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COUNTABLY COMPLEMENTABLE LINEAR ORDERINGS

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ABSTRACT. We say that a countable linear ordering $\mathcal{L}$ is countably complementable if there exists a linear ordering $\overline{\mathcal{L}}$, possibly uncountable, such that for any countable linear ordering $\mathcal{B}$, $\mathcal{L}$ does not embed into $\mathcal{B}$ if and only if $\mathcal{B}$ embeds into $\overline{\mathcal{L}}$. We characterize the linear orderings which are countably complementable. We also show that this property is equivalent to the countable version of the finitely faithful extension property introduced by Hagendorf.

Using similar methods and introducing the notion of weakly countably complementable linear orderings, we answer a question posed by Rosenstein and prove the countable case of a conjecture of Hagendorf, namely, that every countable linear ordering satisfies the countable version of the totally faithful extension property.

1. Introduction

We are interested in the structure $(\mathbb{L}, \preccurlyeq)$, where $\mathbb{L}$ is the set of countable linear orderings, and $\preccurlyeq$ is the embeddability relation. (For more information on this structure, see [Ros82, Chapter 10], [Fra00, Chapters 5, 6 and 7] or [Mon].) Given $\mathcal{L} \in \mathbb{L}$, let

$$\mathbb{L}(\mathcal{L}) = \{\mathcal{B} \in \mathbb{L} : \mathcal{L} \not\preccurlyeq \mathcal{B}\}.$$

In this paper we completely characterize the following class of linear orderings.

Definition 1.1. A countable linear ordering $\mathcal{L}$ is *countably complementable* if there exists a linear ordering $\overline{\mathcal{L}}$, possibly uncountable, such that for every countable linear ordering $\mathcal{B}$,

$$\mathcal{B} \in \mathbb{L}(\mathcal{L}) \iff \mathcal{B} \preccurlyeq \overline{\mathcal{L}},$$

in which case we call $\overline{\mathcal{L}}$ a *complement* for $\mathcal{L}$.

This is a quite natural definition, although to our knowledge it has not been studied before.

In this paper, we show that the class of countably complementable linear orderings coincides with the class of linear orderings that have the finitely faithful extension property, and with the class of linear orderings that have the completely faithful extension property. These two properties, which we define below, were introduced by Hagendorf in [Hag77] (see [Hag79, page 426]) for arbitrary cardinality. In this paper we treat only the countable case.

Definition 1.2. Let $\mathcal{L} \in \mathbb{L}$

(1) $\mathcal{L}$ has the *finitely faithful extension property* if for all $\mathcal{A}, \mathcal{B} \in \mathbb{L}(\mathcal{L})$, there exists $\mathcal{C} \in \mathbb{L}(\mathcal{L})$ such that $\mathcal{A} \preccurlyeq \mathcal{C}$ and $\mathcal{B} \preccurlyeq \mathcal{C}$.
(2) $\mathcal{L}$ has the *completely faithful extension property* if for every set $\{\mathcal{A}_i : i \in \omega\} \subseteq \mathbb{L}(\mathcal{L})$, there exists $\mathcal{C} \in \mathbb{L}(\mathcal{L})$ such that $(\forall i \in \omega)\ \mathcal{A}_i \preccurlyeq \mathcal{C}$.

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(3) $\mathcal{L}$ has the *totally faithful extension property* if for every set $\{\mathcal{A}_i : i \in \omega\} \subseteq \mathbb{L}(\mathcal{L})$ which is totally ordered by embeddability, there exists $\mathcal{C} \in \mathbb{L}(\mathcal{L})$ such that $(\forall i \in \omega)\ \mathcal{A}_i \preccurlyeq \mathcal{C}$.

The completely faithful extension property actually appeared much earlier, in a paper by Fraïssé, [Fra48] in which four conjectures about $\mathbb{L}$ were made (see [Ros82, page 178]). One of them is the celebrated *Fraïssé's Conjecture*, which states that in $\mathbb{L}$, ordered by embeddability, there are no infinite strictly descending sequences and no infinite antichains. This conjecture was later proved by Laver [Lav71]. (See [Clo90, Sho93, Mon06] for proof-theoretic analyses of this statement.) Of the other three conjectures, two have been proved to be true and the other one has been shown to be false. The false conjecture says the following:

> (Conjecture (3) in [Fra48].) For every countable linear orderings $\mathcal{L}, \mathcal{C}_0, \mathcal{C}_1, ...$, if for every $\mathcal{C}$ such that $\forall n(\mathcal{C}_n \preccurlyeq \mathcal{C})$ we have that $\mathcal{L} \preccurlyeq \mathcal{C}$, then there is some n such that $\mathcal{L} \preccurlyeq \mathcal{C}_n$.

Note that this is equivalent to the statement that every countable linear ordering $\mathcal{L}$ has the completely faithful extension property. The first counterexample, $\mathcal{L} = \omega + \omega^*$, was found by Jullien [Jul69] (see [Ros82, Chapter 10]). That ordering does not have even the finitely faithful extension property, as witnessed by $\mathcal{A} = \omega \cdot \omega^*$ and $\mathcal{B} = \omega^* \cdot \omega$. It will follows from Lemma 3.2 that $\omega + \omega^*$ is also an example of a linear ordering which is not countably complementable. A characterization of all such counterexamples follows from Theorem 1.3.

Our main theorem gives four characterizations of linear orderings that are countably complementable.

Moreover, we will give an explicit way of constructing complements for any linear orderings that has one. This construction would be by transfinite recursion and can be recovered from the proof of Proposition 3.9.

Here is our main Theorem. We will prove it in Section 3.

Theorem 1.3. *Let $\mathcal{L}$ be a countable linear ordering. The following are equivalent:*

(CC.1) *$\mathcal{L}$ is countably complementable;*
(CC.2) *$\mathcal{L}$ has the completely faithful extension property;*
(CC.3) *$\mathcal{L}$ has the finitely faithful extension property;*
(CC.4) *$\mathcal{L}$ has no essential segment of the form $\langle \rightarrow \mid \leftarrow \rangle$;*
(CC.5) *Either $\mathcal{L}$ is equimorphic to $\mathbb{Q}$, or $\mathcal{L}$ is scattered and if $\mathcal{F}_0 + ... + \mathcal{F}_n$ is a minimal decomposition of $\mathcal{L}$, then for no $i < n$ we have that $\mathcal{F}_i$ and $\mathcal{F}_{i+1}$ are incomparable, $\mathcal{F}_i$ is indecomposable to the right, and $\mathcal{F}_{i+1}$ is indecomposable to the left.*

See relevant definitions in the background section below.

