THE COMPLEXITY OF COMPUTABLE CATEGORICITY

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Abstract. We show that the index set complexity of the computably categorical structures is Π^1_1-complete, demonstrating that computable categoricity has no simple syntactic characterization. As a consequence of our proof, we exhibit, for every computable ordinal α, a computable structure that is computably categorical but not relatively Δ^0_α-categorical.

1. Introduction

The goal of the present paper is to solve one of the most fundamental and longstanding questions in computable model theory. We show that the index set of computably categorical structures is Π^1_1-complete and hence there cannot be any reasonable or structural characterization of this class.

Mathematics is often concerned with classifying objects like groups and rings in terms of invariants which classify those objects up to isomorphism. For example the familiar use of dimension classifies vector spaces. As we see below, logic gives tools to demonstrate when *no* reasonable invariants or simpler descriptions are possible.

The concern of the present paper grew from the long-term program which seeks to understand the effective (i.e., algorithmic) content of mathematics. Familiar realizations of this program include algorithmic questions in groups such as the word problem, or the search for effective procedures in field theory and ring theory, such as the computational effectiveness of the Hilbert Basis Theorem. These considerations go back to the early 20th century, beginning with the work of Dehn [Deh11], Grete Herrmann [Her26], and Van der Waerden [vdW30]. Starting with Fröhlich and Shepherdson [FS56], Rabin [Rab60], Mal'cev [Mal61, Mal62] (and arguably Turing [Tur36]), the language and techniques of computability theory enable the modern precision possible in these studies. We can now calibrate the level of computability aligned to specific algorithmic questions. Clearly, if we are concerned with algorithms on structures, we should have some method of describing the domains in some kind of effective way. Thus, when we want to study the effective properties of mathematical structures, i.e., a set together with some operations and relations on it, it is natural to start with the *computable structures*, i.e, those mathematical structures having computable presentations. A presentation is *computable* if the set and the operations and relations on it are computable. A computable (presentation of a) field would be one whose domain is a computable set on which the operations of $+, \cdot, ^{-1}$ are computable.

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The first obstacle we encounter when we restrict ourselves to computable structures is that there can be computable presentations which are isomorphic, but not computably isomorphic. This implies that we cannot translate computational properties from one to the other. For instance, there are computable presentations of the countably infinite-dimensional $\mathbb{Q}$-vector space $\mathbb{Q}^\infty$ where all the finite-dimensional subspaces are computable, and computable presentations of $\mathbb{Q}^\infty$ where no finite-dimensional subspace is computable (see [DHK+07]). The reader can see that there is a fundamental tension between the classification tools of classical mathematics, like isomorphism, and those of effective mathematics, like *computable* isomorphism, and hence between invariants and *algorithmic* invariants.

The principal concern of the present paper is the class of structures where there is no such difference, as encapsulated by the following definition.

Definition 1.1. A computable structure $\mathcal{S}$ is *computably categorical* if any two computable presentations $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$ are computably isomorphic.

This concept was originally introduced by Mal'cev [Mal62] under the name 'autostability'[1] and has become an important branch of mathematical logic. Here is our basic problem:

Characterize the computably categorical structures.

For many classes of structures, there is a concise syntactic definition of the computably categorical structures: A computable linear order is computably categorical if and only if it has finitely many adjacencies (Dzgoev and Goncharov [GD80]); a computable Boolean algebra is computably categorical if and only if it has finitely many atoms (Goncharov, and independently La Roche [LR78]); a computable ordered abelian group is computably categorical if and only if it has finite rank (Goncharov, Lempp, and Solomon [GLS03]); a computable tree of finite height is computably categorical if and only if it is of finite type (Lempp, McCoy, R. Miller, and Solomon [LMMS05]); a computable torsion-free abelian group is computably categorical iff it has finite rank (Nurtazin [Nur74]); a computable p-group is computably categorical iff it can be written in one of the following forms: (i) $(\mathbb{Z}(p^\infty))^\ell \oplus G$ for $\ell \in \omega \cup \{\infty\}$ and G finite, or (ii) $(\mathbb{Z}(p^\infty))^n \oplus (\mathbb{Z}_{p^k})^\infty \oplus G$ where G is finite, and $n, k \in \omega$ (Goncharov [Gon80] and Smith [Smi81]); and so on.

Based on these examples, it is natural to hope for a simple characterization of computable categoricity. What form would such a characterization take? A fundamental aim of mathematical logic is to exhibit natural connections between syntax and semantics. In the same way, a fundamental aim of computable structure theory is to connect computational properties of algebraic structures with structural properties. So we would anticipate that a solution might entail a result saying that computably categorical structures had simple descriptions, either in arithmetic or in the language of the structure. That is, an acceptable answer would be a syntactic or structural characterization of computable categoricity.

Groundbreaking results of Goncharov [Gon75, Gon77] showed that if a structure is *sufficiently computable* then there was indeed a *syntactic* characterization of computable categoricity in terms of "effective naming" of the elements of the structure. That is, Goncharov demonstrated that a 2-decidable computable structure was computably categorical iff it had a computably enumerable Scott family of existential formulas, i.e., a computably enumerable family of existential formulas that define the automorphism orbits of the structure.[2] Goncharov's results filtered through another related notion called *relative* computable categoricity.

[1] We thank Alexandra Soskova for providing this reference.

[2] Here and later, we refer the reader to Section 2 for prerequisite definitions.

Definition 1.2. A computable structure $\mathcal{S}$ is *relatively computably categorical* if for any two presentations $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$ (computable or not), there is an isomorphism between them that is computable with the presentations of $\mathcal{A}$ and $\mathcal{B}$ as oracles.

Goncharov [Gon77] demonstrated the equivalence between relative computable categoricity and the existence of a computably enumerable Scott family of existential formulas, and in [Gon75] proved that for 2-decidable structures the notions of computable categoricity and relative computable categoricity coincide. A consequence of Goncharov's work is that relatively computably categorical structures are well-understood. For example, it is straightforward to show that the index set complexity of the relatively computably categorical structures is simply defined in terms of the arithmetical hierarchy: It is Σ^0_3-complete (Downey, Kach, Lempp, and Turetsky [DKLTed]). The point here is that an index set is a listing of members of the class, and *if the class admits a simple description, then its index set should be easily described in the arithmetical hierarchy, as this example shows.* We will return to this point later.

Implicit in Goncharov's papers from the 1970's is the question of whether there is a characterization of computably categorical structures for structures which are not 2-decidable. This is the question we answer here.

Downey, Kach, Lempp, and Turetsky [DKLTed] showed that a 1-decidable structure is computably categorical iff it has a Σ^0_2 infinitary Scott family. Thus, again, we find a simple characterization of computably categorical structures. The pattern generated by the 1- and 2-decidable examples suggests that perhaps a computable structure is computably categorical iff it has a Σ^0_3 infinitary Scott family. As we see below, this is not true.

At the same time there also had been considerable evidence that computable categoricity is an ill-behaved notion and could conceivably have no simple syntactic characterization. Evidence for this has taken many forms: Goncharov [Gon77] constructed a graph witnessing the divergence of computable categoricity and relative computable categoricity; White [Whi03] demonstrated the index set complexity of the computably categorical structures to be Π^0_4-hard and thus strictly more difficult than the index set complexity of the relatively computably categorical structures; Csima, Khoussainov, and Liu [CKL08] constructed a strongly locally finite computably categorical graph with an infinite chain of properly embedded components; R. Miller and Schoutens [MSar] constructed a computably categorical field of infinite transcendence degree over $\mathbb{Q}$; and so on.

During a lecture of Goncharov in 1997 at Kazan, Shore suggested a method to demonstrate that there was *no* reasonable characterization of computably categorical computable structures by proving completeness of the index set at a high level. Shore suggested that it would be enough to show that the index set of computably categorical structures is Π^1_1-complete. We prove this theorem in this paper. Let us explain what this means.

What such an index set result shows is that there is no computationally simpler way of telling if a computable presentation $\mathcal{A}$ is computably categorical than *to check, for any other computable presentation $\mathcal{B}$, if there exists a classical isomorphism between $\mathcal{A}$ and $\mathcal{B}$, then there is a computable one.* Note that this requires checking *all potential classical isomorphisms,* that is all continuum many functions from one domain to the other. We would expect that if there were a simple syntactic characterization of computable categoricity, then such a characterization should produce a simpler way for checking if a structure is computably categorical.

By way of illustration, consider the isomorphism problem for torsion-free abelian groups. The classical group theory literature suggests that there are no reasonable invariants for classifying torsion-free abelian groups up to isomorphism. Mathematical logic gives us a way to *prove that there are no such invariants.* What do we mean by this? Plainly, one invariant for the isomorphism type of G is "the isomorphism type of G". Note that this is a Σ^1_1-class as we need to search through the possible functions which could be isomorphisms for G. But such an "invariant" is hardly useful for understanding isomorphism types, since what we seek is something like dimension for vector spaces which *simplifies the isomorphism problem.* Downey and Montalbán [DM08] showed that the isomorphism problem for torsion-free abelian groups is Σ^1_1-complete, and hence *no invariant can be simpler arithmetically than the isomorphism type itself.* There are no useful invariants. The same reasoning applies here. The index set is as bad as it can possibly be and hence there is no reasonable simpler characterization.

To state our theorem formally, for each $e \in \mathbb{N}$, we let $\mathcal{M}_e$ be the eth (partial) computable structure computed by the eth Turing machine.

Theorem 1. *The index set*

$$I_{cc} := \{e \in \mathbb{N} \colon \mathcal{M}_e \text{ is computably categorical}\}$$

of the computably categorical structures is Π^1_1*-complete.*

The proof of this result is complex and unusual in several ways. In order to demonstrate Theorem 1, and of important independent interest, we show there is no connection between computable categoricity and relative Δ^0_α-categoricity for computable ordinals α.[3] The question of whether computable categoricity implies relative Δ^0_α-categoricity for some fixed α has also been open for some time. Ash [Ash87] showed that a structure is relatively Δ^0_α-categorical if and only if it has a computably enumerable Scott family of Σ^c_α-formulas. This is a nice syntactical characterization of the notion of relative Δ^0_α-categoricity, and implies that the index set of such structures is $\Sigma^0_{\alpha+2}$. It is not hard to see that if every computably categorical structure were relatively Δ^0_α-categorical, then we could decide if $\mathcal{M}_e$ is computably categorical as follows: First check if it is relatively Δ^0_α-categorical, and then check that for any other computable structure $\mathcal{A}$, if there exists a Δ^0_α-isomorphism between $\mathcal{S}$ and $\mathcal{A}$, then there is a computable one. This would be a $\Pi^0_{\alpha+3}$ procedure, which is much weaker that Π^1_1. Thus, the following theorem follows from our main theorem.

Theorem 2. *For every computable ordinal* α*, there is a computable structure that is computably categorical but not relatively* Δ^0_α*-categorical.*

Of course, this result is a strengthening of Chisholm, Fokina, Goncharov, Harizanov, Knight, and Quinn [CFG+09], extending results of Goncharov, Harizanov, Knight, McCoy, Miller and Solomon [GHK+05], where it is shown that Δ^0_α-categoricity does not imply relative Δ^0_α-categoricity.

The necessary argument for Theorem 1 does not need to explicitly exhibit a particular witness to Theorem 2. We will, however, in the course of our proof, explicitly exhibit such a structure for each computable ordinal α. Unfortunately, although this paper demonstrates that computable categoricity has no simple syntactic characterization, it fails to completely settle the connection between computable categoricity and relative hyperarithmetic categoricity. We don't know of another example of analytic completeness of an index set proven without solving the important question (also asked as Question 6.1) below which remains open.

Question 1.3. Is there a computable structure that is computably categorical but not relatively hyperarithmetically categorical?

Finally, we discuss an important issue of uniformity involving computably categorical structures. Given computable presentations $\mathcal{M}_i$ and $\mathcal{M}_j$ of a computably categorical structure $\mathcal{S}$, there is an index e of a computable isomorphism $\Phi_e : \mathcal{M}_i \cong \mathcal{M}_j$. Of course, though it is not difficult to see that $\mathbf{0}''$ always suffices to find such an index e, there is no a priori

[3] Again, refer to Section 2 for prerequisite definitions.

reason that such an index e can be found computably. When demonstrating that the structure built for Theorem 2 is computably categorical in Lemma 4.22, we need the nonuniform information of g restricted to $\mathrm{T}^{\underline{\mu}}$ (the exact meaning of this is unimportant here). The curious reader might wonder whether this nonuniformity is necessary. Since uniform computable categoricity is equivalent to relatively computable categoricity (see Downey, Hirschfeldt, and Khoussainov [DKK03] for definitions and results), this nonuniformity is provably required.

2. Background and Notation

Though we refer the reader to Ash and Knight [AK00] for further background on computable structure theory and computable model theory and to Soare [Soa87] for further background on computability theory, we present much of the necessary background in this section.

