(r, \gamma) \in \mathbb{P}_s \setminus \mathbb{P}_t$ and $|\gamma| \geqslant |\rho_q|$, then for all $\delta \supset \gamma$ of length $k_r(\gamma)$, we have $\zeta_r(\delta) \supset \eta_{q, j_r(\gamma)}$.

Proof. Let $v \in [u, s)$ be the stage at which γ is added to $\operatorname{dom} \alpha_r$. Let o be r's child at stage $v - 1$. Let $j = j_r(\gamma)$. Let $\delta \supset \gamma$ have length $k_r(\gamma)$. At stage v, we pick $\zeta_r(\delta)$ to extend $\eta_{o,j}$.

- By Claim 6.21, applied at stage v, $\eta_{q,j} \subseteq \xi^*_{j,v,|\rho_q|}$.
- We have $\xi^*_{j,v,|\rho_q|} \subseteq \xi_{j,v-1,|\rho_q|}$ (and in fact we have equality, as $m_v \upharpoonright_{|\rho_q|} = m_{v-1} \upharpoonright_{|\rho_q|}$).
- We have $|\rho_o| \geqslant |\gamma| \geqslant |\rho_q|$. Hence $\xi_{j,v-1,|\rho_q|} \subseteq \xi_{j,v-1,|\rho_o|}$.
- By Lemma 6.19, applied at stage $v - 1 < s$, we have $\xi_{j,v-1,|\rho_o|} = \eta_{o,j}$.

Hence overall we get $\eta_{q,j} \subseteq \eta_{o,j} \subset \zeta_r(\delta)$. □

Claim 6.24. If p is called at stage s, then for both $j < 2$, $\eta_{q,j} \subset \eta_{p,j}$.

Proof. Let $\bar{p}$ be the child of r at stage s. If $\bar{p} < p$, then the result follows by induction on s, since $\bar{p} \in \operatorname{dom} \mathbb{P}_{s-1}$, and $\eta_{\bar{p},j} \subset \eta_{p,j}$ (Lemma 6.18). So we assume that r is p's mother. Let δ be p's father. By Claim 6.22, $(r, \delta) \notin \mathbb{P}_t$.

If $\eta_{p,j}$ is chosen to be an extension of $\xi^*_{j,s,|\rho_p|}$, then we note that $|\rho_p| > |\rho_q|$ (Claim 6.20); so by Claim 6.21, $\xi^*_{j,s,|\rho_p|} \supseteq \xi^*_{j,s,|\rho_q|} \supseteq \eta_{q,j}$.

Otherwise, $(r, \delta^+) \in \mathbb{P}_{s-1}$, where $\delta^+ = B_s\restriction_{|\delta|+1}$. We first argue that $|\delta| \geqslant |\rho_q|$. Let v be the stage at which δ^+ is added to $\operatorname{dom}\alpha_r$. Since $(r, \delta) \notin \mathbb{P}_t$, we have $u < v < s$. We have $(r, \delta^+) \in \mathbb{Q}_v$, but $(r, \delta^+) \notin \mathbb{Q}_s$ (Lemma 6.15). Hence $m_s\restriction_{|\delta^+|} \neq m_v\restriction_{|\delta^+|}$. As $m_s\restriction_{|\rho_q|} = m_v\restriction_{|\rho_q|}$, we get $|\delta^+| > |\rho_q|$, so $|\delta| \geqslant |\rho_q|$.

Now if $\eta_{p,j}$ is chosen as an extension of $\xi^*_{j,s,|\delta|}$ (if $j \neq j_r(\rho_p)$), then $\eta_{q,j} \subset \eta_{p,j}$ follows again from Claim 6.21.

Otherwise, $j = j_r(\rho_p)$, and $\eta_{p,j}$ is chosen as an extension of $\zeta_r(\rho_p)$. The desired result now follows from Claim 6.23. □

6.4.5. *The consistency of* $\Xi_j(m)$.

Proposition 6.25. *For both* $j < 2$, $\Xi_{j,s}(m_s) \in 2^{<\omega}$.

Proof. By Lemma 6.4, $\Xi_{j,s-1}(m_s) \in 2^{<\omega}$. So suppose that a new axiom $\mu_p \mapsto \eta_{p,j}$ is enumerated into Ξ_j at stage s, due to p being called; we need to verify that this new axiom does not contradict any other axiom in $\Xi_{j,s}$ which applies to m_s. Suppose that $\mu_q \mapsto \eta_{q,j}$ is another axiom in $\Xi_{j,s}$, and that $\mu_q \subset m_s$; we need to show that $\eta_{q,j} \not\perp \eta_{p,j}$. In fact, $\eta_{q,j} \subset \eta_{p,j}$.

For if $q \in \operatorname{dom}\mathbb{P}_s$, then $q < p$; by Lemma 6.18, $\eta_{q,j} \subset \eta_{p,j}$. And if $q \notin \operatorname{dom}\mathbb{P}_s$, then by Claim 6.24, $\eta_{q,j} \subset \eta_{p,j}$. □

The proof of Proposition 6.25 allows us to pay our other debt: Lemma 6.19 holds at stage s.

Proof of Lemma 6.19. Let $p \in \operatorname{dom}\mathbb{P}_s$. If $p \in \operatorname{dom}\mathbb{P}_{s-1}$, then $m_{s-1}\restriction_{|\rho_p|} = m_s\restriction_{|\rho_p|}$. Also, if a new axiom $\mu \mapsto \zeta$ is enumerated into Ξ_j at stage s, then $|\mu| > |\rho_p|$, as $\mu = \mu_q$ for some $q > p$. Hence $\xi_{j,s,|\rho_p|} = \xi_{j,s-1,|\rho_p|}$, and so $\eta_{p,j} = \xi_{j,s,|\rho_p|}$ follows by induction.