Conditions (CC.4) and (CC.5) are the ones that we see as characterizing the countably complementable linear orderings. Condition (CC.4) says that the linear orderings of the form $\langle \rightarrow \mid \leftarrow \rangle$ are essentially the only reason why a linear ordering could be not countably complementable. Condition (CC.5) characterizes the countably complementable linear orderings, because every scattered linear ordering has a unique minimal finite decomposition into indecomposable linear orderings; see Lemma 2.1 below.

A characterization of a similar sort was given by Jullien [Jul69] for the extendible linear orderings. A linear ordering $\mathcal{L} \in \mathbb{L}$ is *extendible* if every countable partial ordering into which $\mathcal{L}$ does not embed has a linearization into which $\mathcal{L}$ does not embed either. This class of linear orderings was studied by Bonnet and Pouzet [BP82] even in the uncountable case, and then from a logic viewpoint by Downey, Hirschfeldt, Lempp and Solomon [DHLS03] and by Montalbán [Mon06]. We observe that every extendible linear ordering has the finitely faithful extension property, and thus is countably complementable. (If $\mathcal{A}, \mathcal{B} \in \mathbb{L}(\mathcal{L})$, then $\mathcal{L}$ cannot be embedded into the partial ordering that consists of incomparable copies of $\mathcal{A}$ and $\mathcal{B}$, and a linearization of this partial ordering without copies of $\mathcal{L}$ gives an extension of both $\mathcal{A}$ and $\mathcal{B}$ which is in $\mathbb{L}(\mathcal{L})$.) The other direction does not hold. For example, $\omega + \mathbf{1} + \mathbf{1}$ is countably complementable but not extendible.

An effective version of the notion of taking complements of a linear ordering had already appeared in [Mon06], where the author studies the extendibility property from a logic viewpoint. Presented in that paper is an algorithm that, given a computable indecomposable linear ordering $\mathcal{L}$, constructs a (countable) Π^1_1 linear ordering $\mathrm{com}^{CK}(\mathcal{L})$ such that, for every computable linear ordering $\mathcal{A}$, $\mathcal{A} \preccurlyeq \mathrm{com}^{CK}(\mathcal{L}) \iff \mathbf{1}+\mathcal{L}+\mathbf{1} \not\preccurlyeq \mathcal{A}$.

Though Conjecture (3) in [Fra48] was proved to be false, the question of whether it holds in the case where $\mathcal{C}_0 \preccurlyeq \mathcal{C}_1 \preccurlyeq \mathcal{C}_2 \preccurlyeq \ldots$ remained open (see [Ros82, page 178]). We will show that every countable linear ordering has the totally faithful extension property, answering this question in the affirmative. Hagendorf [Hag79] has conjectured that that this is true even without restricting to the countable case.

Here is our second main theorem. We will prove it in section 4.

Theorem 1.4. *Every $\mathcal{L} \in \mathbb{L}$ has the totally faithful extension property.*

The way we prove this is by showing that every countable linear ordering is weakly countably complementable (see definition below) and showing that every weakly countably complementable linear ordering has the totally faithful extension property.

Definition 1.5. A countable linear ordering $\mathcal{L}$ is *weakly countably complementable* if there exist linear orderings $\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_n$, possibly uncountable, such that for every countable linear ordering $\mathcal{B}$,

$$\mathcal{B} \in \mathbb{L}(\mathcal{L}) \iff (\exists i \leq n)\ \mathcal{B}_i \preccurlyeq \overline{\mathcal{L}}_i,$$

in which case we call $\{\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_n\}$ a *complement set* for $\mathcal{L}$.

A construction of a complement set for each countable linear ordering follows from Proposition 4.3.

2. Background

In this section, we introduce our notation and prove some basic lemmas about indecomposable linear orderings. Let $\mathcal{A} = \langle A, \leq_{\mathcal{A}} \rangle$ be a linear ordering. The *reverse* linear ordering of $\mathcal{A}$ is $\mathcal{A}^* = \langle A, \geq_{\mathcal{A}} \rangle$. Let $\mathcal{B} = \langle B, \leq_{\mathcal{B}} \rangle$ be another linear ordering. The *product*, $\mathcal{A} \cdot \mathcal{B}$, of $\mathcal{A}$ and $\mathcal{B}$ is obtained by starting with $\mathcal{B}$ and substituting a copy of $\mathcal{A}$ for each element of $\mathcal{B}$. That is, $\mathcal{A} \cdot \mathcal{B} = \langle A \times B, \leq_{\mathcal{A}\cdot\mathcal{B}} \rangle$, where $\langle x, y \rangle \leq_{\mathcal{A}\cdot\mathcal{B}} \langle x', y' \rangle$ iff either $y <_{\mathcal{B}} y'$, or $y = y'$ and $x \leq_{\mathcal{A}} x'$. The *sum*, $\sum_{i \in \mathcal{A}} \mathcal{B}_i$, of a set $\{\mathcal{B}_i\}_{i \in A}$ of linear orderings indexed by $\mathcal{A}$, is constructed by starting with $\mathcal{A}$ and, for each $i \in A$, substituting a copy of $\mathcal{B}_i$. Thus, $\mathcal{A} \cdot \mathcal{B} = \sum_{i \in \mathcal{B}} \mathcal{A}$. If $\mathcal{A} = \{0 < 1 < \cdots < m-1\}$, we sometimes write $\mathcal{B}_0 + ... + \mathcal{B}_{m-1}$ or $\sum_{i=0}^{m-1} \mathcal{B}_i$ instead of $\sum_{i \in \mathcal{A}} \mathcal{B}_i$.