2.1. Prerequisite Terminology and Results.

Definition 2.1. A computable structure $\mathcal{S}$ is *relatively Δ^0_α-categorical* if between any two presentations $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$ there is a $(\Delta^0_\alpha(\mathcal{A}) \oplus \Delta^0_\alpha(\mathcal{B}))$-computable isomorphism.

Theorem 2.2 (Ash [Ash87])**.** *The following are equivalent for a computable structure $\mathcal{S}$:*

(1) *The structure $\mathcal{S}$ is relatively Δ^0_α-categorical.*
(2) *The orbits of $\mathcal{S}$ are effectively isolated by Σ^c_α-formulas, i.e., there is a computably enumerable family Φ of Σ^c_α-formulas over some fixed $\overline{c} \in S$ such that each $\overline{a} \in S$ satisfies some $\phi \in \Phi$, and if $\overline{a}, \overline{b} \in S$ both satisfy the same $\phi \in \Phi$ then they are automorphic.*
(3) *The Σ^c_α-types of $\mathcal{S}$ are effectively isolated by Σ^c_α-formulas, i.e., there is a computably enumerable family Φ of Σ^c_α-formulas over some fixed $\overline{c} \in S$ such that each $\overline{a} \in S$ satisfies some $\phi \in \Phi$, and if $\overline{a}, \overline{b} \in S$ both satisfy the same $\phi \in \Phi$ then they satisfy the same Σ^c_α-formulas.*

2.2. **Kleene's $\mathcal{O}$ and Feferman and Spector's $\mathcal{O}^*$.** We will use $\mathcal{O}^*$, an extension of Kleene's $\mathcal{O}$ due to Feferman and Spector [FS62]. The salient features of $\mathcal{O}^*$ are:

- There is a c.e. ordering relation $\preccurlyeq$ on the Σ^1_1-set $\mathcal{O}^*$ such that for each $\alpha \in \mathcal{O}^*$, the set $\{\beta \in \mathcal{O}^* : \beta \preccurlyeq \alpha\}$ is linearly ordered by $\preccurlyeq$ and has no hyperarithmetic infinite descending sequences.
- The set $\mathcal{O}^*$ has a $\preccurlyeq$-least element $\mathbf{0}$ (diverging from the standard notation 1). The sets of successor and limit elements are computable, and so is the predecessor function $\mathrm{pred}(\cdot)$ defined on the successor elements.
- The set of $\alpha \in \mathcal{O}^*$ for which $\{\beta \in \mathcal{O}^* : \beta \preccurlyeq \alpha\}$ is well-ordered is $\mathcal{O}$.
- There is a computable sequence of limit elements $\{\alpha_n \in \mathcal{O}^* : n \in \mathbb{N}\}$ such that the set $\{n : \alpha_n \in \mathcal{O}\}$ is Π^1_1-complete.

2.3. **Computably Enumerable Relations.** For each $\alpha \in \mathcal{O}^*$, we will build a structure $\mathcal{A}_\alpha$ in the language

$$\mathcal{L}_\alpha := \{\mathsf{E}\} \cup \{\mathsf{R}_\beta : \beta \preccurlyeq \alpha\} \cup \{\mathsf{M}_m : m \in \mathbb{N}\} \cup \{\mathsf{L}_m : m \in \mathbb{N}\},$$

where E is a binary relation and R_β, M_m, and L_m are unary relations. The binary relation E will represent the edge relation on a tree, with the root being identified as the only element on which E is reflexive. The unary relations R_β, M_m, and L_m will serve to *label* elements of the tree.

As the construction of $\mathcal{A}_\alpha$ and $\mathcal{A}_{\alpha'}$ will be independent for $\alpha \neq \alpha'$, we later fix an $\alpha \in \mathcal{O}^*$ and abbreviate $\mathcal{L}_\alpha$ as $\mathcal{L}$ and $\mathcal{A}_\alpha$ as $\mathcal{A}$. So that the construction better reflects the intuition, we work with the language $\mathcal{L}$ rather than the language of directed graphs. Though this language shift facilitates the intuition, it necessitates the following (non-standard) definitions.

Definition 2.3. A presentation $\mathcal{A}$ of an $\mathcal{L}$-structure is *$\mathcal{L}$-computable* if its domain A is computably enumerable, the relations $\mathsf{E}^{\mathcal{A}}$, $\mathsf{R}^{\mathcal{A}}_{\beta}$, and $\mathsf{M}^{\mathcal{A}}_{m}$ are uniformly computable, and the relations $\mathsf{L}^{\mathcal{A}}_{m}$ are uniformly c.e.

We then need to define the meaning of computable categoricity for $\mathcal{L}$-computable structures. It remains unchanged.

Definition 2.4. An $\mathcal{L}$-computable structure is *computably categorical* if any two $\mathcal{L}$-computable presentations of it are computably isomorphic.

It is not hard to show that, uniformly from an $\mathcal{L}$-computable presentation $\mathcal{A}$, there is a presentation $\mathcal{G}$ of a computable graph such that $\mathcal{A}$ is computably categorical (in the sense above) if and only if $\mathcal{G}$ is computably categorical (in the usual sense). A simple way to do this is to use loops to label nodes of the tree, rather than the unary relations. It is also easy to see that there is an effective enumeration $\{\mathcal{B}_\ell\}_{\ell\in\mathbb{N}}$ of all (partial) $\mathcal{L}$-computable structures.

We also need to define the meaning of embeddings of $\mathcal{L}$-computable structures. Viewing $\mathcal{L}$-structures as computable graphs via the transformation in the previous paragraph, an embedding of computable presentations of $\mathcal{L}$-structures should only require the preservation of the relations and non-relations E, R_β, and M_m, and the relations (but not the non-relations) L_m.

Definition 2.5. An *embedding* $\iota : \mathcal{A} \to \mathcal{B}$ is an (injective) map that preserves all relations and non-relations E, R_β, and M_m, and the relations (but not necessarily the non-relations) L_m.

2.4. **Notation.** Though our notation is mostly standard, we review certain definitions and conventions.

Definition 2.6. Let T be a tree. For a node $\rho \in T$, we denote the parent of ρ by ρ^-. By convention, the parent of the root of T is the root itself.

Throughout, we maintain certain notational conventions.

Convention 2.7. *The symbols α and β will be reserved for elements of $\mathcal{O}^*$; the symbols μ and ν will be reserved for nodes on the priority tree of strategies $\mathcal{TS}$ (described later); and σ, τ, and ρ will be reserved for nodes on any of the trees T_t, $\hat{T}$, and A (also all described later). The symbol t will be reserved for terms. The symbol s will be reserved for stage numbers.*

Elements $\alpha \in \mathcal{O}$ are sometimes treated as ordinals. Thus, for example, we sometimes write Σ^i_α for Σ^i_γ, where γ is the order type of $\{\beta : \beta \prec \alpha\}$.

We emphasize that the exposition will involve many different trees. Though there should be no cause for confusion, we caution the reader of this fact.

Definition 2.8. Fix an $\mathcal{L}_\alpha$-structure $\mathcal{B}$ with domain B. The *root* of $\mathcal{B}$, if it exists, is the Gödel-least $x \in B$ such that $\mathsf{E}(x, x)$. We denote the root by $r_{\mathcal{B}}$, if it exists. The *length* $|\pi|$ of an element $\pi \in B$, if it exists, is the length of the shortest sequence $x_1, \ldots, x_n$ satisfying $\mathsf{E}(x_i, x_{i+1})$ for $1 \leq i < n$, $r_{\mathcal{B}} = x_1$, and $\pi = x_n$. Note that the length of the root is 1.

3. The Trees

As preparation for the proof of Theorem 2, we introduce the trees that will play a crucial role in the construction of the requisite computable structures. We do so in several steps: In Section 3.1, for $\alpha \in \mathcal{O}^*$ and $m \in \mathbb{N}$, we define the tree $\mathrm{T}_{(\alpha,m)}$; in Section 3.2, we define an expansion $\mathcal{T}$ of the tree $\mathrm{T}_{(\alpha,0)}$; in Section 3.3, we prove symmetry properties of $\mathcal{T}$; and in Section 3.4, we prove relative categoricity properties of $\mathcal{T}$.

We start by defining some features and attributes of our trees.

3.1. **The Basic Trees** $\mathrm{T}_{(\alpha,m)}$. We will build a tree $\mathrm{T}_{(\alpha,m)}$ for each $\alpha \in \mathcal{O}^*$ and $m \in \mathbb{N}$. Our aim in defining this tree, roughly speaking, is that we should be able to use it to define a structure in which, for $\beta \prec \alpha$, there are non-automorphic elements which have the same Σ^0_β-type. The tree will consist of finite sequences of terms, which are defined as follows:

Definition 3.1. A *term* is either:

- a pair $t = (\beta, n)$ with $\beta \in \mathcal{O}^*$ and $n \in \mathbb{N}$; or
- a triple $t = (\beta, n, (\beta', n'))$ with $\beta \in \mathcal{O}^*$ a limit, $n, n' \in \mathbb{N}$, and β' is a successor with $\beta' \prec \beta$.

We call β the *rank* of t. We call a term a *successor* or a *limit* according to its rank.

In fact, for each term t we will build a tree T_t (not just for those of the form (α, m)). It will be convenient to identify the term t with the sequence σ of length 1 with $\sigma(0) = t$, so that we may write $\tau^\frown t$ rather than $\tau^\frown\langle t\rangle$. We start by describing T_t in the case when $\mathrm{rank}(t) \in \mathcal{O}$ as the definition can be done by transfinite recursion in this case. This definition is not only easier to understand, but also gives the intuition for the general definition.

(1) If $t = (\mathbf{0}, n)$, then T_t is the tree with one node $(\mathbf{0}, n)$.
(2) If $t = (\beta, n)$ is a successor term, then T_t is the tree with root (β, n) and having one subtree $\mathrm{T}_{t'}$ for each term t' of rank $\mathrm{pred}(\beta)$ except for the term $(\mathrm{pred}(\beta), n)$. More formally,
$$\mathrm{T}_t := \{t\} \cup \{t^\frown\sigma : \sigma \in \mathrm{T}_{t'} \text{ for } t' \text{ a term of rank } \mathrm{pred}(\beta),\ t' \neq (\mathrm{pred}(\beta), n)\}.$$
(3) If $t = (\beta, n)$ is a limit term, then T_t is the tree with root (β, n) and having one subtree $\mathrm{T}_{t'}$ for each successor term t' of rank less than β. More formally,
$$\mathrm{T}_t := \{t\} \cup \{t^\frown\sigma : \sigma \in \mathrm{T}_{t'} \text{ for } t' \text{ a successor term of rank } \prec \beta\}.$$
(4) If $t = (\beta, n, (\beta', n'))$ is a limit term, then T_t is the tree with root $(\beta, n, (\beta', n'))$ and which has one subtree $\mathrm{T}_{t'}$ for each successor term t' of rank less than β except for the term (β', n'). More formally,
$$\mathrm{T}_t := \{t\} \cup \{t^\frown\sigma : \sigma \in \mathrm{T}_{t'} \text{ for } t' \text{ a successor term of rank } \prec \beta,\ t' \neq (\beta', n')\}.$$

To extend the definition to terms with rank not in $\mathcal{O}$, we need to define the trees in a more direct way.

Definition 3.2. A finite sequence $\sigma := \langle t_i\rangle_{0\le i<\ell}$ of terms, where $\ell \geq 1$ and t_i is (β_i, n_i) or $(\beta_i, n_i, (\beta'_i, n'_i))$, is *acceptable* if for all $i < \ell - 1$:

(T1) if β_i is a successor, then $\beta_{i+1} = \mathrm{pred}(\beta_i)$;
(T2) if β_i is a limit, then β_{i+1} is a successor and $\beta_{i+1} \prec \beta_i$;
(T3) if β_i is a successor, then $t_{i+1} \neq (\mathrm{pred}(\beta_i), n_i)$;
(T4) if β_i is a limit and $t_i = (\beta_i, n_i, (\beta'_i, n'_i))$, then $t_{i+1} \neq (\beta'_i, n'_i)$.

We define T_t to be the tree of acceptable strings σ with $\sigma(0) = t$. Of course, we order these strings by initial segment. We write $\mathrm{last}(\sigma)$ to denote $\sigma(|\sigma|-1)$. For each acceptable string of terms σ, we define $\mathrm{rank}(\sigma) = \mathrm{rank}(\mathrm{last}(\sigma))$. We emphasize that the empty string is not part of our tree.

Is not hard to prove, in the case when $\mathrm{rank}(t) \in \mathcal{O}$, that T_t is a well-founded tree of rank $\mathrm{rank}(t)$. Also, let us observe that for all $\sigma \in \mathrm{T}_t$, $\mathrm{T}_{\mathrm{last}(\sigma)} = \{\mathrm{last}(\sigma)^\frown\tau : \sigma^\frown\tau \in \mathrm{T}_t\}$.