Suppose then that p is called at stage s. By Lemma 6.16, $\eta_{p,j} \subseteq \xi_{j,s,|\rho_p|}$. By the proof of Proposition 6.25, if $\mu_q \mapsto \eta_{q,j}$ is any axiom in $\Xi_{j,s}$ such that $\mu_q \subset m_s$, then $\eta_{q,j} \subseteq \eta_{p,j}$. Hence $\xi_{j,s,|\rho_p|} \subseteq \eta_{p,j}$. □

6.4.6. *What happens when* $\operatorname{dom}\alpha_r$ *is extended.* We work toward showing that for both $j < 2$, the functional Γ_j is consistent. This will be proved by induction. We fix $s > 0$ and assume, in this section and the next one, that $\Gamma_{j,s-1}$ is consistent.

In this section we investigate the following scenario. Assume that at stage s, a string γ^+ is added to $\operatorname{dom}\alpha_r$, where $\gamma^+ \neq \rho_r$. So letting $\gamma = \gamma^+\restriction_{|\gamma^+|-1}$, we have $(r, \gamma) \in \mathbb{P}_{s-1}, \mathbb{Q}_s$, and $\gamma^+ = B_s\restriction_{|\gamma|+1}$, indeed $(r, \gamma^+) \in \mathbb{Q}_s$.

Let $j = j_r(\gamma+)$, and let $\delta \supset \gamma^+$ have length $k_r(\gamma^+)$.

Let q be the child of r at stage $s-1$. At stage s, we choose $\zeta_r(\delta) \supset \eta_{q,j}$.

Claim 6.26. $(r, \gamma) = \texttt{parents}(q)$ and $\alpha_r(\gamma^+) \subseteq \alpha_q(\rho_q)$.

Proof. The point is that $\mu_q \subset m_s$, $q \in \operatorname{dom}\mathbb{P}_{s-1}$, but $q \notin \operatorname{dom}\mathbb{P}_s$, so we may apply the analysis of Section 6.4.4. r is the weakest procedure in $\operatorname{dom}\mathbb{P}_s$, and $r \in \operatorname{dom}\mathbb{P}_{s-1}$, so r of Section 6.4.4 coincides with r of the present section. The comments before Claim 6.20 indeed show that γ is the father of q. The fact that

$\mu_q \subset m_s$ implies that $\alpha_q(\rho_q) \subset A_s$. As mentioned, we have $\alpha_r(\gamma^+) \subset A_s$ as well. Since $|\mu_q| > |\gamma|$ (Corollary 6.10) and $|\alpha_r(\gamma^+)| = |\gamma| + 1$, $|\mu_q| = |\alpha_q(\rho_q)|$ we get $\alpha_r(\gamma^+) \subseteq \alpha_q(\rho_q)$. □

Let $(\sigma, \zeta) \mapsto \beta$ be an axiom in $\Gamma_{j,s-1}$, and suppose that $\zeta \subseteq \zeta_r(\delta)$. Since $\eta_{q,j} \subset \zeta_r(\delta)$, we have $\zeta \not\perp \eta_{q,j}$.

Claim 6.27. If $\sigma \not\supseteq \alpha_r(\gamma^+)$, then $\zeta \subseteq \eta_{q,j}$.

Proof. Let p be the procedure such that $\sigma = \alpha_p(\rho_p)$, $\zeta = \eta_{p,j}$, and $\beta = \rho_p$. Suppose that $\eta_{q,j} \subset \eta_{p,j}$.

Let $t < s$ be the stage at which q was called. Since $\eta_{q,j}$ is chosen long at stage t, we get that p is chosen at a stage $u \in (t, s)$. Since $q \in \operatorname{dom} \mathbb{P}_t, \operatorname{dom} \mathbb{P}_{s-1}$, we have $q \in \operatorname{dom} \mathbb{P}_u$, so $q < p$. This implies that $\alpha_q(\rho_q) \subset \alpha_p(\rho_p)$ (Lemma 6.13). Since $\alpha_r(\gamma^+) \subseteq \alpha_q(\rho_q)$, we get $\alpha_r(\gamma^+) \subseteq \alpha_p(\rho_p) = \sigma$. □

Claim 6.28. If $\sigma \not\perp \neg(\alpha_r(\gamma^+))$, then $\Gamma_{j,s-1}(\sigma, \zeta_r(\delta)) \subseteq \gamma$.

Proof. Let $\beta = \Gamma_{j,s-1}(\sigma, \zeta_r(\delta))$. Let $\bar{\sigma} \subseteq \sigma$ and $\zeta \subseteq \zeta_r(\delta)$ be strings such that the axiom $(\bar{\sigma}, \zeta) \mapsto \beta$ is in $\Gamma_{j,s-1}$. We have $\bar{\sigma} \not\perp \neg(\alpha_r(\gamma^+))$, so $\bar{\sigma} \not\supseteq \alpha_r(\gamma^+)$. By Claim 6.27, we have $\zeta \subseteq \eta_{q,j}$.

By the choice of j at stage s, we know that there are no $\sigma' \supseteq \neg(\alpha_r(\gamma^+))$ and $\zeta' \subseteq \eta_{q,j}$ such that $(\sigma', \zeta') \in \operatorname{dom} \Gamma_{j,s}$. Hence we must have $\bar{\sigma} \subset \neg\alpha_r(\gamma^+)$, that is, $\bar{\sigma} \subseteq \alpha_r(\gamma)$.