A linear ordering $\mathcal{L}$ is *indecomposable* if whenever $\mathcal{L} = \mathcal{A}+\mathcal{B}$, either $\mathcal{L} \preccurlyeq \mathcal{B}$ or $\mathcal{L} \preccurlyeq \mathcal{A}$. We allow $\mathcal{A}$ or $\mathcal{B}$ to be $\emptyset$, so $\mathbf{1}$, the linear ordering with one element, is indecomposable. $\mathcal{L}$ is *indecomposable to the right (resp. left)* if $\mathcal{L} \neq \mathbf{1}$ and whenever $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $\mathcal{B} \neq \emptyset$ (resp. $\mathcal{A} \neq \emptyset$), we have that $\mathcal{L} \preccurlyeq \mathcal{B}$ (resp. $\mathcal{L} \preccurlyeq \mathcal{A}$). Sometimes, we say that $\mathcal{L}$ is $\langle \rightarrow \rangle$ (resp. $\langle \leftarrow \rangle$), to mean that $\mathcal{L}$ is indecomposable to the right (resp. left).

Two linear ordering $\mathcal{L}_1$, $\mathcal{L}_2$ are *equimorphic* (denoted $\mathcal{L}_1 \sim \mathcal{L}_2$) if they can be embedded into each other. We say that a linear ordering $\mathcal{L}$ has the form $\langle \rightarrow, \leftarrow \rangle$ if it can be written as $\mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is $\langle \rightarrow \rangle$, $\mathcal{B}$ is $\langle \leftarrow \rangle$, and neither $\mathcal{L} \sim \mathcal{A}$ nor $\mathcal{L} \sim \mathcal{B}$. If, in addition, $\mathcal{A}$ and $\mathcal{B}$ are incomparable, we say that $\mathcal{L}$ is of the form $\langle \rightarrow \mid \leftarrow \rangle$.

We note that all the properties of linear orderings that are discussed in this paper are preserved under equimorphism.

Given $\mathcal{L} = \mathcal{A} + \mathcal{B} + \mathcal{C}$, we say that $\mathcal{B}$ is an *essential segment* of $\mathcal{L}$ if whenever $\mathcal{L} \preccurlyeq \mathcal{A} + \mathcal{B}' + \mathcal{C}$ we have that $\mathcal{B} \preccurlyeq \mathcal{B}'$. For example, in $\omega + \omega + \omega$, each copy of ω is essential, but in $\omega^2 + \omega + \omega^2$ the middle copy of ω is not.

A linear ordering is *scattered* if it contains no copy of $\mathbb{Q}$, the ordering of the rationals. If $\mathcal{L}$ is countable, then $\mathcal{L}$ is scattered if and only if it is not equimorphic to $\mathbb{Q}$, the reason being that every countable linear ordering can be embedded into $\mathbb{Q}$. We say that $\mathcal{L} = \mathcal{A}_0 + ... + \mathcal{A}_n$ is a *minimal decomposition* of $\mathcal{L}$ if each $\mathcal{A}_i$ is indecomposable and n is least possible.

Theorem 2.1 (Laver [Lav71], Jullien [Jul69])**.** *Every scattered linear ordering has a unique minimal decomposition up to equimorphism.*

The existence of finite decompositions is due to Laver, and the uniqueness of minimal decompositions is due to Jullien.

Another very important structural theorem of Laver is the following one.

Theorem 2.2. *Every countable scattered indecomposable linear ordering, different from* $\mathbf{1}$*, can be written as either an* ω*-sum or an* ω^**-sum of indecomposables.*

Now we state a few very simple facts that will be useful later.

Lemma 2.3.
1. *If* $\mathcal{A} + \mathcal{B} \preccurlyeq \mathcal{C} + \mathcal{D}$*, then either* $\mathcal{A} \preccurlyeq \mathcal{C}$ *or* $\mathbf{1} + \mathcal{B} \preccurlyeq \mathcal{D}$*.*
2. *If* $\mathcal{L}$ *is indecomposable and* $\mathcal{L} \preccurlyeq \sum_{i=0}^{n} \mathcal{A}_i$*, then for some* $i \leq n$*,* $\mathcal{L} \preccurlyeq \mathcal{A}_i$*.*
3. *If* $\mathcal{L}$ *is indecomposable to the left and* $\mathcal{L} \preccurlyeq \sum_{i \in \alpha} \mathcal{A}_i$*, where* α *is an ordinal, then for some* $i \in \alpha$*,* $\mathcal{L} \preccurlyeq \mathcal{A}_i$*.*
4. *If* $\mathcal{A} + \mathcal{A} \preccurlyeq \mathcal{A}$*, then* $\mathbb{Q} \preccurlyeq \mathcal{A}$*.*
5. *If* $\mathcal{A}$ *is scattered and indecomposable to the right, then* $\mathcal{A} + \mathbf{1} \not\preccurlyeq \mathcal{A}$*.*

3. Countably complementable linear orderings

This section is dedicated to prove Theorem 1.3.

First, in Section 3.1, we show that all the conditions in Theorem 1.3 are true when $\mathcal{L}$ is equimorphic to $\mathbb{Q}$, and we show that (CC.1) $\Longrightarrow$ (CC.2). The implication (CC.2) $\Longrightarrow$ (CC.3) is trivially true. The implication (CC.4) $\Longrightarrow$ (CC.5) is a particular case of

[Mon06, Lemma 5.9]. The remaining two implications are proved in the following subsections.

3.1. **The simple implications.** We start by proving that Theorem 1.3 holds when $\mathcal{L} \sim \mathbb{Q}$. So, later, we can assume $\mathcal{L}$ is scattered and use all the structural results we know about scattered linear orderings.

Lemma 3.1. *All the conditions in Theorem 1.3 are satisfied if* $\mathcal{L} \sim \mathbb{Q}$.

Proof. To prove (CC.1) consider $\mathbb{Z}^{\omega_1}$, where $\mathbb{Z}$ is the ordering of the integers. $\mathbb{Z}^{\omega_1}$ consists of the set of maps $f\colon \omega_1 \to \mathbb{Z}$ such that $f(\xi) = 0$ for all but finitely many $\xi \in \omega_1$, where $f \leq g$ if for the greatest $\xi \in \omega_1$ where $f(\xi) \neq g(\xi)$ we have $f(\xi) \leq_{\mathbb{Z}} g(\xi)$. It is well known that $\mathbb{Q} \not\preccurlyeq \mathbb{Z}^{\omega_1}$ and that for every countable scattered $\mathcal{B}$, we have $\mathcal{B} \preccurlyeq \mathbb{Z}^{\omega_1}$. (We actually have $\mathcal{B} \preccurlyeq \mathbb{Z}^{\mathrm{rk}(\mathcal{B})+1}$, where $\mathrm{rk}(\mathcal{B})$ is the Hausdorff rank of $\mathcal{B}$.) Thus, $\mathbb{Z}^{\omega_1}$ is a complement for $\mathbb{Q}$.