We now use the tree T_t in order to define a structure $\mathcal{T}_t$ with domain T_t. This structure has a binary relation E that reflects the edge relation on the tree: For all $\sigma, \tau \in \mathrm{T}_t$, $\mathsf{E}(\tau, \sigma)$ if and only if $\tau = \sigma^-$ (in particular, the root is linked to itself). The structure also has a myriad of unary relations that we use to label the nodes of T_t. The unary relations are of two types:

- Height Labels: For each $\beta \in \mathcal{O}^*$, we have a *height relation* R_β. Informally, these relations specify the height of an element σ. More formally, the relation $\mathsf{R}_\beta(\sigma)$ holds if and only if $\beta = \mathrm{rank}(\sigma)$.
- Marker Label: For each integer $j \in \mathbb{N}$, we have a *marker relation* M_j. We have that $\mathsf{M}_j(\sigma)$ holds iff $\mathrm{rank}(\sigma)$ is a non-successor and $\mathrm{last}(\sigma) \in \{(\beta, j), (\beta, j, (\beta', n'))\}$. When $\mathrm{rank}(\sigma)$ is a successor, no relation M_j holds of σ.

Our aim in defining $\mathcal{T}_t$ has been to ensure that certain elements are hard to distinguish from each other (while being non-automorphic) and that Lemma 3.3 below also holds.

Lemma 3.3. *Let $\beta \in \mathcal{O}$. Then:*

1. *If $m \neq m'$, then $\mathcal{T}_{(\beta,m)}$ does not embed into $\mathcal{T}_{(\beta,m')}$.*
2. *If β is a limit, then $\mathcal{T}_{(\beta,m)}$ does not embed into $\mathcal{T}_{(\beta,n,(\beta',m'))}$ for any $\beta' \prec \beta$ and any $n, m' \in \mathbb{N}$.*
3. *If β is a limit and $(\beta', m') \neq (\beta'', m'')$, then, for any $n \in \mathbb{N}$, $\mathcal{T}_{(\beta,m,(\beta'',m''))}$ does not embed into $\mathcal{T}_{(\beta,n,(\beta',m'))}$.*

We note here that for any limit β, any successor $\beta' \prec \beta$ and any $n, m \in \mathbb{N}$, $\mathcal{T}_{(\beta,n,(\beta',m))}$ does embed into $\mathcal{T}_{(\beta,n)}$.

Proof. We show the three statements simultaneously by induction on β.

If $\beta = 0$, the marker relation M_m holds of the unique element in $T_{(\beta,m)}$ but not of the unique element in $T_{(\beta,m')}$. Consequently, no embedding of $\mathcal{T}_{(\beta,m)}$ into $\mathcal{T}_{(\beta,m')}$ exists.

If β is a successor, towards a contradiction, fix an embedding $\iota : \mathcal{T}_{(\beta,m)} \to \mathcal{T}_{(\beta,m')}$. Let $\sigma_0 := \langle(\beta, m)\rangle \in T_{(\beta,m)}$ and $\sigma_1 := \langle(\beta, m), (\mathrm{pred}(\beta), m')\rangle \in T_{(\beta,m)}$. Then $\iota(\sigma_0)$ must be $\langle(\beta, m')\rangle \in T_{(\beta,m')}$ as R_β holds of $\sigma_0 \in T_{(\beta,m)}$ and $\langle(\beta, m')\rangle \in T_{(\beta,m')}$ is the unique element of $T_{(\beta,m')}$ that R_β holds of. As the edge relation E holds of the pair (σ_0, σ_1), it must be that the edge relation E holds of the pair $(\iota(\sigma_0), \iota(\sigma_1))$. Thus, the rank of $\mathrm{last}(\iota(\sigma_1))$ must be $\mathrm{pred}(\beta)$. If $\mathrm{pred}(\beta)$ is a limit, then $\mathsf{M}_{m'}$ holds of σ_1, so it must also hold of $\iota(\sigma_1)$, and thus $\mathrm{last}(\iota(\sigma_1))$ is either $(\mathrm{pred}(\beta), m')$ or $(\mathrm{pred}(\beta), m', (\beta'', m''))$ for some β'', m''. Because of the edge relation E, the embedding ι induces an embedding of $T_{\mathrm{last}(\sigma_1)}$ into $T_{\mathrm{last}(\iota(\sigma_1))}$. The inductive hypothesis (either (1) or (2), depending on whether $\mathrm{pred}(\beta)$ is a limit) implies $\mathrm{last}(\iota(\sigma_1)) = \mathrm{last}(\sigma_1)$. This is a contradiction as $\langle(\beta, m'), (\mathrm{pred}(\beta), m')\rangle \notin T_{(\beta,m')}$. Consequently, no embedding of $\mathcal{T}_{(\beta,m)}$ into $\mathcal{T}_{(\beta,m')}$ exists.

If β is a limit, note first that (1) holds because in $\mathcal{T}_{(\beta,m)}$, the label M_m holds of the unique element for which R_β holds, while this is not true of $\mathcal{T}_{(\beta,m')}$. Towards a contradiction to (2) or (3), fix an embedding $\iota : \mathcal{T}_{(\beta,m)} \to \mathcal{T}_{(\beta,n,(\beta',m'))}$ (for (2)) or an embedding $\iota : \mathcal{T}_{(\beta,m,(\beta'',m''))} \to \mathcal{T}_{(\beta,n,(\beta',m'))}$ (for (3)). By considering the images of $\sigma_0 := \langle(\beta, m)\rangle$ and $\sigma_1 := \langle(\beta, m), (\beta', m')\rangle$ or $\sigma_2 := \langle(\beta, m, (\beta'', m''))\rangle$ and $\sigma_3 := \langle(\beta, m, (\beta'', m'')), (\beta', m')\rangle$ under ι, a similar contradiction is reached in each case. Consequently, no embedding of $\mathcal{T}_{(\beta,m)}$ into $\mathcal{T}_{(\beta,n,(\beta',m'))}$ or $\mathcal{T}_{(\beta,m,(\beta'',m''))}$ into $\mathcal{T}_{(\beta,n,(\beta',m'))}$ exists. □

In a strong sense, the proof above exploits all of the obstacles to an embedding. In particular, we have:

Remark 3.4. For successors $\alpha \in \mathcal{O}$, the structures $\mathcal{T}_{(\alpha,m)} - \mathcal{T}_{(\mathrm{pred}(\alpha),n)}$ and $\mathcal{T}_{(\alpha,n)} - \mathcal{T}_{(\mathrm{pred}(\alpha),m)}$ are isomorphic.

Definition 3.5. Let $\sigma = \langle t_i\rangle_{0 \le i < \ell} \in \mathcal{T}_t$, where t_i is (β_i, n_i) or $(\beta_i, n_i, (\beta'_i, n'_i))$. We define the *backbone of* σ, denoted $\mathrm{bb}(\sigma)$, to be the sequence $\langle\gamma_i\rangle_{0 \le i < \ell}$, where $\gamma_i = (\beta_i)$ if β_i is a successor and $\gamma_i = (\beta_i, n_i)$ otherwise.

Thus the backbone of σ specifies the sequence of height and marker labels placed on initial segments of σ.

Definition 3.6. Let $\sigma = \langle t_i \rangle_{0 \le i \le \ell} \in \mathcal{T}_t$, where t_i is (β_i, n_i) or $(\beta_i, n_i, (\beta'_i, n'_i))$. We define the *weak rank of* σ, denoted $\mathrm{wr}(\sigma)$, to be $\min\{\beta_\ell\} \cup \{\beta'_i : t_i = (\beta_i, n_i, (\beta'_i, n'_i))\}$.

Thus the weak rank of σ is the least ordinal which occurs in any of the terms of σ.

3.2. **The Expanded Trees $\hat{\mathrm{T}}$.** Although we have not proved it yet, when $\alpha \in \mathcal{O}$ is a limit $\mathcal{T}_\alpha$ is not relatively Δ^0_α-categorical. Our task is now to take the structure $\mathcal{T}_t$ and to modify it so as to ensure that it is computably categorical when $\mathrm{rank}(t) \in \mathcal{O}$. Thus, in the construction of the structure for Theorem 2, we will build a structure $\mathcal{A}_t$ which is a fattening of $\mathcal{T}_t$, but where (an isomorphic copy of) T_t is a Π^0_2-subset of A_t (where A_t is the domain of $\mathcal{A}_t$). Furthermore, the part of the structure which is not in T_t will be, in a sense we will describe, symmetric with respect to T_t. So, from a structural viewpoint, the larger structure $\mathcal{A}_t$ will not be too different from $\mathcal{T}_t$. However, the larger structure $\mathcal{A}_t$ will be computably categorical when $\mathrm{rank}(t) \in \mathcal{O}$.

Let us first informally describe the shape of the larger structure $\mathcal{A}_t$. The construction for Theorem 2 will have a tree of strategies, denoted $\mathcal{TS}$, where every node is associated with some requirement and the children of each node reflect the outcomes of the requirement. Though standard, the precise definition of $\mathcal{TS}$ is not yet important.

Each node $\mu \in \mathcal{TS}$ will be responsible for enumerating certain nodes to our new fattening $\mathcal{A}_t$ of $\mathcal{T}_t$. We will have to use a new kind of acceptable term that reflects the node in $\mathcal{TS}$ responsible for it.

Convention 3.7. *Fix a limit $\alpha \in \mathcal{O}^*$ for the remainder of the construction. Fix an effective enumeration $\{\alpha_i\}_{i \in \mathbb{N}}$ of the set $\{\beta \in \mathcal{O}^* : \beta \preccurlyeq \alpha\}$ with $\alpha_0 = \alpha$ and for which $\alpha_j = \mathrm{pred}(\alpha_i)$ implies $j \le i$.*

We note that such effective enumerations exist (uniformly in α): Given any enumeration of all $\beta \in \mathcal{O}^*$ satisfying $\beta \preccurlyeq \alpha$, the enumeration can be modified so that when a successor β appears, we compute successive predecessors until either an already enumerated element is reached or a limit is reached, whereupon we enumerate this (finite) sequence in reverse order (not including the already enumerated element, if appropriate). Of course, a limit must eventually be reached because there are no infinite computable (hyperarithmetic) descending sequences, so this sequence is necessarily finite.

We will build a modified version of $\mathcal{T}_{(\alpha,0)}$. For the rest of this section, we abbreviate $\mathcal{T}_{(\alpha,0)}$ as $\mathcal{T}$.

It is worth highlighting the fact that the root of the tree of strategies is $\emptyset$, and so is of length 0, as opposed to the root of $\mathcal{T}$, which is of length 1.

Definition 3.8. Let $\mu \in \mathcal{TS}$ be a node on the tree of strategies. Let $\ell := |\mu|$.

A *μ-term* is either

- a triple $t = (\alpha_i, n, \mu)$ with (α_i, n) an acceptable term, $i, n \le \ell$, and $\ell \in \{i, n\}$; or
- a quadruple $t = (\alpha_i, n, (\alpha_{i'}, n'), \mu)$ with $(\alpha_i, n, (\alpha_{i'}, n'))$ an acceptable term, $i, i', n, n' \le \ell$, and $\ell \in \{i, i', n, n'\}$.

We define $\mathrm{term}(t) := (\alpha_i, n)$ or $\mathrm{term}(t) := (\alpha_i, n, (\alpha_{i'}, n'))$ depending on the form of t.

A *$\underline{\mu}$-term* is a ν-term for some $\nu \subseteq \mu$.

Let P be a path through $\mathcal{TS}$. A *P-term* is a ν-term for some $\nu \subset P$. A *$\mathcal{TS}$-term* is a ν-term for some $\nu \in \mathcal{TS}$.

Definition 3.9. A finite sequence $\sigma := \langle t_i \rangle_{0 \le i < \ell}$ of $\mathcal{TS}$-terms, where t_i is (β_i, n_i, μ_i), or $(\beta_i, n_i, (\beta'_i, n'_i), \mu_i)$, is *acceptable* if for all $i < \ell - 1$:

- if β_i is not a limit, then $\beta_{i+1} = \mathrm{pred}(\beta_i)$;
- if β_i is a limit, then β_{i+1} is a successor and $\beta_{i+1} \prec \beta_i$;
- if μ_i and μ_{i+1} are comparable and β_i is a successor, then we have that $t_{i+1} \neq (\mathrm{pred}(\beta_i), n_i, \mu_{i+1})$;
- if μ_i and μ_{i+1} are comparable and β_i is a limit and $t_i = (\beta_i, n_i, (\beta_i', n_i'), \mu_i)$, then $t_{i+1} \neq (\beta_i', n_i', \mu_{i+1})$.

We define $\hat{\mathrm{T}}$ to be the tree of acceptable sequences σ of $\mathcal{TS}$-terms with $\sigma(0) = (\alpha, 0, \emptyset)$. Of course, we order these strings by initial segment.