The axiom $(\alpha_q(\rho_q), \eta_{q,j}) \mapsto \rho_q$ is in $\Gamma_{j,s-1}$, and is compatible with the axiom $(\bar{\sigma}, \zeta) \mapsto \beta$. By our assumption that $\Gamma_{j,s-1}$ is consistent, we have $\beta \not\perp \rho_q$. As $\bar{\sigma} \subseteq \alpha_r(\gamma)$, we have $|\beta| = |\bar{\sigma}| \leqslant |\gamma|$. Hence $\beta \subseteq \rho_q\restriction_{|\gamma|} = \gamma$. □

Claim 6.29. Let $o \in \operatorname{dom} \mathbb{P}_{<\omega}$. If $\eta_{o,j} \subseteq \zeta_r(\delta)$, then either $\rho_o \subseteq \gamma$, or $\alpha_o(\rho_o) \perp \neg(\alpha_r(\gamma^+))$.

Proof. Let u be the stage at which o is called. We have $u \neq s$, since at stage s, no new procedure is called. Since $|\eta_{o,j}| > \#(t-1)$ and $\#(s) \geqslant |\zeta_r(\delta)|$, we have $u < s$.

Thus, the axiom $(\alpha_o(\rho_o), \eta_{o,j}) \mapsto \rho_o$ is in $\Gamma_{j,s-1}$. Suppose that $\eta_{o,j} \subseteq \zeta_r(\delta)$ and that $\alpha_o(\rho_o) \not\perp \neg(\alpha_r(\gamma^+))$. Claim 6.28, applied to $\sigma = \alpha_o(\rho_o)$, implies that

$$\rho_o \subseteq \Gamma_{j,s-1}(\sigma, \zeta_r(\gamma)) \subset \gamma$$

as required. □

6.4.7. *Consistency of* Γ_j.

Lemma 6.30. *Suppose that p is called at stage s; let $q \in \operatorname{dom} \mathbb{P}_{<\omega}$, and let $j < 2$. If $\eta_{q,j} \subseteq \eta_{p,j}$ then either $\alpha_p(\rho_p) \perp \alpha_q(\rho_q)$, or $\rho_q \subseteq \rho_p$.*

Proof. We prove the lemma by induction on s. Let $q \in \operatorname{dom} \mathbb{P}_{<\omega}$. We first prove the following:

($\otimes$) For all $n < \omega$, if $\eta_{q,j} \subseteq \xi^*_{j,s,n}$, then either $A_s\restriction_n \perp \alpha_q(\rho_q)$, or $\rho_q \subseteq B_s\restriction_n$.

To see this, suppose indeed that $\eta_{q,j} \subseteq \xi^*_{j,s,n}$. Then there is some $\bar{p} \in \operatorname{dom} \mathbb{P}_{<s}$ such that $\mu_{\bar{p}} \subset m_s$ and such that $|\rho_{\bar{p}}| \leqslant n$ and $\eta_{q,j} \subseteq \eta_{\bar{p},j}$. By the assumption that the lemma holds before stage s, either $\alpha_{\bar{p}}(\rho_{\bar{p}}) \perp \alpha_q(\rho_q)$, or $\rho_q \subseteq \rho_{\bar{p}}$. In the first case, we note that $\mu_{\bar{p}} \subset m_s$ implies that $\alpha_{\bar{p}}(\rho_{\bar{p}}) \subset A_s$, so $A_s\restriction_n \supseteq \alpha_{\bar{p}}(\rho_{\bar{p}})$. In the second case, we note that $\mu_{\bar{p}} \subset m_s$ implies that $\rho_{\bar{p}} \subset B_s$, so $\rho_q \subseteq \rho_{\bar{p}} \subseteq B_s\restriction_n$.

Now we verify that the lemma holds at stage s. Suppose that p is called at stage s, and let $(r, \beta) = \texttt{parents}(p)$. Suppose that $\eta_{q,j} \subseteq \eta_{p,j}$. Assuming that $p \neq q$, reasoning about the length of $\eta_{q,j}$ as in the proof of Lemma 6.29, we can conclude that $q \in \operatorname{dom} \mathbb{P}_{<s}$. There are two cases.

If $\eta_{p,j}$ is chosen to be an extension of $\xi^*_{j,s,|\rho_p|}$ or of $\xi^*_{j,s,|\beta|}$, then let $n = |\rho_p|$ or $n = |\beta|$ accordingly. By the freeness of the extension, and by the fact that $q \in \operatorname{dom} \mathbb{P}_{<s}$, we have $\eta_{q,j} \subseteq \xi^*_{j,s,n}$. We have $B_s\restriction_n \subseteq \rho_p$ and $A_s\restriction_n \subseteq \alpha_p(\rho_p)$. By ($\otimes$), either $A_s\restriction_n \perp \alpha_q(\rho_q)$, or $\rho_q \subseteq B_s\restriction_n$. Hence either $\alpha_p(\rho_p) \perp \alpha_q(\rho_q)$, or $\rho_q \subseteq \rho_p$.

Otherwise, for $\beta^+ = B_s\restriction_{|\beta|+1}$ and $\alpha^+ = A_s\restriction_{|\beta|+1}$, we have $\rho_p \supset \beta^+$, $|\rho_p| = k_r(\beta^+)$, $j = j_r(\rho_p)$; we have $\alpha_p(\rho_p) \supseteq \neg\alpha_r(\beta^+)$, and $\eta_{p,j}$ is chosen to be a free extension of $\zeta_r(\rho_p)$. Again by the freeness of the extension, $\eta_{q,j} \subseteq \eta_{p,j}$ implies that $\eta_{q,j} \subseteq \zeta_r(\delta)$, where $\delta = B_s\restriction_{k_r(\beta^+)}$. Now by Claim 6.29, either $\neg\alpha_r(\beta^+) \perp \alpha_q(\rho_q)$, whence $\alpha_p(\rho_p) \perp \alpha_q(\rho_q)$, or $\rho_q \subseteq \beta \subset \rho_p$, as required. □

If we observe the definition of Γ_j, we get that $\Gamma_{j,s}$ is consistent. This completes the induction started in the previous section, and proves:

Proposition 6.31. *For both $j < 2$, Γ_j is a consistent functional.*

6.4.8. *What happens to ξ_j when $\zeta_r(\gamma)$ is activated.* In this section we investigate a third scenario – in a sense, an extension of the second scenario. Suppose that at a stage s, a procedure p is called, and suppose that, letting $(r, \gamma) = \texttt{parents}(p)$ and $\gamma^+ = B_s\restriction_{|\gamma|+1}$, we have $(r, \gamma^+) \in \mathbb{P}_{s-1}$.