Conditions (CC.2) and (CC.3) follow from the fact that countable sums of countable scattered linear orderings are also countable and scattered.

For condition (CC.4), we note that if $\mathcal{L} = \mathcal{A} + \mathcal{B} + \mathcal{C}$ and $\mathcal{B}$ is a segment of the form $\langle \rightarrow \mid \leftarrow \rangle$, then, since $\mathbb{Q} \not\preccurlyeq \mathcal{B}$, $\mathbb{Q}$ embeds into either $\mathcal{A}$ or $\mathcal{C}$. Then, since $\mathcal{L} \preccurlyeq \mathbb{Q}$, the segment $\mathcal{B}$ cannot be essential.

Condition (CC.5) is trivially satisfied. □

Lemma 3.2. *Every countably complementable linear ordering has the completely faithful extension property.*

Proof. Let $\mathcal{L}$ be countably complementable, and let $\overline{\mathcal{L}}$ be a complement for $\mathcal{L}$. Consider a sequence $\{\mathcal{A}_i : i \in \omega\} \subseteq \mathbb{L}(\mathcal{L})$. For each i, there exists a subset $\mathcal{B}_i \subseteq \overline{\mathcal{L}}$ which is isomorphic to $\mathcal{A}_i$. Let $\mathcal{C} = \bigcup_{i\in\omega} \mathcal{B}_i$. Then $\mathcal{C} \in \mathbb{L}(\mathcal{L})$ and $(\forall i \in \omega)\ \mathcal{A}_i \preccurlyeq \mathcal{C}$. □

3.2. **Non-countably complementable linear orderings.** Now we prove that (CC.3) implies (CC.4).

Lemma 3.3. *If* $\mathcal{L}$ *is* $\langle \rightarrow \mid \leftarrow \rangle$*, it does not have the finitely faithful extension property.*

Proof. Write $\mathcal{L}$ as $\mathcal{D}+\mathcal{E}$, where $\mathcal{D}$ is indecomposable to the right and $\mathcal{E}$ is indecomposable to the left, and $\mathcal{D}$ and $\mathcal{E}$ are indecomposable and incomparable. Write $\mathcal{D}$ as $\sum_{i\in\omega} \mathcal{D}_i$ and $\mathcal{E}$ as $\sum_{i\in\omega^*} \mathcal{E}_i$. Let

$$\mathcal{A} = \sum_{n\in\omega} (\mathcal{E} + \mathcal{D}_n) \quad \text{and} \quad \mathcal{B} = \sum_{n\in\omega^*} (\mathcal{D} + \mathcal{E}_n).$$

First, note that $\mathcal{D}$ does not embed into any proper initial segment of $\mathcal{A}$; since $\mathcal{D}$ is indecomposable and cannot be embedded in $\mathcal{E}$, or any of the $\mathcal{D}_n$'s. Therefore $\mathcal{L} \not\preccurlyeq \mathcal{A}$. Similarly, $\mathcal{E}$ does not embed into any proper final segment of $\mathcal{B}$; hence $\mathcal{L} \not\preccurlyeq \mathcal{B}$.

Let $\mathcal{C}$ be such that $\mathcal{A} \preccurlyeq \mathcal{C}$ and $\mathcal{B} \preccurlyeq \mathcal{C}$, and let $\mathcal{A}'$ and $\mathcal{B}'$ be subsets of $\mathcal{C}$ isomorphic to $\mathcal{A}$ and $\mathcal{B}$, respectively. If every element of $\mathcal{A}'$ is less than every element of $\mathcal{B}'$, then $\mathcal{L} = \mathcal{D} + \mathcal{E} \preccurlyeq \mathcal{A} + \mathcal{B} \preccurlyeq \mathcal{C}$. Otherwise, there exists a final segment of $\mathcal{A}'$ that lies to the right of some initial segment of $\mathcal{B}'$. Since $\mathcal{E}$ embeds into every final segment of $\mathcal{A}$, and $\mathcal{D}$ embeds into every initial segment of $\mathcal{B}$, we have that $\mathcal{L} \preccurlyeq \mathcal{C}$. In any case $\mathcal{C} \notin \mathbb{L}(\mathcal{L})$, so $\mathcal{L}$ does not have the finitely faithful extension property. □

Lemma 3.4. *Let $\mathcal{L} = \mathcal{D} + \mathcal{E} + \mathcal{F}$, where $\mathcal{E}$ is an essential segment of $\mathcal{L}$. If $\mathcal{E}$ does not have the finitely faithful extension property, neither does $\mathcal{L}$.*