For each such acceptable sequence, we define $\mathrm{rank}(\sigma) := \mathrm{rank}(\mathrm{last}(\sigma))$ and $\mathrm{term}(\sigma) := \mathrm{term}(\mathrm{last}(\sigma))$. We emphasize that the empty string is not an element of our tree.

We then use the tree $\hat{\mathrm{T}}$ in order to define a structure $\hat{\mathcal{T}}$ with domain $\hat{\mathrm{T}}$, exactly as before. For all $\sigma, \tau \in \hat{\mathrm{T}}$, $\mathsf{E}(\tau, \sigma)$ if and only if $\tau = \sigma^-$. The relation $\mathsf{R}_\beta(\sigma)$ holds if and only if $\beta = \mathrm{rank}(\mathrm{last}(\sigma))$. The relation $\mathsf{M}_j(\sigma)$ holds if and only if $\mathrm{rank}(\sigma)$ is a non-successor and $\mathrm{last}(\sigma) \in \{(\beta, j, \mu), (\beta, j, (\beta', n'), \mu)\}$. Though this tree might look rather messy, the restriction of it to an appropriate subset is not.

Definition 3.10. Fix a node $\mu \in \mathcal{TS}$ and a path $P \subseteq \mathcal{TS}$. Define the tree $\mathrm{T}^{\underline{\mu}}$ of $\underline{\mu}$-terms to be the tree

$$\mathrm{T}^{\underline{\mu}} := \Big\{\sigma \in \hat{\mathrm{T}} : (\forall i)\ \big[\sigma(i) \text{ is a } \underline{\mu}\text{-term}\big]\Big\}.$$

Define the tree T^P of P-terms to be the tree

$$\mathrm{T}^P := \Big\{\sigma \in \hat{\mathrm{T}} : (\forall i)\ [\sigma(i) \text{ is a } P\text{-term}]\Big\}.$$

We also let $\mathcal{T}^P$ be the corresponding induced substructure of $\hat{\mathcal{T}}$.

We make a few quick observations. The tree $\mathrm{T}^{\underline{\mu}}$ is finite for all $\mu \in \mathcal{TS}$. It is not too difficult to see that $\mathcal{T}$ is isomorphic to $\mathcal{T}^P$. Further, it is immediate that $\mathrm{T}^P = \bigcup_{\mu \subset P} \mathrm{T}^{\underline{\mu}}$.

To say that any $\mathcal{T}'$ is a substructure of $\hat{\mathcal{T}}$ means that the domain is a subset of $\hat{\mathrm{T}}$ and that for any elements of the domain, a relation holds in $\mathcal{T}'$ if and only if it holds in $\hat{\mathcal{T}}$. The structure $\mathcal{A}$ we build will be an expansion of a substructure of $\hat{\mathcal{T}}$ to the language $\mathcal{L}$, and will have as its domain a c.e. subset A of $\hat{\mathrm{T}}$. This domain will contain $\mathrm{T}^{\mathcal{TP}}$, where $\mathcal{TP}$ is the true path of the construction. Being an expansion to $\mathcal{L}$, the structure $\mathcal{A}$ will also have a new kind of label, *temporary labels*, that are specified by unary relations L_m for $m \in \mathbb{N}$. These labels will not be computable, but rather uniformly c.e.; that is, for the structure to be computable, we only demand the relations L_m be uniformly c.e. (recall Definition 2.3).

3.3. Symmetry with Respect to $\mathcal{T}^P$.

Having defined the requisite trees, we establish symmetry properties that guarantee the existence and nonexistence of embeddings and isomorphisms between subtrees of expansions of $\hat{\mathcal{T}}$.

Definition 3.11. We define the backbone and weak rank for an acceptable sequence of $\mathcal{TS}$-terms just as before. Let $\sigma = \langle t_i \rangle_{0 \le i \le \ell} \in \hat{\mathcal{T}}$, where t_i is (β_i, n_i, μ_i) or $(\beta_i, n_i, (\beta_i', n_i'), \mu_i)$.

We define the *backbone of* σ, denoted $\mathrm{bb}(\sigma)$, to be the sequence $\langle \gamma_i \rangle_{0 \le i \le \ell}$, where $\gamma_i = (\beta_i)$ if β_i is a successor and $\gamma_i = (\beta_i, n_i)$ otherwise. If two sequences have the same backbone we shall also say that they are *similar*.

We define the *weak rank of* σ, denoted $\mathrm{wr}(\sigma)$, to be $\min\{\beta_\ell\} \cup \{\beta_i' : t_i = (\beta_i, n_i, (\beta_i', n_i'), \mu_i)\}$.

Definition 3.12. Let $\mathcal{A}$ be an expansion of a substructure of $\hat{\mathcal{T}}$ to $\mathcal{L}$ with domain A, and let P be a path through $\mathcal{TS}$. We say that $\mathcal{A}$ is *symmetric* with respect to T^P if

(S1) $\mathrm{T}^P \subseteq A$,

(S2) For all similar $\sigma, \sigma' \in \mathrm{T}^P$ and for all τ with $\sigma^\frown\tau(0) \notin \mathrm{T}^P$, we have that

$$\sigma^\frown\tau \in \mathcal{A} \iff \sigma'^\frown\tau \in \mathcal{A},$$

and $\sigma^\frown\tau$ and $\sigma'^\frown\tau$ have the same labels.

The structure $\mathcal{A}$ we construct will satisfy the following properties:

(P1) The structure $\mathcal{A}$ is symmetric with respect to $\mathrm{T}^{\mathcal{TP}}$.
(P2) The nodes in $\mathrm{T}^{\mathcal{TP}}$ have infinitely many temporary labels. All the nodes in $\mathrm{T}^{\mathcal{TP}}$ have exactly the same temporary labels.
(P3) The nodes in $A \setminus \mathrm{T}^{\mathcal{TP}}$ have only finitely many temporary labels.

We now show that if $\mathcal{A}$ satisfies the above properties, then the lemma above, which we know holds for $\mathcal{T}^{\mathcal{TP}}$, still holds about $\mathcal{A}$.

Given a node $\sigma \in A$, let

$$A_\sigma := \{\sigma^\frown\tau : \sigma^\frown\tau \in A\}$$

and let $\mathcal{A}_\sigma$ be the restriction of $\mathcal{A}$ to A_σ.

Lemma 3.13. *Fix a structure $\mathcal{A}$ satisfying P1, P2 and P3. Fix distinct $\sigma, \sigma' \in \mathrm{T}^{\mathcal{TP}}$ with common rank $\beta := \mathrm{rank}(\sigma) = \mathrm{rank}(\sigma')$. Then:*

(1) *If $\beta \in \mathcal{O}$ and $\mathrm{term}(\sigma) \neq \mathrm{term}(\sigma')$, then $\mathcal{A}_\sigma$ does not embed into $\mathcal{A}_{\sigma'}$ unless $\mathrm{term}(\sigma)$ and $\mathrm{term}(\sigma')$ are of the form $(\beta, m, (\beta', m'))$ and (β, m), respectively, for some $\beta' < \beta$ and $m, m' \in \mathbb{N}$.*
(2) *If $\beta \in \mathcal{O}^*$, $\mathrm{term}(\sigma) \neq \mathrm{term}(\sigma')$, and $\mathcal{A}$ is hyperarithmetic, then there is no hyperarithmetic embedding of $\mathcal{A}_\sigma$ into $\mathcal{A}_{\sigma'}$ unless $\mathrm{term}(\sigma)$ and $\mathrm{term}(\sigma')$ are of the form $(\beta, m, (\beta', m'))$ and (β, m), respectively, for some $\beta' < \beta$ and $m, m' \in \mathbb{N}$.*
(3) *If $\beta \in \mathcal{O}^* \setminus \mathcal{O}$, σ and σ' are similar, and there is no $\beta' \in \mathcal{O}$, $m, m' \in \mathbb{N}$ such that $(\beta, m, (\beta', m')) \in \{\mathrm{term}(\sigma), \mathrm{term}(\sigma')\}$, then $\mathcal{A}_\sigma$ and $\mathcal{A}_{\sigma'}$ are isomorphic.*

Proof. For (1), we fix an embedding $\iota : \mathcal{A}_\sigma \to \mathcal{A}_{\sigma'}$. As x and $\iota(x)$ must have the same cardinality of labels, Properties P2 and P3 imply ι sends elements of $\mathrm{T}^{\mathcal{TP}}$ to elements of $\mathrm{T}^{\mathcal{TP}}$. Thus ι induces an embedding $\mathcal{T}_{\mathrm{term}(\sigma)} \to \mathcal{T}_{\mathrm{term}(\sigma')}$. By Lemma 3.3, this implies $\mathrm{term}(\sigma) = (\beta, m, (\beta', m'))$ and $\mathrm{term}(\sigma') = (\beta, m)$.

For (2), suppose $\mathrm{term}(\sigma)$ and $\mathrm{term}(\sigma')$ are not of the specified form, and fix an embedding $\iota : \mathcal{A}_\sigma \to \mathcal{A}_{\sigma'}$. By the same reasoning as in part (1), we see that ι induces an embedding $\hat{\iota} : \mathcal{T}_{\mathrm{term}(\sigma)} \to \mathcal{T}_{\mathrm{term}(\sigma')}$ which is computable in $\iota \oplus \mathrm{T}^{\mathcal{TP}}$. Further, by Properties *P2* and *P3*, $\mathrm{T}^{\mathcal{TP}}$ is computable in $\mathcal{A}''$. Thus $\hat{\iota}$ is computable in $\iota \oplus \mathcal{A}''$.

Observe that the proof of Lemma 3.3 was effective: From t_0, t_0' not of the specified form and an embedding $\hat{\iota} : \mathcal{T}_{t_0} \to \mathcal{T}_{t_0'}$, we effectively obtain terms t_1, t_1' not of the specified form with $\mathrm{rank}(t_1) = \mathrm{rank}(t_1') < \beta$ and $\iota' \upharpoonright \mathcal{T}_{t_1} : \mathcal{T}_{t_1} \to \mathcal{T}_{t_1'}$ an embedding. Repeating this, $\hat{\iota}$ computes a sequence $t_0, t_1, \ldots$ with $\mathrm{rank}(t_0) > \mathrm{rank}(t_1) > \ldots$. Since $\mathcal{O}^*$ contains no infinite, descending, hyperarithmetic sequence, it follows that $\hat{\iota}$ is not hyperarithmetic. Thus ι must not be hyperarithmetic.

For (3), we first define an isomorphism $\pi : \mathcal{A}_\sigma \cap \mathrm{T}^{\mathcal{TP}} \cong \mathcal{A}_{\sigma'} \cap \mathrm{T}^{\mathcal{TP}}$. Then we will argue that by symmetry, π extends to an isomorphism $\mathcal{A}_\sigma \cong \mathcal{A}_{\sigma'}$. It is tempting to define π recursively in β; however, since β is not well-founded, we must instead define $\pi(\tau)$ recursively in $|\tau|$. Our inductive hypothesis will be the following:

- π preserves edge relation and non-relation;
- τ and $\pi(\tau)$ are similar;
- if there are $\gamma \in \mathcal{O}^*$, $\gamma' \in \mathcal{O}$, $m, m' \in \mathbb{N}$ such that $(\gamma, m, (\gamma', m')) \in \{\mathrm{term}(\tau), \mathrm{term}(\pi(\tau))\}$, then $\mathrm{term}(\tau) = \mathrm{term}(\pi(\tau))$; and

- if $\mathrm{rank}(\tau) \in \mathcal{O}$, then $\mathrm{term}(\tau) = \mathrm{term}(\pi(\tau))$.

To summarize the last two points: If $\mathrm{term}(\tau) \neq \mathrm{term}(\pi(\tau))$, then all elements of $\mathcal{O}^*$ mentioned in either term are in $\mathcal{O}^* \setminus \mathcal{O}$. The idea is the following: By the proof of Lemma 3.3, if $\mathrm{term}(\tau) \neq \mathrm{term}(\pi(\tau))$, then there must be some t with $\mathrm{term}(\tau^\frown t) \neq \mathrm{term}(\pi(\tau^\frown t))$. Since τ with $\mathrm{rank}(\tau) = 0$ are labeled by their term, the proof derives a contradiction using well-foundedness. To avoid this problem, we arrange that for such t, $\mathrm{rank}(t) \in \mathcal{O}^*\setminus\mathcal{O}$ — we send the "incompatibility" down an infinitely descending chain in $\mathcal{O}^*$, so that it never reaches rank 0 and destroys our isomorphism. The crucial step is Case 2b below, where we choose $\gamma' \in \mathcal{O}^*\setminus\mathcal{O}$.

Base case: We define $\pi(\sigma) = \sigma'$. By assumption, this definition satisfies the inductive hypothesis.