Let $t < s$ be the stage at which γ^+ was added to $\operatorname{dom} \alpha_r$.

Claim 6.32. $|\gamma|$ is the greatest n such that $A_t\restriction_n = A_s\restriction_n$.

Proof. by definition, we have $\alpha_r(\gamma^+) \subset A_t$, hence $\alpha_r(\gamma) \subset A_t$; since $(r, \gamma^+) \notin \mathbb{Q}_s$, and $\gamma^+ \subset B_s$, we have $\alpha_r(\gamma^+) \not\subset A_s$. On the other hand, $\alpha_r(\gamma) \subset A_s$. □

Claim 6.33. If $q \in \operatorname{dom} \mathbb{P}_{<s}$ and $\mu_q \subset m_s$, then $q \in \operatorname{dom} \mathbb{P}_{<t}$.

Proof. We have $\gamma^+ \subset B_s$ and $\neg(\alpha_r(\gamma^+)) \subset A_s$. Let u be the least stage such that for all $v \in [u, s]$ we have $\gamma^+ \subset B_v$ and $\neg(\alpha_r(\gamma^+)) \subset A_v$. We have $u > t$ because $\alpha_r(\gamma^+) \subset A_t$.

The minimality of u implies that $m_{u-1}\restriction_{|\gamma|+1} \neq m_u\restriction_{|\gamma|+1}$.

The procedure r is the weakest procedure in $\operatorname{dom} \mathbb{P}_t$. To prove the claim, we show that if q is a child of r which is called at some stage $v \in (t, s)$, then $\mu_q \not\subset m_s$.

Let q be such a child. There are two cases.

First suppose that $v < u$. Let β be the father of q. By Lemma 6.15, β is the longest string such that $(r, \beta) \in \mathbb{Q}_v$. Now γ is the longest string such that $(r, \gamma) \in \mathbb{Q}_u$. Hence, if $\beta \neq \gamma$, then $m_v\restriction_{|\beta|+1} \neq m_u\restriction_{|\beta|+1}$. If $\beta = \gamma$, then we already concluded that $m_{u-1}\restriction_{|\beta|+1} \neq m_u\restriction_{|\beta|+1}$; since $v < u$, we get $m_v\restriction_{|\beta|+1} \neq m_u\restriction_{|\beta|+1}$. Now $|\mu_q| > |\beta|$ (Lemma 6.9), and $\mu_q \subset m_v$, so $\mu_q \not\subset m_u$. Hence $\mu_q \not\subset m_s$.

The other case is that $v \geqslant u$. In this case we know that the father of q is γ. Since p is a child of r which is called at stage s, and $v < s$, we know that q must be cancelled at a stage $w \in (v, s]$. Since $w \geqslant u$, we know that $B_w\restriction_{|\gamma|+1} = \gamma^+$; and since $w > t$, we know that $(r, \gamma^+) \in \mathbb{P}_{w-1}$. So it is impossible that q is cancelled at step 2 of stage w: such a cancellation can only occur in order to add γ^+ to $\operatorname{dom}\alpha_r$. Hence $\mu_q \not\subset m_w$, so $\mu_q \not\subset m_s$. ☐

Again for brevity, let $\xi_{j,s} = \Xi_{j,s}(m_s) = \bigcup_k \xi_{j,s,k}$ and $\xi^*_{j,s} = \Xi_{j,s-1}(m_s) = \bigcup_k \xi^*_{j,s,k}$.

Claim 6.33 implies that for both $j < 2$, $\xi^*_{j,s} \subseteq \xi_{j,t}$.

Claim 6.34. Let $j < 2$. Suppose that $(\sigma, \zeta) \in \operatorname{dom}\Gamma_{j,s-1}$ and either:

- $\zeta \subseteq \xi_{j,t}$; or
- $\zeta \subseteq \zeta_r(\delta)$ for some $\delta \supset \gamma^+$ of length $k_r(\gamma^+)$.

Then $(\sigma, \zeta) \in \operatorname{dom}\Gamma_{j,t}$.

Proof. Let q be any procedure which is called at a stage $v \in (t, s)$. $\eta_{q,j}$ is chosen to be a free extension at stage v. $|\xi_{j,t}|, |\zeta_r(\delta)| \leqslant \#(t)$. Hence we cannot have $\eta_{q,j} \subseteq \xi_{j,t}$ or $\eta_{q,j} \subseteq \zeta_r(\delta)$. ☐

6.4.9. *The main lemma.*

Lemma 6.35. *Let $p \in \operatorname{dom}\mathbb{P}_{<\omega}$ and $j < 2$. There are no $\sigma \supset \alpha_p(\rho_p)$ and $\zeta \subseteq \eta_{p,j}$ such that $(\sigma, \zeta) \in \operatorname{dom}\Gamma_j$.*

Proof. Let $(\sigma, \zeta) \in \operatorname{dom}\Gamma_j$; there is some $q \in \operatorname{dom}\mathbb{P}_{<\omega}$ such that $\sigma = \alpha_q(\rho_q)$ and $\zeta = \eta_{q,j}$. Suppose that $\zeta \subseteq \eta_{p,j}$. By Lemma 6.30, either $\sigma \perp \alpha_p(\rho_p)$ or $\rho_p \subseteq \rho_q$. In the first case, certainly $\sigma \not\supset \alpha_p(\rho_p)$. In the second case, $|\sigma| = |\rho_q| \leqslant |\rho_p| = |\alpha_p(\rho_p)|$ so again we cannot have $\sigma \supset \alpha_p(\rho_p)$. ☐

Here is the main lemma.