Proof. Let $\mathcal{A}, \mathcal{B} \in \mathbb{L}(\mathcal{E})$ be such that $\mathcal{C} \notin \mathbb{L}(\mathcal{E})$ for any $\mathcal{C}$ with $\mathcal{A} \preccurlyeq \mathcal{C}$ and $\mathcal{B} \preccurlyeq \mathcal{C}$. Since $\mathcal{E}$ is essential in $\mathcal{L}$, we have that $\mathcal{L} \not\preccurlyeq \mathcal{D} + \mathcal{A} + \mathcal{F}$, because otherwise we would have $\mathcal{E} \preccurlyeq \mathcal{A}$. Analogously, $\mathcal{L} \not\preccurlyeq \mathcal{D} + \mathcal{B} + \mathcal{F}$. Now, let $\mathcal{C}$ be such that both $\mathcal{D} + \mathcal{A} + \mathcal{F} \preccurlyeq \mathcal{C}$ and $\mathcal{D} + \mathcal{B} + \mathcal{F} \preccurlyeq \mathcal{C}$. We want to show that $\mathcal{C} \notin \mathbb{L}(\mathcal{L})$. Choose embeddings of $\mathcal{D} + \mathcal{A} + \mathcal{F}$ and $\mathcal{D} + \mathcal{B} + \mathcal{F}$ into $\mathcal{C}$. Then express $\mathcal{C}$ as $\mathcal{C}_0 + \mathcal{C}_1 + \mathcal{C}_2$, where $\mathcal{C}_0$ contains the image of $\mathcal{D}$ in at least one of those embeddings, $\mathcal{C}_2$ contains the image of $\mathcal{F}$ in at least one of them, and $\mathcal{C}_1$ contains the images of both $\mathcal{A}$ and $\mathcal{B}$. By the choice of $\mathcal{A}$ and $\mathcal{B}$, we have that $\mathcal{E} \preccurlyeq \mathcal{C}_1$. But then $\mathcal{L} = \mathcal{D} + \mathcal{E} + \mathcal{F} \preccurlyeq \mathcal{C}_0 + \mathcal{C}_1 + \mathcal{C}_2 = \mathcal{C}$, so $\mathcal{L}$ does not have the finitely faithful extension property. □

The implication (CC.3) $\Longrightarrow$ (CC.4) in Theorem 1.3 follows from the two lemmas above.

3.3. **Constructing the complements.** Now we show that (CC.5) implies (CC.1). Moreover, we explicitly construct a complement for each countable linear ordering that satisfies (CC.5). The idea of the construction in following lemma is taken from [Mon06, 6.5], where the effective version is considered.

Lemma 3.5. *Every scattered indecomposable countable linear ordering $\mathcal{L}$ is countably complementable.*

Moreover, if $\mathcal{L}$ is $\langle \to \rangle$ and $\mathcal{L} = \sum_{i\in\omega} \mathcal{L}_i$, where, for every i, $\mathcal{L}_i$ is indecomposable and $\overline{\mathcal{L}_i^{+1}}$ is a complement for $\mathcal{L}_i + \mathbf{1}$, then

$$\overline{\mathcal{L}} = \left(\sum_{i\in\omega^*} \overline{\mathcal{L}_i^{+1}} \right) \cdot \omega_1^*$$

is a complement for $\mathcal{L}$, and

$$\overline{\mathcal{L}^{+1}} = \left(\sum_{i\in\omega^*} \overline{\mathcal{L}_i^{+1}} \right) \cdot \omega_1^* \cdot \omega$$

is a complement for $\mathcal{L} + \mathbf{1}$.

Proof. We first show that $\overline{\mathcal{L}}$ is a complement for $\mathcal{L}$. Suppose, toward a contradiction, that $\mathcal{L} \preccurlyeq \overline{\mathcal{L}}$. Then, since $\mathcal{L}$ is indecomposable to the right, $\mathcal{L} \preccurlyeq \sum_{i\in\omega^*} \overline{\mathcal{L}_i^{+1}}$. Moreover, for some $i \in \omega^*$, $\mathcal{L} \preccurlyeq \overline{\mathcal{L}_i^{+1}}$. But then $\mathcal{L}_i + \mathbf{1} \preccurlyeq \overline{\mathcal{L}_i^{+1}}$, which is a contradiction.

Suppose that $\mathcal{B}$ is countable and $\mathcal{L} \not\preccurlyeq \mathcal{B}$. It cannot be the case that, for every i, $\mathcal{L}_i + \mathbf{1}$ embeds into every final segment of $\mathcal{B}$, because otherwise we could construct an embedding of $\mathcal{L}$ into $\mathcal{B}$. Thus, there exist some final segment $\mathcal{B}_0$ of $\mathcal{B}$ and some $j \in \omega$ such that $\mathcal{L}_j + \mathbf{1} \not\preccurlyeq \mathcal{B}_0$. Let $\mathcal{B}^0$ be such that $\mathcal{B} = \mathcal{B}^0 + \mathcal{B}_0$. Now, since $\mathcal{L} \not\preccurlyeq \mathcal{B}^0$, we can write $\mathcal{B}^0$ as $\mathcal{B}^1 + \mathcal{B}_1$ in such way that for some j, $\mathcal{L}_j + \mathbf{1} \not\preccurlyeq \mathcal{B}_1$. Continuing in this fashion we can decompose $\mathcal{B}$ as $\sum_{\xi\in\beta} \mathcal{B}_\xi$ for some $\beta < \omega_1$, where for every $\xi \in \beta$ there is some $j_\xi \in \omega$ such that $\mathcal{L}_{j_\xi} + \mathbf{1} \not\preccurlyeq \mathcal{B}_\xi$. Then, for every ξ, $B_\xi \preccurlyeq \overline{\mathcal{L}_{j_\xi}^{+1}} \preccurlyeq \sum_{i\in\omega^*} \overline{\mathcal{L}_j^{+1}}$. Therefore, $\mathcal{B} \preccurlyeq \overline{\mathcal{L}}$.

We now prove that $\overline{\mathcal{L}^{+1}}$ is the complement of $\mathcal{L} + \mathbf{1}$.

First, we note that $\mathcal{L}+\mathbf{1} \not\preccurlyeq \overline{\mathcal{L}^{+1}}$, because otherwise we would have that $\mathcal{L} \preccurlyeq \overline{\mathcal{L}}$. Second, suppose that $\mathcal{B}$ is countable and $\mathcal{L}+\mathbf{1} \not\preccurlyeq \mathcal{B}$. If $\mathcal{B}$ has a last element, then $\mathcal{L} \not\preccurlyeq \mathcal{B}$ and hence $\mathcal{B} \preccurlyeq \overline{\mathcal{L}} \preccurlyeq \overline{\mathcal{L}^{+1}}$. So suppose that $\mathcal{B}$ has no last element, and write $\mathcal{B}$ as $\sum_{m\in\omega} \mathcal{B}_m$. Then, we have that $\mathcal{L} \not\preccurlyeq \mathcal{B}_m$ for every m, and hence that $\mathcal{B}_m \preccurlyeq \overline{\mathcal{L}}$. It follows that $\mathcal{B} \preccurlyeq \overline{\mathcal{L}^{+1}}$. □

Lemma 3.6. *If $\mathcal{D}+\mathbf{1}$ and $\mathbf{1}+\mathcal{E}$ are countably complementable, then so is $\mathcal{L}=\mathcal{D}+\mathbf{1}+\mathcal{E}$.*

Moreover, If $\overline{\mathcal{D}^{+1}}$ is a complement for $\mathcal{D}+\mathbf{1}$ and $\overline{{}^{1+}\mathcal{E}}$ is a complement for $\mathbf{1}+\mathcal{E}$, then

$$\overline{\mathcal{L}} = \overline{\mathcal{D}^{+1}} + \overline{{}^{1+}\mathcal{E}}$$

is a complement for $\mathcal{L}$.