Inductive step: Suppose that for all ρ with $\sigma \subseteq \rho \subseteq \tau$, we have defined $\pi(\rho)$ in a fashion satisfying the inductive hypothesis; we must now define $\pi(\tau^\frown t)$ for all t with $\tau^\frown t \in \mathrm{T}^{\mathfrak{TP}}$ in a manner that induces a bijection between $\{\tau^\frown t : \tau^\frown t \in \mathrm{T}^{\mathfrak{TP}}\}$ and $\{\pi(\tau)^\frown t : \pi(\tau)^\frown t \in \mathrm{T}^{\mathfrak{TP}}\}$. This will ensure that π continues to preserve edge relation and non-relation.

We have several cases:

(1) If $\mathrm{term}(\tau) = \mathrm{term}(\pi(\tau))$, then $\{t : \tau^\frown t \in \mathrm{T}^{\mathfrak{TP}}\} = \{t : \pi(\tau)^\frown t \in \mathrm{T}^{\mathfrak{TP}}\}$, and we define $\pi(\tau^\frown t) = \pi(\tau)^\frown t$ for all such t. Clearly the inductive hypothesis is preserved.

(2) If $\mathrm{term}(\tau) \neq \mathrm{term}(\pi(\tau))$, then by the inductive hypothesis $\mathrm{rank}(\tau) \in \mathcal{O}^* \setminus \mathcal{O}$. There are several cases, depending on $\mathrm{rank}(\tau)$:

   (a) If $\mathrm{rank}(\tau) = \gamma + 2$ for some $\gamma \in \mathcal{O}^*$, fix any bijection $f : \{t : \tau^\frown t \in \mathrm{T}^{\mathfrak{TP}}\} \to \{t : \pi(\tau)^\frown t \in \mathrm{T}^{\mathfrak{TP}}\}$; such bijections exist because both sets are countable. We define $\pi(\tau^\frown t) = \pi(\tau)^\frown f(t)$ for all such t. Since $\mathrm{rank}(\tau^\frown t) = \gamma + 1 \in \mathcal{O}^*$ is not a limit, the inductive hypothesis is preserved.

   (b) If $\mathrm{rank}(\tau) = \gamma+1$ for some limit $\gamma \in \mathcal{O}^*\setminus\mathcal{O}$, fix $\gamma' \in \mathcal{O}^*\setminus\mathcal{O}$ a successor with $\gamma' < \gamma$, and let $m, n \in \mathbb{N}$ be such that $\mathrm{term}(\tau) = (\gamma+1, m)$ and $\mathrm{term}(\pi(\tau)) = (\gamma+1, n)$. By assumption, $m \neq n$. Define π as follows:

$$\pi(\tau^\frown t) = \begin{cases} \pi(\tau)^\frown(\gamma, m) & \text{if } \mathrm{term}(t) = (\gamma, m, (\gamma', 0)), \\ \pi(\tau)^\frown(\gamma, m, (\gamma', k)) & \text{if } \mathrm{term}(t) = (\gamma, m, (\gamma', k+1)), \\ \pi(\tau)^\frown(\gamma, n, (\gamma', 0)) & \text{if } \mathrm{term}(t) = (\gamma, n), \\ \pi(\tau)^\frown(\gamma, n, (\gamma', k+1)) & \text{if } \mathrm{term}(t) = (\gamma, n, (\gamma', k)), \\ \pi(\tau)^\frown t & \text{otherwise.} \end{cases}$$

   Then, since $\gamma, \gamma' \in \mathcal{O}^* \setminus \mathcal{O}$, the inductive hypothesis is preserved.

   (c) If $\mathrm{rank}(\tau)$ is a limit, there are several cases, depending on $\mathrm{term}(\tau)$ and $\mathrm{term}(\pi(\tau))$:

      (i) If $\mathrm{term}(\tau) = (\gamma, m, (\gamma'_0, n_0))$ and $\mathrm{term}(\pi(\tau)) = (\gamma, m, (\gamma'_1, n_1))$, then by the inductive hypothesis, $\gamma'_0, \gamma'_1 \notin \mathcal{O}$. By assumption, $(\gamma'_0, n_0) \neq (\gamma'_1, n_1)$. By definition of acceptable terms, neither γ'_0 nor γ'_1 are limits. There are several cases, depending on the relationship of $\gamma'_0, \gamma'_1, n_0$ and n_1:

         (A) If $\gamma'_0 \neq \gamma'_1$, define π as follows:

$$\pi(\tau^\frown t) = \begin{cases} \pi(\tau)^\frown(\gamma'_0, k) & \text{if } \mathrm{term}(t) = (\gamma'_0, k) \text{ and } k < n_0, \\ \pi(\tau)^\frown(\gamma'_0, k-1) & \text{if } \mathrm{term}(t) = (\gamma'_0, k) \text{ and } k > n_0, \\ \pi(\tau)^\frown(\gamma'_1, k) & \text{if } \mathrm{term}(t) = (\gamma'_1, k) \text{ and } k < n_1, \\ \pi(\tau)^\frown(\gamma'_1, k+1) & \text{if } \mathrm{term}(t) = (\gamma'_1, k) \text{ and } k \geq n_1, \\ \pi(\tau)^\frown t & \text{otherwise.} \end{cases}$$

(B) If $\gamma_0' = \gamma_1'$ and $n_0 < n_1$, define π as follows:

$$\pi(\tau^\frown t) = \begin{cases} \pi(\tau)^\frown(\gamma_0', k) & \text{if } \mathrm{term}(t) = (\gamma_0', k) \text{ and } k < n_0, \\ \pi(\tau)^\frown(\gamma_0', k-1) & \text{if } \mathrm{term}(t) = (\gamma_0', k) \text{ and } n_0 < k \leq n_1, \\ \pi(\tau)^\frown(\gamma_0', k) & \text{if } \mathrm{term}(t) = (\gamma_0', k) \text{ and } k > n_1, \\ \pi(\tau)^\frown t & \text{otherwise.} \end{cases}$$

(C) The case for $\gamma_0' = \gamma_1'$ and $n_0 > n_1$ is similar to case (A).

Since γ_0' and γ_1' are not limits, $\tau^\frown t$ and $\pi(\tau^\frown t)$ are similar. Since $\gamma_0', \gamma_1' \notin \mathcal{O}$, the remainder of the inductive hypothesis is preserved.

(ii) The case for $\mathrm{term}(\tau) = (\gamma, m, (\gamma', n))$ and $\mathrm{term}(\pi(\tau)) = (\gamma, m)$ is similar to case (i).

(iii) The case for $\mathrm{term}(\tau) = (\gamma, m)$ and $\mathrm{term}(\pi(\tau)) = (\gamma, m, (\gamma', n))$ is similar to case (i).

So $\pi : \mathcal{A}_\sigma \cap \mathrm{T}^{\mathcal{TP}} \to \mathcal{A}_{\sigma'} \cap \mathrm{T}^{\mathcal{TP}}$ is a bijection which preserves edge relation and non-relation, and such that τ and $\pi(\tau)$ are similar for all τ. By Property P2, π is an isomorphism.

To extend π to $\mathcal{A}_\sigma$, for $\tau \in \mathcal{A}_\sigma \setminus \mathrm{T}^{\mathcal{TP}}$, let $\tau = \rho^\frown\zeta$ with ρ maximal such that $\rho \in \mathrm{T}^{\mathcal{TP}}$. Note that $\sigma \subseteq \rho$. Define $\pi(\tau) = \pi(\rho)^\frown\zeta$. Since ρ and $\pi(\rho)$ are similar, Property P1 implies that $\pi : \mathcal{A}_\sigma \to \mathcal{A}_{\sigma'}$ is an isomorphism. $\square$

3.4. **Relative Δ^0_α-Categoricity.** Fix a computable structure $\mathcal{A}$ satisfying P1, P2 and P3. For $\beta < \alpha$, if $\beta \in \mathcal{O}$, then $\mathcal{A}$ is not relatively Δ^0_β-categorical. Though demonstrating this is not required for either Theorem 1 or Theorem 2, we do so as the computations help explain the motivation and purpose of the requisite properties. It also allows us to offer an explicit structure for Theorem 2 rather than the existence proof that Theorem 1 yields.

Not only does $\mathcal{A}$ not have a computably enumerable Σ^c_β-Scott family, it has no Σ^i_β-Scott family of any computational complexity. The reason, in essence, is that the Σ^i_β-types of $\langle(\alpha, 0), (\beta+1, 0)\rangle$ and $\langle(\alpha, 0), (\beta+1, 1)\rangle$ coincide, as we will show in Lemma 3.14 below. As $\mathrm{T}_{\beta+1,0}$ and $\mathrm{T}_{\beta+1,1}$ are not isomorphic, this implies $\mathcal{A}$ has no Σ^i_β-Scott family.

We remark that if $\alpha \in \mathcal{O}^* \backslash \mathcal{O}$, then $\mathcal{A}$ is not relatively hyperarithmetically categorical. This is of little interest because, by Lemma 3.13 above, $\mathcal{A}$ is not even Δ^0_γ-categorical for any computable ordinal γ. Briefly, fix $\mathcal{TP}$-terms t_0 and t_1 with $\mathrm{term}(t_0) = (\beta, n)$ and $\mathrm{term}(t_1) = (\beta, m)$ for $\beta \in \mathcal{O}^* \setminus \mathcal{O}$ a successor and $n \neq m$. Let $\mathcal{A}' = \mathcal{A} \setminus \mathcal{A}_{\langle(\alpha,0,\emptyset),t_0\rangle} \sqcup \mathcal{A}_{\langle(\alpha,0,\emptyset),t_1\rangle}$ — so $\mathcal{A}'$ is made from $\mathcal{A}$ by replacing $\mathcal{A}_{\langle(\alpha,0,\emptyset),t_0\rangle}$ with a second copy of $\mathcal{A}_{\langle(\alpha,0,\emptyset),t_1\rangle}$. Then $\mathcal{A} \cong \mathcal{A}'$, but any isomorphism would send $\mathcal{A}_{\langle(\alpha,0,\emptyset),t_0\rangle}$ to some $\mathcal{A}_\tau$ with $\mathrm{term}(\tau) \neq t_0$. Thus there can be no hyperarithmetic isomorphism.

Lemma 3.14. *Let $\overline{\sigma} = (\sigma_1, \ldots, \sigma_k), \overline{\sigma}' = (\sigma_1', \ldots, \sigma_k')$ be tuples from $\mathcal{A}$ such that:*

- *σ_i and σ_i' are similar for every i; and*
- *for every i, j, $|\sigma_i \wedge \sigma_j| = |\sigma_i' \wedge \sigma_j'|$.*

For each i, let ρ_i be maximal (possibly empty) such that $\sigma_i = \tau_i{}^\frown\rho_i$ and $\sigma_i' = \tau_i'{}^\frown\rho_i$ for some τ_i and τ_i'. Let $\beta = \min\{\mathrm{wr}(\tau_1), \ldots, \mathrm{wr}(\tau_k), \mathrm{wr}(\tau_1'), \ldots, \mathrm{wr}(\tau_k')\}$.

If every $\tau_i, \tau_i' \in \mathrm{T}^{\mathcal{TP}}$, then for any $\beta_0 < \beta$, the (parameter-free) $\Sigma^i_{\beta_0}$-types of $\overline{\sigma}$ and $\overline{\sigma}'$ coincide.

Proof. By induction on β_0: For $\beta_0 = 0$, note that quantifier-free types can only specify the labels of the elements and equality or inequality. The former coincide since, by hypothesis, the tuples are pairwise similar, and each τ_i and τ_i' is drawn from $\mathrm{T}^{\mathcal{TP}}$ and $\mathcal{A}$ satisfies Properties P1 and P2. The latter coincide since the various meets have the same lengths.

For $\beta_0 > 0$, let $\psi(\overline{x}, \overline{y})$ be a Π^i_γ-formula for some $\gamma < \beta_0$, and suppose $\mathcal{A} \models \exists \overline{x}\, \psi(\overline{x}, \overline{\sigma})$. We must show that $\mathcal{A} \models \exists \overline{x}\, \psi(\overline{x}, \overline{\sigma}')$ — the other direction will then follow by symmetry. So fix $\overline{z} \in \mathcal{A}$ such that $\mathcal{A} \models \psi(\overline{z}, \overline{\sigma})$, and partition $\overline{z}$ into $\overline{z}_0 \in \mathrm{T}^{\mathcal{TP}}$ and $\overline{z}_1 \notin \mathrm{T}^{\mathcal{TP}}$. Without loss of generality, assume that $\{\overline{z}_0, \overline{z}_1, \overline{\sigma}\}$ is downward closed; so $\{\overline{z}_0, \overline{z}_1, \overline{\sigma}\}$ is a tree. We will define a tree-map f on $\{\overline{z}_0, \overline{z}_1, \overline{\sigma}\}$, such that $\mathcal{A} \models \psi(f(\overline{z}_0\overline{z}_1), \overline{\sigma}')$.