Lemma 6.36. *Let $s < \omega$. For every $k > 0$ there is some $j < 2$ such that there are no $\sigma \supseteq \neg(A_s\restriction_k)$ and $\zeta \subseteq \xi_{j,s}$ such that $(\sigma, \zeta) \in \operatorname{dom}\Gamma_{j,s}$.*

Proof. The lemma is proved by induction on s. The lemma clearly holds for $s = 0$. Let $s > 0$, and suppose the lemma holds at every $t < s$.

We first note that it is sufficient to show, for every $k > 0$, that there is some $j < 2$ such that there are no $\sigma \supseteq \neg(A_s\restriction_k)$ and $\zeta \subseteq \xi_{j,s}$ such that $(\sigma, \zeta) \in \operatorname{dom}\Gamma_{j,s-1}$. The reason is that such (σ, ζ) cannot be added to $\operatorname{dom}\Gamma_j$ at stage s, because at stage s we only enumerate such axioms such that $\sigma \subset A_s$.

If there is a procedure which is called at stage s, call that procedure p; let $(r, \gamma) = \texttt{parents}(p)$, and let $\gamma^+ = B_s\restriction_{|\gamma|+1}$.

There are three cases:

(1) No new procedure is called at stage s.
(2) $(r, \gamma^+) \notin \mathbb{P}_{s-1}$.
(3) $(r, \gamma^+) \in \mathbb{P}_{s-1}$.

For each case, we use the inductive hypothesis relative to some previous stage t. In cases (1) and (2), let $t = s - 1$. In case (3), let t be the stage at which γ^+ is added to $\operatorname{dom}\alpha_r$.

In all of the cases, let n be the length of the longest common initial segment of A_s and A_t.

Let $k \leqslant n$. Since $A_t\restriction_k = A_s\restriction_k$, $\neg(A_s\restriction_k) = \neg(A_t\restriction_k)$. By induction, there is some $j < 2$ for which there are no $\sigma \supseteq \neg(A_s\restriction_k)$ and no $\zeta \subseteq \xi_{j,t}$ such that $(\sigma, \zeta) \in \operatorname{dom}\Gamma_{j,t}$.

We claim that the same j witnesses the lemma at stage s for k. Let $\zeta \subseteq \xi_{j,s}$ and $\sigma \supseteq \neg(A_s\restriction_k)$. We argue that $(\sigma, \zeta) \notin \operatorname{dom}\Gamma_{j,s-1}$. Assume, for contradiction, that $(\sigma,\zeta) \in \operatorname{dom}\Gamma_{j,s-1}$; so $|\zeta| \leqslant \#(s-1)$.

In case (1), we have $\xi_{j,s} = \xi^*_{j,s} \subseteq \xi_{j,s-1}$, so $\zeta \subseteq \xi_{j,t}$; so by induction, $(\sigma,\zeta) \notin \operatorname{dom}\Gamma_{j,s-1}$.

In case (2), We have $\xi_{j,s} = \eta_{p,j}$, which is chosen, at stage s, to be a free extension of $\xi^*_{j,s,|\rho_p|}$. The assumption $|\zeta| \leqslant \#(s-1)$ implies that $\zeta \subseteq \xi^*_{j,s,|\rho_p|} \subseteq \xi^*_{j,s} \subseteq \xi_{j,s-1}$. So again by induction, $(\sigma,\zeta) \notin \operatorname{dom}\Gamma_{j,s-1}$.

In case (3), there are two sub-cases, depending on the value of j. If $j \neq j_r(\gamma^+)$, then the argument is similar to that of case (2). We have $\xi_{j,s} = \eta_{p,j}$ is chosen as a free extension of $\xi^*_{j,s,|\gamma|}$; $|\zeta| \leqslant \#(s-1)$ implies that $\zeta \subseteq \xi^*_{j,s,|\gamma|} \subseteq \xi^*_{j,s}$. By Claim 6.33, we have $\xi^*_{j,s} \subseteq \xi_{j,t}$, and by Claim 6.34, $(\sigma,\zeta) \in \operatorname{dom}\Gamma_{j,t}$, contradicting the induction assumption.

Suppose that $j = j_r(\gamma^+)$. Then $\xi_{j,s} = \eta_{p,j}$ is chosen to be a free extension of $\zeta_r(\delta)$, for $\delta = B_s\restriction_{k_r(\gamma^+)}$. Again, $|\zeta| \leqslant \#(s-1)$ implies that $\zeta \subseteq \zeta_r(\delta)$. By Claim 6.34, $(\sigma,\zeta) \in \operatorname{dom}\Gamma_{j,t}$.

By Claim 6.32, $n = |\gamma|$; so $k \leqslant n$ and $\sigma \supseteq \neg(A_t\restriction_k)$ implies that $\sigma \not\supseteq \alpha_r(\gamma^+)$. By Claim 6.27, we have $\zeta \subseteq \eta_{q,j}$, where q is the child of r at stage $t-1$; since $\mu_q \subset m_t$, we have $\eta_{q,j} \subseteq \xi_{j,t}$. So $\zeta \subseteq \xi_{j,t}$, contradicting the induction assumption.