Proof. First, we note that $\mathcal{L} \not\preccurlyeq \overline{\mathcal{D}^{+1}} + \overline{{}^{1+}\mathcal{E}}$, because otherwise we would have that either $\mathcal{D}+\mathbf{1} \preccurlyeq \overline{\mathcal{D}^{+1}}$ or $\mathbf{1}+\mathcal{E} \preccurlyeq \overline{{}^{1+}\mathcal{E}}$. Now consider $\mathcal{B} \in \mathbb{L}(\mathcal{L})$. Let $\mathcal{B}_0 = \{x \in \mathcal{B} : \mathcal{D}+\mathbf{1} \not\preccurlyeq \mathcal{B}_{(\leq x)}\}$, and let $\mathcal{B}_1$ be such that $\mathcal{B}=\mathcal{B}_0+\mathcal{B}_1$. Clearly, $\mathcal{D}+\mathbf{1} \not\preccurlyeq \mathcal{B}_0$. Also, if $\mathbf{1}+\mathcal{E} \preccurlyeq \mathcal{B}_1$, there would be some $x \in \mathcal{B}_1$ such that $\mathcal{D} \preccurlyeq \mathcal{B}_{(<x)}$ and $\mathcal{E} \preccurlyeq \mathcal{B}_{(>x)}$ contradicting that $\mathcal{D}+\mathbf{1}+\mathcal{E} \not\preccurlyeq \mathcal{B}$. Thus, $\mathcal{B}_0 \preccurlyeq \overline{\mathcal{D}^{+1}}$ and $\mathcal{B}_1 \preccurlyeq \overline{{}^{1+}\mathcal{E}}$, and hence $\mathcal{B} \preccurlyeq \overline{\mathcal{L}}$. □

Lemma 3.7. *If $\mathcal{L}+\mathbf{1}$ is countably complementable, then so is $\mathcal{L}+\mathbf{1}+\mathbf{1}$.*

Moreover, if $\overline{\mathcal{L}^{+1}}$ is a complement for $\mathcal{L}+\mathbf{1}$,then

$$\overline{\mathcal{L}^{+1}} + \mathbf{1}$$

is a complement for $\mathcal{L}+\mathbf{1}+\mathbf{1}$.

Proof. Clearly, $\mathcal{L}+\mathbf{1}+\mathbf{1} \not\preccurlyeq \overline{\mathcal{L}^{+1}}+\mathbf{1}$, because otherwise we would have that $\mathcal{L}+\mathbf{1} \preccurlyeq \overline{\mathcal{L}^{+1}}$. Now consider $\mathcal{B} \in \mathbb{L}(\mathcal{L}+\mathbf{1}+\mathbf{1})$. If $\mathcal{B}$ has no last element, then $\mathcal{L}+\mathbf{1} \not\preccurlyeq \mathcal{B}$, hence $\mathcal{B} \preccurlyeq \overline{\mathcal{L}^{+1}} \preccurlyeq \overline{\mathcal{L}^{+1}}+\mathbf{1}$. So suppose that $\mathcal{B}=\mathcal{A}+\mathbf{1}$. Then $\mathcal{L}+\mathbf{1} \not\preccurlyeq \mathcal{A}$, hence $\mathcal{A} \preccurlyeq \overline{\mathcal{L}^{+1}}$ and $\mathcal{B} \preccurlyeq \overline{\mathcal{L}^{+1}}+\mathbf{1}$. □

Lemma 3.8. *If $\mathcal{D}$ and $\mathcal{E}$ are indecomposable, $\mathcal{E}$ is $\langle\leftarrow\rangle$, and $\mathcal{D} \preccurlyeq \mathcal{E}$, then $\mathcal{L}=\mathcal{D}+\mathcal{E}$ is countably complementable.*

Moreover, if $\overline{\mathcal{D}}$ is a complement for $\mathcal{D}$ and $\overline{{}^{1+}\mathcal{E}}$ is a complement for $\mathbf{1}+\mathcal{E}$, then

$$\overline{\mathcal{L}} = \overline{\mathcal{D}} + \overline{{}^{1+}\mathcal{E}}$$

is a complement for $\mathcal{L}$.

Proof. First, note that $\mathcal{L} \not\preccurlyeq \overline{\mathcal{L}}$, because $\mathcal{D} \not\preccurlyeq \overline{\mathcal{D}}$ and $\mathbf{1}+\mathcal{E} \not\preccurlyeq \overline{{}^{1+}\mathcal{E}}$.

Consider $\mathcal{F} \in \mathbb{L}(\mathcal{L})$. Let

$$F_0 = \{x \in F : \mathcal{E} \preccurlyeq \mathcal{F}_{(>x)}\} \quad \text{and} \quad F_1 = \{x \in F : \mathcal{D} \preccurlyeq \mathcal{F}_{(<x)}\}.$$

Clearly, no element of F can be in both F_0 and F_1, because that would imply that $\mathcal{D}+\mathcal{E} \preccurlyeq \mathcal{F}$. First, suppose that there exists some $x \in F$ such that $x \notin F_0 \cup F_1$. Then $\mathcal{D} \not\preccurlyeq \mathcal{F}_{(<x)}$ and $\mathbf{1}+\mathcal{E} \not\preccurlyeq \mathcal{F}_{(\geq x)}$, hence $\mathcal{F} \preccurlyeq \overline{\mathcal{D}} + \overline{{}^{1+}\mathcal{E}}$. Finally, suppose that F_0 is the complement of F_1. Then $\mathcal{F}=\mathcal{F}_0+\mathcal{F}_1$ and

$$F_0 = \{x \in F : \mathcal{D} \not\preccurlyeq \mathcal{F}_{(<x)}\} \quad \text{and} \quad F_1 = \{x \in F : \mathcal{E} \not\preccurlyeq \mathcal{F}_{(>x)}\}.$$