For $\zeta \in \{\overline{z}_0, \overline{z}_1, \overline{\sigma}\}$, if $\zeta \subseteq \sigma_i$, define $f(\zeta) = \sigma_i' \upharpoonright |\zeta|$. Since $|\sigma_i \wedge \sigma_j| = |\sigma_i' \wedge \sigma_j'|$, this is well defined, even if there are multiple i with $\zeta \subseteq \sigma_i$. If $\zeta \not\subseteq \sigma_i$ for any i, assume that we have already defined $f(\zeta^-)$. There are several cases, based on $\mathrm{wr}(\zeta)$, $\mathrm{rank}(\zeta)$ and whether $\zeta \in \overline{z}_1$:

(1) If $\zeta \in \overline{z}_0$, $\mathrm{wr}(\zeta) \geq \beta_0$ and $\mathrm{rank}(\zeta)$ is a successor, define $f(\zeta) = f(\zeta^-)^\frown(\mathrm{rank}(\zeta), m, \mu)$ for some large m (and the unique $\mu \subset \mathcal{TP}$ for which this is a μ-term).
(2) If $\zeta \in \overline{z}_0$, $\mathrm{wr}(\zeta) \geq \beta_0$ and $\mathrm{rank}(\zeta)$ is a limit with $\mathrm{term}(\zeta) = (\mathrm{rank}(\zeta), m)$ or $\mathrm{term}(\zeta) = (\mathrm{rank}(\zeta), m, (\beta', m'))$, define $f(\zeta) = f(\zeta^-)^\frown(\mathrm{rank}(\zeta), m, (\beta'', m''), \mu)$ for some $\beta'' > \gamma$ and large m'' (and the unique $\mu \subset \mathcal{TP}$ for which this is a μ-term).
(3) If $\mathrm{wr}(\zeta) < \beta_0$, define $f(\zeta) = f(\zeta^-)^\frown\mathrm{last}(\zeta)$.
(4) If $\zeta \in \overline{z}_1$, define $f(\zeta) = f(\zeta^-)^\frown\mathrm{last}(\zeta)$.

We must show that $f(\zeta) \in \mathcal{A}$. For $\zeta \in \overline{\sigma}$, this is immediate, since $f(\zeta) \in \overline{\sigma}'$. To show this for $\zeta \in \overline{z}_0$, we must show that $f(\zeta)$ is an acceptable sequence: Then it will be an element of $\mathrm{T}^{\mathcal{TP}} \subseteq \mathcal{A}$. The only case to consider is when $\mathrm{wr}(\zeta) < \beta_0 \leq \mathrm{wr}(\zeta^-)$.

If $\zeta^- \not\subseteq \sigma_i$ for any i, then $\mathrm{term}(f(\zeta^-)) = (\mathrm{rank}(\zeta), m)$ for m chosen large relative to $\mathrm{term}(\zeta)$, or $\mathrm{term}(f(\zeta^-)) = (\mathrm{rank}(\zeta), m, (\beta', m'))$ for m' chosen large relative to $\mathrm{term}(\zeta)$. Either way, we see that $f(\zeta)$ is acceptable.

If $\zeta^- \subseteq \sigma_i$ and $\beta_0 > \mathrm{rank}(\zeta)$, then $\mathrm{rank}(\zeta^-) \geq \mathrm{wr}(\zeta^-) \geq \beta > \beta_0 > \mathrm{rank}(\zeta)$, so it must be that $\mathrm{rank}(\zeta^-)$ is a limit. By definition of β, either $\mathrm{term}(f(\zeta^-)) = (\mathrm{rank}(\zeta^-), m)$ or $\mathrm{term}(f(\zeta^-)) = (\mathrm{rank}(\zeta^-), m, (\beta', m'))$ for some $\beta' \geq \beta > \beta_0 > \mathrm{rank}(\zeta)$, so $f(\zeta)$ is acceptable.

If $\zeta^- \subseteq \sigma_i$ and $\beta_0 \leq \mathrm{rank}(\zeta)$, then since $\mathrm{wr}(\zeta) < \beta_0 \leq \mathrm{wr}(\zeta^-)$, it must be that $\mathrm{term}(\zeta) = (\mathrm{rank}(\zeta), m, (\mathrm{wr}(\zeta), m'))$. By definition of acceptable sequences, since $f(\zeta^-)$ is acceptable, $f(\zeta)$ is acceptable.

Next, consider $\zeta \in \overline{z}_1$. Let $\zeta = \zeta'^\frown\zeta''$ with ζ' maximal such that $\zeta' \in \overline{z}_0$. Note that by construction, ζ' and $f(\zeta')$ are similar, so by Property P1, $f(\zeta) = f(\zeta')^\frown\zeta'' \in \mathcal{A}$.

Since f is a length-preserving tree-map, it preserves meets. As mentioned before, ζ and $f(\zeta)$ are similar by construction. Finally, let $\tau \subseteq \zeta$ be maximal with $\mathrm{wr}(\tau) \geq \beta_0$ and $\tau \in \mathrm{T}^{\mathcal{TP}}$, and let $\zeta = \tau^\frown\rho$. Then $f(\zeta) = f(\tau)^\frown\rho$, $\mathrm{wr}(f(\tau)) > \gamma$ and $f(\tau) \in \mathrm{T}^{\mathcal{TP}}$. By the inductive hypothesis, it follows that $(\overline{z}, \overline{\sigma})$ and $(f(\overline{z}), f(\overline{\sigma})) = (f(\overline{z}), \overline{\sigma}')$ have the same Σ^i_γ-types, and so $\mathcal{A} \models \psi(f(\overline{z}), \overline{\sigma}')$. □

Proposition 3.15. *The structure $\mathcal{A}$ has no Σ^i_β-Scott family for any $\beta < \alpha$.*

Proof. Fix a parameter set $\overline{\tau}$ for a potential Σ^i_β-Scott family, and assume without loss of generality that $\overline{\tau}$ is downward closed in $\mathcal{A}$. Fix any two $\sigma, \sigma' \in \mathrm{T}^{\mathcal{TP}}$ distinct from elements of $\overline{\tau}$ with $|\sigma_1| = |\sigma_2| = 2$ and $\mathrm{rank}(\sigma_1) = \mathrm{rank}(\sigma_2) = \beta + 1$. Then the Σ^i_β-types of σ and σ' over $\overline{\tau}$ are determined by the parameter-free Σ^i_β-types of $(\sigma, \overline{\tau})$ and $(\sigma', \overline{\tau})$, so by Lemma 3.14 they are the same.

Since $\mathcal{A}_\sigma$ is not isomorphic to $\mathcal{A}_{\sigma'}$, σ and σ' are not in the same orbit. Thus they witness the failure of the potential Σ^i_β-Scott family. □

4. Computable Categoricity

Recall that we fixed a limit $\alpha \in \mathcal{O}^*$. Uniformly in α, we build $\mathcal{A}$ which is an expansion of a substructure of $\hat{\mathcal{T}}$ as described in the previous section. The objective is to make $\mathcal{A}$ computably categorical if $\alpha \in \mathcal{O}$.

The construction is a priority construction. We describe the requirements, the outcomes, and the tree of strategies in Section 4.1; the action of an instance of the Ξ-requirement in Section 4.2; the outcome of an instance of a Φ_ℓ-requirement in Section 4.3; the construction in Section 4.4; and the verification in Section 4.5.

A global feature of the construction will be a (computably enumerable) *bag* of labels. At each stage s, this bag will contain a subset of the temporary labels L_m for $m \in \mathbb{N}$. In the limit, the infinitely many labels possessed by nodes in $\mathrm{T}^{\mathcal{TP}}$ (recall P2) will be precisely the labels in the bag. The nodes in $\mathcal{A} \setminus \mathrm{T}^{\mathcal{TP}}$ will have at most finitely many labels (recall P3), and if $\sigma \in \mathcal{A} \setminus \mathrm{T}^{\mathcal{TP}}$ but $\sigma^- \in \mathrm{T}^{\mathcal{TP}}$, σ will have at least one label not in the bag.

We fix an effective enumeration $\{\mathcal{B}_\ell\}_{\ell\in\mathbb{N}}$ of all (partial) computable $\mathcal{L}$-structures. For convenience, we assume that if $\pi \in \mathcal{B}_\ell$, then $|\pi|$ exists and is witnessed by elements with Gödel number not greater than π.

4.1. **The Requirements, Outcomes, and Tree of Strategies.** In order to build $\mathcal{A}$, we satisfy two types of requirements:

Ξ**:** The structure $\mathcal{A}$ satisfies properties P1, P2 and P3.

Φ_ℓ**:** If $\mathcal{A}$ and $\mathcal{B}_\ell$ are isomorphic, then they are computably isomorphic.

Of course, we only have to ensure that the Φ_ℓ requirements are satisfied in the case that $\alpha \in \mathcal{O}$.

There is an unusual relationship between these types of requirements that allowed us to simplify the exposition in the previous section: A node working for the Ξ-requirement needs to build part of $\mathcal{A}$ but has no need for multiple outcomes; a node working for a Φ_ℓ-requirement has a need for multiple outcomes but has no need to build part of $\mathcal{A}$. This allows every node μ on the tree of strategies $\mathcal{TS}$ to be shared between the Ξ-requirement and a Φ_ℓ-requirement, with the former dictating the action and the latter dictating the outcome.[4]

If μ is a node on the tree of strategies with $|\mu| = \ell$, then this node will be concerned with the satisfaction of Φ_ℓ and will have finitary and expansionary outcomes. The rough idea is that μ will have finitary outcome while it waits to see $\mathcal{B}_\ell$ match everything that had already been enumerated into $\mathcal{A}$ by the end of the last expansionary stage, and then once it sees this happen it will have expansionary outcome. In fact, we shall have to use a number of different expansionary outcomes depending on the level of evidence for the existence of the isomorphism.

For the sake of satisfying Φ_ℓ, μ has a set of outcomes which depends upon the maximum length of sequences in $T^{\underline{\mu}}$. If this maximum length is m then μ has outcomes:

$$\begin{aligned}\mathtt{Out}(\mu) := \{{}_m\mathtt{pe}_\emptyset\} \cup \{{}_{m-1}\mathtt{pe}_{\overline{k}} : \overline{k} \in \mathbb{N}^1\} \cup \{{}_{m-2}\mathtt{pe}_{\overline{k}} : \overline{k} \in \mathbb{N}^2\} \cup \cdots \\ \cup \{{}_1\mathtt{pe}_{\overline{k}} : \overline{k} \in \mathbb{N}^{m-1}\} \cup \{\mathtt{se}_{\overline{k}} : \overline{k} \in \mathbb{N}^m\} \cup \{\mathtt{f}_{\overline{k}} : \overline{k} \in \mathbb{N}^{m+1}\}.\end{aligned}$$

The true outcome will be ${}_m\mathtt{pe}_\emptyset$ if $\mathcal{A}$ and $\mathcal{B}_\ell$ are isomorphic, although the converse will not necessarily hold. All outcomes of the form ${}_n\mathtt{pe}_{\overline{k}}$ are referred to as *primary expansionary* outcomes, and more specifically as ${}_n\mathtt{pe}$ outcomes. Outcomes of the form $\mathtt{se}_{\overline{k}}$ are referred to as *secondary expansionary* outcomes; outcomes of the form $\mathtt{f}_{\overline{k}}$ are referred to as *finitary* outcomes. All of these outcomes are ordered lexicographically according to their right suffix, but using the reverse ordering on the natural numbers. Thus the leftmost outcome is ${}_m\mathtt{pe}_\emptyset$, while, e.g., ${}_{m-1}\mathtt{pe}_{(5)}$ is to the left of ${}_{m-2}\mathtt{pe}_{(5,1)}$, which is to the left of ${}_{m-1}\mathtt{pe}_{(4)}$, which is to the left of ${}_{m-2}\mathtt{pe}_{(4,3)}$.

These outcomes may initially look a little complex, but they are really very simple. The way in which to understand them and their ordering is roughly as follows. We start with the idea

[4]It might be more natural to devote, say, even levels to nodes which work for Ξ and odd levels to nodes which work for Φ_ℓ-requirements. Doing so, however, would complicate Definition 3.8 as we only want nodes μ associated with Ξ there.

that there should be a set of outcomes ${}_m\mathtt{pe} < {}_{m-1}\mathtt{pe} < \cdots < {}_1\mathtt{pe} < \mathtt{se} < \mathtt{f}$, where $\mathtt{f}$ will be the finitary outcome indicating that we are not observing any evidence of an isomorphism between $\mathcal{A}$ and $\mathcal{B}_\ell$, and with the other outcomes corresponding to different levels of evidence for this isomorphism.[5] The outcome ${}_m\mathtt{pe}$ indicates the highest level of evidence for the existence of an isomorphism, and then ${}_{m-1}\mathtt{pe}$ indicates a slightly lower level of evidence, and so on, in a way which shall be made precise later. The outcome $\mathtt{se}$ indicates the lowest level of evidence other than $\mathtt{f}$. We then have to modify this idea, however, because we want that if an outcome $\mathtt{o}$ is visited, and then an outcome to the left of $\mathtt{o}$ is visited, $\mathtt{o}$ cannot later be revisited. We want this because it will simplify our analysis of the interaction between nodes on the tree of strategies. So rather than μ having outcome ${}_{m-1}\mathtt{pe}$ at some stage, for example, we let k be the number of times that μ has had outcome ${}_m\mathtt{pe}_\emptyset$ and give it outcome ${}_{m-1}\mathtt{pe}_{(k)}$. This means that whenever μ has outcome ${}_m\mathtt{pe}_\emptyset$ we start using what one might think of as a new version of ${}_{m-1}\mathtt{pe}$, which is to the left of the old ones. Thus the right suffix does nothing more than allow us to ensure that nodes on the tree of strategies which have already been visited, are not visited again once the path has been to the left (so the right suffix can basically be ignored once this feature of the construction is given).