Now let $k > n$. There is some $j < 2$ such that $\xi_{j,s} = \xi^*_{j,s}$ or $\xi_{j,s}$ is chosen, at stage s, as a free extension of an initial segment of $\xi^*_{j,s}$: both $j < 2$ would do in cases (1) and (2), and in case (3), we choose $j \neq j_r(\gamma^+)$. We claim that such j witnesses the lemma at stage s for k. Suppose that $\sigma \supseteq \neg(A_s\restriction_k)$. Then $\sigma \supset A_s\restriction_n$. Let $\zeta \subseteq \xi_{j,s}$, and suppose, for contradiction, that $(\sigma,\zeta) \in \operatorname{dom}\Gamma_{j,s-1}$. The choice of j implies that $\zeta \subseteq \xi^*_{j,s}$.

Now let $q \in \operatorname{dom}\mathbb{P}_{<s}$ be a procedure such that $\mu_q \subset m_s$ and $\eta_{q,j} = \xi^*_{j,s}$. We claim that $|\mu_q| \leqslant n$. In cases (1) and (2), this follows from the fact that $\mu_q \subseteq m_s$ and $m_s\restriction_{n+1} \neq m_{s-1}\restriction_{n+1}$. In case (3), Claim 6.33 states that $q \in \operatorname{dom}\mathbb{P}_{<t}$, so $|\mu_q| \leqslant n$ follows from the fact that $m_s\restriction_{n+1} \neq m_t\restriction_{n+1}$.

$|\mu_q| \leqslant n$ implies that $|\alpha_q(\rho_q)| \leqslant n$; $\mu_q \subset m_s$ implies that $\alpha_q(\rho_q) \subset A_s$, so $\alpha_q(\rho_q) \subseteq A_s\restriction_n \subset \sigma$. So $\zeta \subseteq \eta_{q,j}$ contradicts Lemma 6.35. ∎

Corollary 6.37. *Let $s > 0$. Suppose that $q \neq p_0$, $q \in \operatorname{dom}\mathbb{P}_{s-1}$ and $\mu_q \subset m_s$. Let $(p,\beta) =$ `parents`(q); let $\beta^+ = B_s\restriction_{|\beta|+1}$ and $\alpha^+ = A_s\restriction_{|\beta|+1}$. Then there is some $j < 2$ such that there are no $\sigma \supseteq \neg(\alpha^+)$ and $\zeta \subseteq \eta_{q,j}$ such that $(\sigma,\zeta) \in \operatorname{dom}\Gamma_{j,s-1}$.*

Proof. Let $k = |\beta| + 1$; so $\alpha^+ = A_s\restriction_k$. Since $q \in \operatorname{dom}\mathbb{P}_{s-1}$ and $\mu_q \subset m_s$, we have, for both $j < 2$, $\eta_{q,j} \subseteq \xi^*_{j,s} \subseteq \xi_{j,s}$. The result now follows from the main Lemma 6.36 for k. ∎

6.4.10. *Uncancelled procedures.* The following is immediate:

Lemma 6.38. *The following are equivalent for $(p,\beta) \in \mathbb{P}_{<\omega}$:*

1. *There are infinitely many s such that $(p,\beta) \in \mathbb{Q}_s$;*
2. *For almost all s, $(p,\beta) \in \mathbb{Q}_s$;*
3. *p is never cancelled, $\beta \subset B$ and $\alpha_p(\beta) \subset A$.*

Let $\mathbb{Q}_\infty$ denote the collection of pairs (p, β) which satisfy the conditions of Lemma 6.38. We note that if $(p, \beta) \in \mathbb{Q}_\infty$ then $(p, \gamma) \in \mathbb{Q}_\infty$ for all γ such that $\rho_p \subseteq \gamma \subseteq \beta$, and that if p is a procedure which is never cancelled, then $(p, \rho_p) \in \mathbb{Q}_\infty$.

In this section, let p be some procedure which is never cancelled. Note that if $p \in \operatorname{dom} \mathbb{P}_s$, then no procedure stronger than p requires attention at stage s, as the strongest such would cancel p.

Claim 6.39. There are only finitely many strings β such that $(p, \beta) \in \mathbb{Q}_\infty$.

Proof. This is because we assume that $A \not\leqslant_T B$. If the lemma fails, then for every $\beta \subset B$ such that $|\beta| \geqslant |\rho_p|$ we have $(p, \beta) \in \mathbb{Q}_\infty$, which implies that $\alpha_p(\beta) \subset A$; thus $A = \bigcup\{\alpha_p(\beta) : \rho_p \subseteq \beta \subset B\}$ which shows that $A \leqslant_T B$. □

Since p is never cancelled, we know that $(p, \rho_p) \in \mathbb{Q}_\infty$ (and so $\rho_p \subset B$). Let β be the longest initial segment of B such that $(p, \beta) \in \mathbb{Q}_\infty$. Let $\beta^+ = B\restriction_{|\beta|+1}$.

If $\beta^+ \in \operatorname{dom} \alpha_p$, let s_0 be a stage such that $(p, \beta^+) \in \mathbb{P}_{s_0}$; otherwise, let s_0 be a stage such that $(p, \beta) \in \mathbb{P}_{s_0}$. Also, choose s_0 sufficiently large such that $m_{s_0}\restriction_{|\beta|+1} \subset m$.

Claim 6.40. p doesn't require attention after stage s_0.

Proof. If $(p, \beta^+) \in \mathbb{P}_{<\omega}$, the conclusion follows from the fact that for all $s \geqslant s_0$, $\beta^+ = B_s\restriction_{|\beta|+1}$. If $(p, \beta^+) \notin \mathbb{P}_{<\omega}$, the conclusion follows from the fact that if p required attention at some stage $s \geqslant s_0$, then since $(p, \beta) \in \mathbb{Q}_s$, it would try to add $B_s\restriction_{|\beta|+1} = \beta^+$ to $\operatorname{dom} \alpha_r$; and then we'd have $(p, \beta^+) \in \mathbb{P}_{<\omega}$, contrary to assumption. □

It follows that for all $s \geqslant s_0$, if after step 1 of stage s, p does not have a child, then such a child is called for p at stage s. So for all $s \geqslant s_0$, p has a child in $\operatorname{dom} \mathbb{P}_s$.