If $\mathcal{E} \not\preccurlyeq \mathcal{F}$, then $\mathcal{F} \preccurlyeq \overline{^{1+}\mathcal{E}} \preccurlyeq \overline{\mathcal{L}}$, so suppose that $\mathcal{E} \preccurlyeq \mathcal{F}$. Note that $\mathbf{1}+\mathcal{E} \not\preccurlyeq \mathcal{F}_1$, and hence $\mathcal{F}_1 \preccurlyeq \overline{^{1+}\mathcal{E}}$. We will show now that $\mathcal{F}_0 \preccurlyeq \overline{\mathcal{D}}$. We cannot have $\mathcal{E} \preccurlyeq F_0$, because that would imply $\mathcal{D}+\mathbf{1} \preccurlyeq \mathcal{E}+\mathbf{1} \preccurlyeq \mathcal{E} \preccurlyeq \mathcal{F}_0$, and for no $x \in F_0$ we have $\mathcal{D} \preccurlyeq \mathcal{F}_{(<x)}$. So, $\mathcal{E} \preccurlyeq \mathcal{F}_1$. This implies that $\mathcal{D} \not\preccurlyeq \mathcal{F}_0$, because otherwise $\mathcal{L} \preccurlyeq \mathcal{F}$. So, we have that $\mathcal{F}_0 \preccurlyeq \overline{\mathcal{D}}$, and hence $\mathcal{F} \preccurlyeq \overline{\mathcal{L}}$. □

By reversing the orderings in the lemma above, we get that If $\mathcal{D}$ and $\mathcal{E}$ are indecomposable, $\mathcal{D}$ is $\langle\rightarrow\rangle$, and $\mathcal{E} \preccurlyeq \mathcal{D}$, then $\mathcal{L} = \mathcal{D}+\mathcal{E}$ is countably complementable.

Proposition 3.9. *Let $\mathcal{F}_0 + ... + \mathcal{F}_n$ be a minimal finite decomposition of $\mathcal{L}$ such that for no i we have that $\mathcal{F}_i + \mathcal{F}_{i+1}$ is $\langle\rightarrow \mid \leftarrow\rangle$. Then $\mathcal{L}$ is countably complementable.*

Proof. We use induction on n. If $n = 0$, then by Lemma 3.5, $\mathcal{L} = \mathcal{F}_0$ is countably complementable.

Suppose $n > 0$, and assume that the proposition is true for all $m < n$.

If $\mathcal{F}_i = \mathbf{1}$ for some i with $0 < i < n$, then $\mathcal{L} = (\mathcal{F}_0 + ... + \mathcal{F}_{i-1}) + \mathbf{1} + (\mathcal{F}_{i+1} + ... + \mathcal{F}_n)$. Using Lemma 3.6 and the induction hypothesis, we get that $\mathcal{L}$ is countably complementable.

If for some $i < n$ either $\mathcal{F}_{i+1}$ is $\langle\rightarrow\rangle$, or $\mathcal{F}_i$ is $\langle\leftarrow\rangle$, then $\mathcal{L}$ is equimorphic to $(\mathcal{F}_0 + ... + \mathcal{F}_i) + \mathbf{1} + (\mathcal{F}_{i+1} + ... + \mathcal{F}_n)$. So, again, using Lemma 3.6 and the induction hypothesis, we get that $\mathcal{L}$ is countably complementable.

This leaves only the possibility of having $n = 1$ and $\mathcal{L} = \mathcal{F}_0 + \mathcal{F}_1$. If $\mathcal{L} = \mathbf{1}+\mathbf{1}$, then it is countably complementable. If either $\mathcal{F}_0 = \mathbf{1}$ and $\mathcal{F}_1$ is $\langle\leftarrow\rangle$ or $\mathcal{F}_1 = \mathbf{1}$ and $\mathcal{F}_0$ is $\langle\rightarrow\rangle$, then $\mathcal{L}$ is countably complementable because of Lemma 3.5.

The only case remaining is $\mathcal{L} = \mathcal{F}_0 + \mathcal{F}_1$ where $\mathcal{F}_0$ is $\langle\rightarrow\rangle$, $\mathcal{F}_1$ is $\langle\leftarrow\rangle$, and either $\mathcal{F}_0 \preccurlyeq \mathcal{F}_1$ or $\mathcal{F}_1 \preccurlyeq \mathcal{F}_0$. In this case, $\mathcal{L}$ is countably complementable by Lemma 3.8. □

4. Weakly countably complementable linear orderings

Recall that $\mathcal{L} \in \mathbb{L}$ is weakly countable complementable if there exist linear orderings $\overline{\mathcal{L}}_0, ..., \overline{\mathcal{L}}_n$ such that $(\forall \mathcal{B} \in \mathbb{L})\ \mathcal{B} \in \mathbb{L}(\mathcal{L}) \iff (\exists i \leq n)\ \mathcal{B}_i \preccurlyeq \overline{\mathcal{L}}_i$.

We will prove in this section that every countable linear ordering has the totally faithfull extension property.

Lemma 4.1. *If $\mathcal{L}$ is weakly countable complementable, it has the totally faithful extension property.*

Proof. Let $\{\mathcal{A}_i : i \in \omega\} \in \mathbb{L}(\mathcal{L})$ be a set which is totally ordered by embeddability. Let $\{\overline{\mathcal{L}}_0, ..., \overline{\mathcal{L}}_n\}$ be a complement set for $\mathcal{L}$. Then for each $i \in \omega$, there exists $j \leq n$ such that $\mathcal{A}_i \preccurlyeq \overline{\mathcal{L}}_j$. Since $\{\mathcal{A}_i : i \in \omega\}$ is totally ordered by embeddability, there is some $j \leq n$ such that $\mathcal{A}_i \preccurlyeq \overline{\mathcal{L}}_j$ for every $i \in \omega$. Let $\mathcal{C} \subseteq \overline{\mathcal{L}}_j$ be the union of the images of all the embeddings $\mathcal{A}_i \preccurlyeq \overline{\mathcal{L}}_j$. So, we have that $(\forall i)\ \mathcal{A}_i \preccurlyeq \mathcal{C}$ but $\mathcal{L} \not\preccurlyeq \mathcal{C}$. □