The tree of strategies $\mathcal{TS}$ is the set of finite sequences μ such that, for all $n < |\mu|$, $\mu(n) \in \mathtt{Out}(\mu \restriction n)$. It is worth noting that since the root of $\mathcal{TS}$ is $\emptyset$, it has length 0, unlike the root of $\mathcal{A}$.

4.2. **The Action of the Ξ-Requirement at a Node μ.** In Definition 3.10, we defined the tree $\mathrm{T}^{\underline{\mu}} \subset \hat{\mathrm{T}}$. The Ξ-requirement at μ is responsible for the elements of $\mathrm{T}^{\underline{\mu}}$ for which higher priority requirements are not already responsible.

Definition 4.1. If $\mu = \emptyset$ then $R^\mu := \mathrm{T}^{\underline{\mu}}$ and otherwise $R^\mu := \mathrm{T}^{\underline{\mu}} \setminus \mathrm{T}^{\underline{\mu^-}}$.

Thus R^μ is precisely the set of sequences of $\underline{\mu}$-terms containing at least one μ-term. These are the elements of $\mathcal{A}$ for which μ is responsible. Note that R^μ is not a tree as it is not downward closed.

The node μ works, of course, according to the assumption that it lies on the true path. This means that when μ is visited it assumes that any μ' which has already been visited cannot be visited again if it lies to the left of μ or if the path has been to the left of μ' after it was visited. As described before [actually not, but to be added], any μ' gives new temporary labels to all the elements of $\mathcal{A}$ that it is responsible for each time that it is visited, as well as giving these elements all the labels which have been added to the bag. It also adds to the bag temporary labels which it gave to elements of $\mathcal{A}$ at previous stages (as a matter of fact we put a delay into this process so that each of the *two* most recent temporary labels it has given to each of its elements are not yet added to the bag). Importantly though, this means that, at any point, the most recent labels which μ' has given to the elements it is responsible for have not been added to the bag. So if μ believes that μ' will never be visited again, then it also believes that these most recent labels will *never* be added to the bag of labels. When μ is visited, labels which have not been added to the bag, but which have been given to some element of $\mathcal{A}$ by some μ' which is either to the left of μ or such that the path has been to the left of μ' after it was visited, are called μ-*dead* labels.

So the basic picture we have in mind is roughly as follows. With its choice of outcomes, μ delays the nodes above its expansionary outcomes from rejuvenating their labels (adding their old temporary labels to the bag and giving new ones to the elements they are responsible

[5]As for why we require different outcomes for different amounts of evidence, for now we can only say that it is essential for making $\mathcal{A}$ computably categorical. Remark 4.21 identifies exactly where in the verification these are required.

for) until it has seen $\mathcal{B}_\ell$ match the elements they have added along with the corresponding labels. Therefore, for the part of $\mathcal{A}$ which is built by these nodes, it will not be difficult to build the computable isomorphism. On the other hand, the elements added by those μ' to the left or right, or $\supset \mu$ but which cannot be visited again, will have μ-dead labels (on some initial segment anyway). Again then, these will not be a problem as we look to construct the computable isomorphism, since they will be easily distinguished. The node μ, however, cannot delay the rejuvenation of labels for those elements in the part of $\mathcal{A}$ built by the nodes $\subseteq \mu$, which is $\mathrm{T}^{\underline{\mu}}$. So μ will not try to work out which of these elements is which. At the end of the construction the computable isomorphism is built, given the finite amount of necessary information regarding the image of $\mathrm{T}^{\underline{\mu}}$. Since these elements are indistinguishable as far as μ is concerned (beyond, of course, the information given by the backbone of each sequence), during the construction it only tries to establish where elements of $\mathcal{A}$ should be mapped in $\mathcal{B}_\ell$ 'modulo' $\mathrm{T}^{\underline{\mu}}$, and this motivates the following definition.

Definition 4.2. Acceptable sequences $\rho, \rho' \in \hat{\mathrm{T}}$ are *symmetric with respect to* $\mathrm{T}^{\underline{\mu}}$, denoted $\rho \equiv_\mu \rho'$, if there are sequences σ, σ', τ such that $|\tau| > 0$ and:

- $\rho = \sigma^\frown\tau$,
- $\rho' = \sigma'^\frown\tau$,
- $\sigma, \sigma' \in \mathrm{T}^{\underline{\mu}}$,
- $\tau(0)$ is not a $\underline{\mu}$-term, and
- σ is similar to σ'.

Having established the prerequisite terminology, we informally describe the action of the Ξ-requirement at a node μ. First Ξ enumerates all the elements of R^μ into $\mathcal{A}$, knowing that all other elements of $\mathrm{T}^{\underline{\mu}}$ were already enumerated by higher priority requirements. Then, Ξ tries to ensure satisfaction of the property P1 by enumerating new elements into $\mathcal{A}$ and copying labels as necessary. Finally, Ξ performs its label rejuvenation by giving labels to all the elements in R^μ in such a way that at every single stage they all have different labels 'modulo $\mathrm{T}^{\underline{\mu}}$', and that they all end up with the same labels after infinitely many rejuvenations. The precise details are as follows. By a *large* number, we mean a number which is (strictly) larger than any m such that a label L_m has previously been given to any element of $\mathcal{A}$. Stages at which μ is visited are also referred to as μ-stages.

Label Rejuvenation. When we *rejuvenate* the labels at a set of nodes $\{\sigma_i\}_{i\in I} \subset \mathcal{A}$, simultaneously we:

(1) Let L_{m_0} be the second largest temporary label which currently holds of each σ_i. Enumerate L_{m_0} into the bag of labels.
(2) Take a large number m and add the temporary label L_m to each σ_i (that is, make $\mathsf{L}_m(\sigma_i)$ hold).
(3) For every label L_n in the bag of labels with $n < m_0$, make $\mathsf{L}_n(\sigma_i)$ hold of each σ_i.

The Instructions. At any stage s when μ is visited, the following action is taken.

(C1) If this is the first μ-stage, enumerate the elements of R^μ into $\mathcal{A}$ and give them all the rank and marker labels that they have in $\hat{\mathcal{T}}$. Add to each element of R^μ two large temporary labels in such a way that, for any $\sigma, \sigma' \in R^\mu$, if $\mathrm{last}(\sigma) = \mathrm{last}(\sigma')$ then σ and σ' have the same temporary labels, and otherwise they do not share any temporary labels. This ends the action for the construction as a whole at this stage. In particular, the current path does not extend past μ (although μ is still considered to have an outcome).
(C2) At every other μ-stage, we look for strings σ, σ', τ such that:

(a) $\sigma \in \mathrm{T}^{\underline{\mu^-}}$ and $\sigma' \in R^{\mu}$,
(b) σ is similar to σ',
(c) $\tau(0)$ is not a $\underline{\mu}$-term, and
(d) $\sigma^\frown\tau \in \mathcal{A}[s-1]$ but $\sigma'^\frown\tau \notin \mathcal{A}[s]$.

For each such σ, σ', and τ, we enumerate $\sigma'^\frown\tau$ into $\mathcal{A}$. We add all the labels on $\sigma^\frown\tau$ to $\sigma'^\frown\tau$. We say that $\sigma'^\frown\tau$ is *copied* from $\sigma^\frown\tau$.

We then rejuvenate the labels for all elements of R^{μ} in such a way that σ and σ' are rejuvenated simultaneously if and only if $\mathrm{last}(\sigma) = \mathrm{last}(\sigma')$.

4.3. **The Outcome of the Φ_ℓ-Requirement at a Node μ.** The aim of the Φ_ℓ-requirement at μ, where $\ell = |\mu|$, is to build a computable isomorphism between $\mathcal{A}$ and $\mathcal{B}_\ell$. Of course, this is only possible if $\mathcal{A}$ and $\mathcal{B}_\ell$ are classically isomorphic. The choice of outcomes will attempt to prevent $\mathcal{A}$ and $\mathcal{B}_\ell$ from being classically isomorphic. If this fails, then in the limit the strategy will have built a computable isomorphism between $\mathcal{A}$ and $\mathcal{B}_\ell$ (using finitely much nonuniform information about the image of $\mathrm{T}^{\underline{\mu}}$).

Let m be the length of the longest strings in $\mathrm{T}^{\underline{\mu}}$. Whenever μ is visited we form a vector $\overline{k}$ as follows. This vector describes the outcomes which μ has had at previous stages. Let k_1 be the number of stages at which μ has been visited and has had outcome ${}_m\mathtt{pe}_\emptyset$, and let $\overline{k_1} = (k_1)$. Given $\overline{k_i}$, if $i < m$ then let k_{i+1} be the number of times that μ has been visited and has had outcome ${}_{m-i}\mathtt{pe}_{\overline{k_i}}$. If $i = m$ then let k_{i+1} be the number of times that μ has been visited and has had outcome $\mathtt{se}_{\overline{k_i}}$. Let $\overline{k_{i+1}} = \overline{k_i}^\frown k_{i+1}$. Let $\overline{k} = \overline{k_{m+1}}$. If $\mathcal{A}$ and $\mathcal{B}_\ell$ do not appear isomorphic (in a precise manner described soon), then the outcome is $\mathtt{f}_{\overline{k}}$. If $\mathcal{A}$ and $\mathcal{B}_\ell$ do appear isomorphic, then we assess the amount of evidence towards this fact, and then decide which of the other outcomes μ should have at this stage.

More formally, the first time the node μ is visited, the outcome is ${}_m\mathtt{pe}_\emptyset$. Recall that, as specified in (C1) above, the stage is then ended without visiting further nodes in $\mathcal{TS}$.

At every other μ-stage, we assess whether E appears to be an edge relation on $\mathcal{B}_\ell$ for a tree. If $\mathcal{B}_\ell[s]$ contains more than one x with $\mathsf{E}(x,x)$, or contains cycles, the outcome of μ is $\mathtt{f}_{\overline{k}}$. Otherwise, let s_0 be the last expansionary stage, primary or secondary, i.e., the last μ-stage at which it did not have finitary outcome. We decide whether $\mathcal{A}$ and $\mathcal{B}_\ell$ seem isomorphic depending on the presence or absence of a *matching*. To obtain a matching, we will give elements of $\mathcal{B}_\ell$ *tags* for elements of $\mathcal{A}[s_0] \setminus \mathrm{T}^{\underline{\mu}}$.[6] These tags are described by a (partial) *tagging function*

$$f\colon \mathcal{A}[s_0] \setminus \mathrm{T}^{\underline{\mu}} \to \mathcal{B}_\ell$$

mapping ρ to f_ρ. The construction may redefine f_ρ (i.e., tags may be moved). The tagging function f is local to μ, being completely independent of other nodes' tagging functions, but we abuse notation and write f_ρ rather than $f^{\mu}_{\rho}[s]$. For $\rho \in \mathcal{A}[s_0] \setminus \mathrm{T}^{\underline{\mu}}$, when we define $f_\rho := \pi \in \mathcal{B}_\ell$, we say that we *place the ρ-tag* on π. When μ is visited, some of the tags will already have been placed at previous μ-stages. The objective will be to place all the tags which are not presently on some element of $\mathcal{B}_\ell$, i.e., to make f total on $\mathcal{A}[s_0] \setminus \mathrm{T}^{\underline{\mu}}$. It may also be necessary to make various values f_ρ undefined. This can be done only if the ρ-tag proves itself to be wrongly placed (as explained later).

So the basic idea is that f will establish the required isomorphism. Since values f_ρ may be redefined, however, how do we ensure that the given isomorphism will be computable?

[6]Note that *tags* are distinct from *labels*. Labels are part of the first-order structure while tags are local to a strategy, are not part of the first-order structure, and serve only to help determine the outcome. Also, note that $\mathcal{A}[s_0]$ refers to the structure built at the *end* of stage s_0.