Claim 6.41. There is some child of p which is never cancelled.

Proof. If $(p, \beta^+) \in \mathbb{P}_{<\omega}$, let $k = k_p(\beta^+)$. Otherwise, let $k = \#(s_0) + 1$. Let $s_1 > s_0$ be a stage such that for all $s \geqslant s_1$, $m_s\restriction_k \subset m$. If q is a child of p which is called after stage s_1, then $|\rho_q| \leqslant k$, and so the stability of $m_s\restriction_k$ after stage s_1 ensures that q is never cancelled. □

6.4.11. *The end.* We know that p_0 is never cancelled (Lemma 6.7). By recursion, given a procedure p_e which is never cancelled, we let, by Claim 6.41, p_{e+1} be the child of p_e which is never cancelled. Note that $e_{p_e} = e$.

Let $j < 2$. Let $X_j = \Xi_j(m)$. For every e, we have $\mu_{p_e} \subset m$, so $\eta_{p_e,j} \subset X_j$. Since $\eta_{p_e,j} \subset \eta_{p_{e+1},j}$, we have $X_j \in 2^\omega$. We have $X_j \leqslant_T m \leqslant_T C$.

For every $e < \omega$, since $\mu_{p_e} \subset m$, we have $\rho_{p_e} \subset B$ and $\alpha_{p_e}(\rho_{p_e}) \subset A$. Since $\eta_{p_e,j} \subset X_j$, and the axiom $(\alpha_{p_e}(\rho_{p_e}), \eta_{p_e,j}) \mapsto \rho_{p_e}$ is in Γ_j, we have $\rho_{p_e} \subseteq \Gamma_j(A, X_j)$. Since $|\rho_{p_e}| < |\rho_{p_{e+1}}|$, we have $B = \bigcup_e \rho_{p_e}$. Hence $B \subseteq \Gamma_j(A, X_j)$. Since Γ_j is consistent, we have $B = \Gamma_j(A, X_j)$. Hence $B \leqslant_T A \oplus X_j$.

To complete the proof of the theorem, we need to show that for all e, the requirement R_e is met. To see this, we look at two cases. Let $p = p_e$; define β, β^+ and s_0 as in the section above. Let $q = p_{e+1}$, and let $s_1 > s_0$ be a stage by which q has been called.

If $(p, \beta^+) \in \mathbb{P}_{<\omega}$, let $j = j_p(\beta^+)$. We know that $X_j \supset \eta_{q,j} \supset \zeta_p(\rho_q)$, that $\beta^+ \subset B$, and that $\Psi_e(\zeta_p(\rho_q)) \perp \beta^+$. Hence $\Psi_e(X_j) \neq B$.

Now suppose that $(p, \beta^+) \notin \mathbb{P}_{<\omega}$, and suppose, for contradiction, that the requirement R_e fails: $\Psi_e(X_0) = \Psi_e(X_1) = B$. Since B is not computable, for both $j < 2$ there is some string $\tau_j \supset \eta_{q,j}$ such that $\Psi_e(\tau_j) \perp B$. Again for both $j < 2$, let $\sigma_j \subset X_j$ be a string such that $\sigma_j \supset \eta_{q,j}$ and such that $\Psi_e(\sigma_j) \perp \Psi_e(\tau_j)$. Let $s \geqslant s_1, |\sigma_0|, |\sigma_1|, |\tau_0|, |\tau_1|$.

By Corollary 6.37, let $j < 2$ be such that there are no $\sigma \supseteq \neg(A_s \restriction_{|\beta|+1})$ and $\zeta \subseteq \eta_{q,j}$ such that $(\sigma, \zeta) \in \operatorname{dom} \Gamma_{j,s-1}$. Let k be large at stage s. Let $\gamma \supset \beta^+$ have length k. If $\gamma \not\perp \Psi_e(\sigma_j)$, let $\zeta_\gamma = \tau_j$; since $|\gamma| = k > |\Psi_e(\tau_j)|$, we have $\Psi_e(\tau_j) \perp \gamma$. Otherwise, we let $\zeta_\gamma = \sigma_j$. Then in either case, we have $\Psi_e(\zeta_\gamma) \perp \gamma$. All the conditions hold for p to require attention at stage s, contradicting Claim 6.40.

7. Join property below non-generalized-low$_2$ degrees.

In this section we prove Theorem 1.7. That is, we wish to show that every non-generalized-low$_2$ degree satisfies the join property i.e. if $\boldsymbol{d}$ is not GL_2 and $\boldsymbol{0} < \boldsymbol{a} < \boldsymbol{d}$, then there exists $\boldsymbol{b} < \boldsymbol{d}$ such that $\boldsymbol{a} \vee \boldsymbol{b} = \boldsymbol{d}$.

So suppose we are given D of degree which is not GL_2, and A which is of non-zero degree strictly below that of D. We want to construct $B <_T D$ such that $A \oplus B \equiv_T D$. Moreover, we will build B to be 1-generic. This will imply that $B \not\equiv_T D$, as 1-generic sets are GL_1.

The general plan is that we construct B by finite approximations, first trying to satisfy a genericity requirement, then coding one bit of D, then trying another genericity requirement, and then another bit of D, etc... The property of non-GL_2 sets we will use is the following: For every $D \oplus \emptyset'$-computable function f_1, there is a D computable function f_2 not dominated by f_1 (see [Ler83]). We will use this function to bound our searches when we are trying to force inside some c.e. set. To get $D \leq_T A \oplus B$, we will have to use a trick, due to Slaman and Steel [SS89]: We will try to satisfy the genericity requirements in a way that can be decoded by A so that A can read off the bits of B that are coding D.