Lemma 4.2. *If $\mathcal{D}$, $\mathcal{D}+\mathbf{1}$, $\mathcal{E}$, and $\mathbf{1}+\mathcal{E}$ are weakly countably complementable, then so is $\mathcal{L} = \mathcal{D}+\mathcal{E}$.*

Moreover, if $\overline{\mathbb{D}}$, $\overline{\mathbb{D}^{+1}}$, $\overline{\mathbb{E}}$ *and* $\overline{{}^{1+}\mathbb{E}}$ *are complement sets for* $\mathcal{D}$, $\mathcal{D}+\mathbf{1}$, $\mathcal{E}$, *and* $\mathbf{1}+\mathcal{E}$, *respectively, then*

$$\{\mathcal{A}+\mathcal{B} : \mathcal{A}\in\overline{\mathbb{D}}, \mathcal{B}\in\overline{{}^{1+}\mathbb{E}}\}\cup\{\mathcal{A}+\mathcal{B} : \mathcal{A}\in\overline{\mathbb{D}^{+1}}, \mathcal{B}\in\overline{\mathbb{E}}\}$$

is a complement set for $\mathcal{L}$.

Proof. First, if $\mathcal{A}\in\overline{\mathbb{D}}$ and $\mathcal{B}\in\overline{{}^{1+}\mathbb{E}}$, then $\mathcal{L}\not\preccurlyeq\mathcal{A}+\mathcal{B}$ because $\mathcal{D}\not\preccurlyeq\mathcal{A}$ and $\mathbf{1}+\mathcal{E}\not\preccurlyeq\mathcal{B}$. Analogously, if $\mathcal{A}\in\overline{\mathbb{D}^{+1}}$ and $\mathcal{B}\in\overline{\mathbb{E}}$, then $\mathcal{L}\not\preccurlyeq\mathcal{A}+\mathcal{B}$.

Now consider $\mathcal{C}\in\mathbb{L}(\mathcal{L})$. Let

$$\mathcal{C}_0=\{x\in\mathcal{C} : \mathcal{E}\not\preccurlyeq\mathcal{C}_{(>x)}\}\quad\text{and}\quad\mathcal{C}_1=\{x\in\mathcal{C} : D\not\preccurlyeq\mathcal{C}_{(<x)}\}.$$

These two sets cover $\mathcal{C}$ because otherwise we would have that $\mathcal{L}\preccurlyeq\mathcal{C}$. If there exists $x\in\mathcal{C}_0\cap\mathcal{C}_1$, then $\mathcal{D}\not\preccurlyeq\mathcal{C}_{(<x)}$ and $\mathbf{1}+\mathcal{E}\not\preccurlyeq\mathcal{C}_{(\geq x)}$, in which case $\mathcal{C}\preccurlyeq\mathcal{A}+\mathcal{B}$ for some $\mathcal{A}\in\overline{\mathbb{D}}$ and $\mathcal{B}\in\overline{{}^{1+}\mathbb{E}}$. So suppose that $\mathcal{C}_0$ and $\mathcal{C}_1$ are disjoint. Then $\mathcal{C}=\mathcal{C}_0+\mathcal{C}_1$ and

$$\mathcal{C}_0=\{x\in\mathcal{C} : \mathcal{D}\preccurlyeq\mathcal{C}_{(<x)}\}\quad\text{and}\quad\mathcal{C}_1=\{x\in\mathcal{C} : E\preccurlyeq\mathcal{C}_{(>x)}\}.$$

Note that $\mathcal{D}+\mathbf{1}\not\preccurlyeq\mathcal{C}_0$ and $\mathbf{1}+\mathcal{E}\not\preccurlyeq\mathcal{C}_1$. Also, it cannot be the case that both $\mathcal{D}\preccurlyeq\mathcal{C}_0$ and $\mathcal{E}\preccurlyeq\mathcal{C}_1$. If $\mathcal{D}\not\preccurlyeq\mathcal{C}_0$, then $\mathcal{C}\preccurlyeq\mathcal{A}+\mathcal{B}$ for some $\mathcal{A}\in\overline{\mathbb{D}}$ and $\mathcal{B}\in\overline{{}^{1+}\mathbb{E}}$. Similarly, if $\mathcal{E}\not\preccurlyeq\mathcal{C}_1$, then $\mathcal{C}\preccurlyeq\mathcal{A}+\mathcal{B}$ for some $\mathcal{A}\in\overline{\mathbb{D}^{+1}}$ and $\mathcal{B}\in\overline{\mathbb{E}}$. □

Proposition 4.3. *Every* $\mathcal{L}\in\mathbb{L}$ *is weakly countably complementable.*

Proof. If $\mathcal{L}\sim\mathbb{Q}$ or $\mathcal{L}$ is finite, then $\mathcal{L}$ is countably complementable, so suppose that $\mathcal{L}$ is infinite and scattered, and let $\mathcal{F}_0+...+\mathcal{F}_n$ be a minimal decomposition of $\mathcal{L}$. By grouping the terms of this decomposition write $\mathcal{L}$ as $\mathcal{A}_0+...+\mathcal{A}_k$ so that for each i, $\mathcal{A}_i$ is infinite and of the form $\mathbf{1}+\mathbf{1}+...+\mathbf{1}+\mathcal{B}_i+\mathbf{1}+...+\mathbf{1}$, where $\mathcal{B}_i$ is either $\langle\leftarrow\rangle$ or $\langle\rightarrow\rangle$. By noting that if $\mathcal{B}$ is $\langle\rightarrow\rangle$ then $\mathbf{1}+\mathcal{B}\sim\mathcal{B}$, and using Lemmas 3.5 and 3.7, we get that, for each $i\leq k$, the orderings $\mathcal{A}_i$, $\mathbf{1}+\mathcal{A}_i$ and $\mathcal{A}_i+\mathbf{1}$ are countably complementable. By induction on k and the lemma above, $\mathcal{L}$ is weakly countably complementable. □

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