Define

$$\sigma_n = \underbrace{000\cdots}_{n \text{ zeros}}1.$$

We can assume that A is not computably enumerable (for instance by considering either it or its complement).

Given any $\sigma \in 2^{<\omega}$ and any $e, s \in \omega$, let $g(\sigma, e, s)$ be defined in the following way. First, let n be the least such that:

$$n \in A \;\Rightarrow\; \nexists \tau \in W_{e,s} \text{ with } \tau \supseteq \sigma^\frown\sigma_n;$$
$$n \notin A \;\Rightarrow\; \exists \tau \in W_{e,s} \text{ with } \tau \supseteq \sigma^\frown\sigma_n.$$

Note that the set of n such that $\exists \tau \in W_{e,s}$ with $\tau \supseteq \sigma * \sigma_n$ is computable, and hence different from A. So, there has to be an n as above. If $n \in A$ define $g(\sigma, e, s) = \sigma * \sigma_n$. If $n \notin A$ then let $g(\sigma, e, s)$ be the first string enumerated into $W_{e,s}$ with $\tau \supseteq \sigma * \sigma_n$. The function $g(\sigma, e, s)$ is computable in A.

Given any $f : \omega \to \omega$ we define a set B_f that uses the function f to bound the searches of the function $g(\sigma, e, s)$. Let $B_f = \bigcup_s \tau_{f,s}$ be defined as follows:
Stage 0. Define $\tau_{f,0} = D(0)$.

Stage $t+1$. Define $\tau_{f,t+1} = g(\tau_{f,t}, e, f(t))^\frown D(t+1)$, where $t = \langle e, s\rangle$.

Note that B_f is computable in $D \oplus f$. We will show later how, if we use a large enough function f, we get that B_f is 1-generic. That D is computable in $A \oplus B$ follows using precisely the same argument as originally provided by Slaman and Steel when proving that $\mathbf{0}'$ satisfies the join property. The point is that using B, A can reconstruct the whole sequence $\tau_{f,0} \subset \tau_{f,1} \subset \tau_{f,2} \subset ...$ as follows. Given $\tau_{f,k}$, let n be such that $\tau_{f,k}{}^\frown\sigma_n \subseteq B$. If $n \in A$, then we know that $g(\tau_{f,t}, e, f(t)) = \tau_{f,k}{}^\frown\sigma_n$. If $n \notin A$, then $g(\tau_{f,t}, e, f(t))$ is the least $\tau \in W_{e,s}$ with $\tau \supseteq \tau_{f,k}{}^\frown\sigma_n$. Then, we get $\tau_{f,k+1}$ by adding, at the end of the string $g(\tau_{f,t}, e, f(t))$, the bit $D(k+1)$, that we can read off from B. So $D \leq_T A \oplus B$.

Given σ and e, let $\langle n, s\rangle$ be the least pair such that one of the following conditions hold.

$$n \in A,\ \nexists\tau \in W_e \text{ with } \tau \supseteq \sigma^\frown\sigma_n, \text{ and } s = 0,$$
$$n \notin A,\ \exists\tau \in W_e \text{ with } \tau \supseteq \sigma^\frown\sigma_n, \text{ and } \tau \in W_{e,s}.$$

Note that the set of n such that $\exists\tau \in W_e$ with $\tau \supseteq \sigma * \sigma_n$ is c.e., and hence different from A. So, there has to exist an n and an s as above. Furthermore, $A \oplus \emptyset'$ can find them. Let $g^*(\sigma, e)$ be the $A \oplus \emptyset'$-computable function that gives us such s. So, we have that if $t \geq g^*(\sigma, e)$, then $g(\sigma, e, t)$ is an extension of σ that either forces inside W_e or forces outside W_e as we need for the genericity requirements.

Now for any t the set $\Pi_t = \{\tau_{f,t} \mid f : \omega \to \omega\}$ is finite and computable in D, so let f_0 be an increasing function computable in $D \oplus \emptyset'$ such that, for all t and all $\tau \in \Pi_t$, $f_0(t)$ is greater than $g^*(\tau, e)$, where $t = \langle e, s\rangle$. Let h be a computable and increasing function such that, for all t and all $e \leq t$, there exists s with $t < \langle e, s\rangle < h(t)$. Define $f_1(t) = f_0(h(t))$ for all t. Since D is not GL_2, we may let f_2 be an increasing function computable in D which is not dominated by f_1 and then define $B = B_{f_2} \leq_T D$.

It remains to show that B is 1-generic. In order to see this, fix $e \in \omega$ and let $t > e$ be such that $f_2(t) > f_1(t)$. Then there exists $t' = \langle e, s\rangle$ such that $t < t' < h(t)$. We have that $f_2(t') > f_2(t) > f_1(t) = f_0(h(t)) > f_0(t')$, so that $f_2(t') > g^*(\tau_{f_2,t'}, e)$ as required.

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E-mail address: Rod.Downey@msor.vuw.ac.nz
URL: http://www.mcs.vuw.ac.nz/∼downey

School of Mathematics, Statistics and Computer Science, Victoria University, P.O. Box 600, Wellington, New Zealand

E-mail address: greenberg@msor.vuw.ac.nz
URL: http://homepages.mcs.vuw.ac.nz/∼greenberg/

School of Mathematics, Statistics and Computer Science, Victoria University, P.O. Box 600, Wellington, New Zealand

E-mail address: andy@aemlewis.co.uk
URL: http://aemlewis.co.uk/

School of Mathematics, University of Leeds, Leeds, UK

E-mail address: antonio@math.uchicago.edu
URL: www.math.uchicago.edu/∼antonio

Department of Mathematics, University of Chicago, 5734 S. University ave., Chicago, IL 60637, USA