Suppose for a moment that g is an isomorphism from $\mathcal{A}$ to $\mathcal{B}_\ell$.[7] It is useful to concentrate initially on what will happen to ρ such that $\rho \notin \mathrm{T}^{\underline{\mu}}$ and $\rho^- \in \mathrm{T}^{\underline{\mu}}$. As we have described already, since μ is not able to delay the rejuvenation of labels given to sequences in $\mathrm{T}^{\underline{\mu}}$, it works 'modulo' $\mathrm{T}^{\underline{\mu}}$. More precisely, this means that it aims to place the ρ-tag on $g(\rho')$ for some $\rho' \equiv_\mu \rho$, in such a way that all the tags for distinct $\rho' \equiv_\mu \rho$ are placed on different elements of $\mathcal{B}_\ell$. When the ρ-tag is placed on $g(\rho')$ for $\rho' \equiv_\mu \rho$ we say that it is placed *correctly*. When the ρ-tag is not placed correctly, it will be placed on π such that π^- is not in $g(\mathrm{T}^{\underline{\mu}})$, and will eventually be proved to be in the wrong place, and so will be moved. So, if we are given the g-image of $\mathrm{T}^{\underline{\mu}}$ (which is a finite amount of information), then the first time we see the ρ-tag placed on π with π^- in this image, we know that it is correctly placed, and will not subsequently move. We then also have to deal with ρ such that $\rho^- \notin \mathrm{T}^{\underline{\mu}}$. The tag for ρ of this kind will only be placed above where we have already placed the ρ^--tag, and will not move unless the tag for an initial segment proves to be wrongly placed. Thus once the tag for ρ with $\rho^- \in \mathrm{T}^{\underline{\mu}}$ is correctly placed, the isomorphism above this will be built uniformly in a computable fashion. In the definition below we assume that E appears to be an edge relation on $\mathcal{B}_\ell$ for a tree, so that π^- is defined for $\pi \in \mathcal{B}_\ell$.

Definition 4.3. We define the backbone of $\pi \in \mathcal{B}_\ell$ in the obvious way. First, $\mathrm{bb}(r_{\mathcal{B}_\ell}) = \langle(\alpha, 0)\rangle$ (recall Definition 2.8). Suppose we are given $\mathrm{bb}(\pi^-)$. If π does not have precisely one rank label, then $\mathrm{bb}(\pi)$ is undefined. If it has the unique rank label R_β and β is a successor then we define $\mathrm{bb}(\pi) := \mathrm{bb}(\pi^-)^\frown(\beta)$. If it has the unique rank label R_β and β is a not a successor, then $\mathrm{bb}(\pi)$ is undefined unless it has a unique marker label M_j. If it does have a unique marker label M_j then we define $\mathrm{bb}(\pi) := \mathrm{bb}(\pi^-)^\frown(\beta, j)$.

The precise instructions at stage s, once μ has decided whether E appears to be an edge relation on $\mathcal{B}_\ell$ for a tree, are as follows. As above, let $s_0 < s$ be the last expansionary stage. First, take each $\rho \in \mathcal{A}[s_0] \setminus \mathrm{T}^{\underline{\mu}}$ such that f_ρ was undefined at the beginning of stage s, in the order in which they were enumerated into $\mathcal{A}$, and perform the following. Let m be largest such that the temporary label L_m has been given to ρ. Look for an element $\pi \in \mathcal{B}_\ell$ which has no tag on it and such that:

(1) $\mathrm{bb}(\pi) = \mathrm{bb}(\rho)$;
(2) π has the temporary label L_m;
(3) (a) if $\rho^- \notin \mathrm{T}^{\underline{\mu}}$, then π^- has the ρ^--tag on it;
    (b) if $\rho^- \in \mathrm{T}^{\underline{\mu}}$ then no proper initial segment of π is tagged or has a μ-dead label;

If there is such a node, let f_ρ be the Gödel-least eligible node π.

Having performed these instructions for all the appropriate ρ, we then have to decide which values f_ρ should be made undefined. If $f_\rho = \pi$ then f_ρ is made undefined only if we see a *proof that the tag is wrongly placed.* This requires that either $\rho^- \in \mathrm{T}^{\underline{\mu}}$ and one of the following hold:

- some proper initial segment of π receives a new tag, or
- some proper initial segment of π has a μ-dead label;

or $\rho^- \notin \mathrm{T}^{\underline{\mu}}$ and the ρ^--tag is proved to be wrongly placed. For all ρ, if we see a proof that the ρ-tag is wrongly placed, we make f_ρ undefined.

Having performed these instructions, we are then ready to decide whether this should be an expansionary stage. A *matching* occurs if f_ρ is defined for all $\rho \in \mathcal{A}[s_0] \setminus \mathrm{T}^{\underline{\mu}}$ and, for each such ρ, the temporary labels of ρ are a subset of the temporary labels of f_ρ. If we do not have a matching, the outcome is $\mathtt{f}_{\overline{k}}$, where $\overline{k}$ is as defined previously. If there is a matching and the

[7] In fact $\mathcal{A}$ will be rigid, but we never directly prove this because we do not need to. It always suffices just to consider an arbitrary isomorphism from $\mathcal{A}$ to $\mathcal{B}_\ell$ (when one exists) and to work with that.

stage s_0 was an ${}_n\mathtt{pe}$-stage (for any $n \leq m$), then the outcome is $\mathtt{se}_{\overline{k}\upharpoonright m}$. Otherwise we have to decide for which $n \leq m$ (if any) this should be an ${}_n\mathtt{pe}$-stage.

For each $1 \leq n \leq m$, let $s_1 < s$ be the last stage which was an ${}_{n'}\mathtt{pe}$-stage for some $n' \geq n$. We say s is *eligible as an* ${}_n\mathtt{pe}$*-stage* if:

(i) $n = 1$ or there has been an ${}_{n-1}\mathtt{pe}$-stage $> s_1$, and
(ii) there exists an expansionary stage s_2 with $s_1 < s_2 < s$ such that:
  (a) all the tags for $\rho \in \mathcal{A}[s_1] \setminus \mathrm{T}^{\underline{\mu}}$ with $\rho^- \in \mathrm{T}^{\underline{\mu}}$ and $|\rho| \leq n+1$ have been placed at stage s_2 and have not been proved wrong since then, and
  (b) for each $1 \leq j \leq n$, all the elements in $\mathcal{B}_\ell[s_2]$ of length j except for at most $|\mathrm{T}^{\underline{\mu}} \upharpoonright j|$-many, presently have a tag on some (not necessarily proper) initial segment.

In the above $\mathrm{T}^{\underline{\mu}} \upharpoonright j$ is the set of strings in $\mathrm{T}^{\underline{\mu}}$ of length j. If there exists a greatest $n \leq m$ such that s is eligible as an ${}_n\mathtt{pe}$-stage then the outcome is ${}_n\mathtt{pe}_{\overline{k}\upharpoonright m-n}$. Otherwise, the outcome is $\mathtt{se}_{\overline{k}\upharpoonright m}$.

This completes the instructions for deciding the outcome of μ.

Now let us *informally* describe the idea behind ${}_n\mathtt{pe}$-stages and why they should exist. Suppose that g is an isomorphism from $\mathcal{A}$ to $\mathcal{B}_\ell$ and that there are infinitely many expansionary stages for μ. We will later be able to show that no tags are ever placed in $g(\mathrm{T}^{\underline{\mu}})$. Let s_0 be a stage of the construction after which all elements of $\mathrm{T}^{\underline{\mu}}$ have appeared in $\mathcal{A}$ and all elements of $g(\mathrm{T}^{\underline{\mu}})$ have appeared in $\mathcal{B}_\ell$. Suppose that $\rho \notin \mathrm{T}^{\underline{\mu}}$, $\rho^- \in \mathrm{T}^{\underline{\mu}}$ and $|\rho| = n+1$. Let $s_1 > s_0$ be an ${}_n\mathtt{pe}$-stage at which ρ has already been enumerated into $\mathcal{A}$. Suppose that $s > s_1$ is an ${}_n\mathtt{pe}$-stage, and let s_2 be as in (ii) of the conditions for an ${}_n\mathtt{pe}$-stage above. Then at stage s the tag for ρ must be placed on some $g(\tau)$ which is a one element extension of an element of $g(\mathrm{T}^{\underline{\mu}})$, since all other elements of $\mathcal{B}_\ell[s_2]$ of length $n+1$ have a tag placed on some proper initial segment. It cannot then be moved from this position, and since there are infinitely many expansionary stages this means that $\mathcal{A}_\rho$ embeds into $\mathcal{A}_\tau$ (after the ρ-tag is placed on $g(\tau)$, the tags for extensions of ρ all get placed without subsequently being moved, and establish this embedding). We then wish to show that $\mathrm{last}(\rho) = \mathrm{last}(\tau)$, so that $\tau \equiv_\mu \rho$ and the ρ-tag is correctly placed. In the case that $\rho \in \mathrm{T}^{\mathcal{TP}}$ this will follow from Lemma 3.13 (there is a detail here which we discuss further below). If $\rho \notin \mathrm{T}^{\mathcal{TP}}$ then it has a finite number of temporary labels given to it, the largest two of which will not be added to the bag. The fact that τ has either of these labels therefore means that in this case we also have $\mathrm{last}(\rho) = \mathrm{last}(\tau)$. In summary, at all sufficiently large ${}_n\mathtt{pe}$-stages, all tags for ρ with $\rho^- \in \mathrm{T}^{\underline{\mu}}$ and $|\rho| = n+1$ will be correctly placed (and will remain so).

Why should there exist infinitely many ${}_n\mathtt{pe}$-stages? The basic idea is as follows, and of course the proper proof will be given in the verification. Suppose inductively that there are infinitely many ${}_{n'}\mathtt{pe}$-stages for $n' < n$. For now, suppose quite simply that no tags are ever placed in $g(\mathrm{T}^{\underline{\mu}})$. As described above this means that, for all $\rho \notin \mathrm{T}^{\underline{\mu}}$ with $\rho^- \in \mathrm{T}^{\underline{\mu}}$ and $|\rho| \leq n$, the tag for ρ is eventually correctly placed. Thus there will be no difficulty in satisfying (iib) of the conditions for an ${}_n\mathtt{pe}$-stage. Suppose that there exists a last ${}_n\mathtt{pe}$-stage s. It suffices to show that, for all $\rho \in \mathcal{A}[s] \setminus \mathrm{T}^{\underline{\mu}}$ with $|\rho| = n+1$ and $\rho^- \in \mathrm{T}^{\underline{\mu}}$, the value f_ρ reaches a limit. These ρ do not have their labels rejuvenated after stage s, and the two largest temporary labels given to them are never added to the bag. After some stage the correct positions for these tags will have all of their finite set of temporary labels. For each of these ρ, any ρ'-tag placed in one of the correct positions for ρ must satisfy $\rho' \equiv_\mu \rho$, since otherwise it cannot have either of the two largest labels given to ρ. So the correct positions will always be available as positions for placing the tags after some point. Thus if f_ρ was moved infinitely often, one of the correct positions would eventually become the Gödel least available option, and the ρ-tag would eventually be correctly placed and not moved, a contradiction.

We promised to say a little bit more about the use of Lemma 3.13 in the argument a couple of paragraphs above, that the tag for ρ as specified must be placed correctly at all sufficiently late ${}_n\mathsf{pe}$-stages. Roughly this works as follows. Let ρ and τ be as in the discussion above. Recall that both of ρ^- and τ^- are in $\mathrm{T}^{\underline{\mu}}$ and that we were considering the case $\rho \in \mathrm{T}^{\mathcal{TP}}$. Although Lemma 3.13(a) permits an embedding when $\mathrm{last}(\rho)$ is of the form $(\beta, n, (\beta', n'), \mu'')$ and $\mathrm{last}(\tau)$ is (β, n, μ') for the unique $\mu' \subseteq \mu''$ for which there is a μ'-term (β, n, μ'), if the tag for ρ is placed on $g(\tau)$ for τ of this form at an ${}_n\mathsf{pe}$-stage, then the pigeon hole principle tells us that we must have permanently placed the tag for some $\tau' \equiv_\mu \tau$ on $g(\sigma)$ for some σ which has the right backbone, but with $\mathrm{last}(\sigma) \neq (\beta, n, \mu')$. Given that we have infinitely many expansionary stages, we get an embedding of $\mathcal{A}_{\tau'}$ into $\mathcal{A}_\sigma$, which does now directly contradict Lemma 3.13. One might then think that it is the existence of infinitely many expansionary stages which is threatened by the exceptional case in Lemma 3.13(a). If there are only finitely many expansionary stages, however, any ρ for which we look to place a tag after the last expansionary stage only receives finitely many labels. In that case ρ and τ as described would be clearly distinguished by their two largest labels, so that the ρ-tag could not be placed on $g(\tau)$ when $\mathrm{last}(\tau) \neq \mathrm{last}(\rho)$.

In the above, we simply assumed here that no tags are placed in $\mathrm{T}^{\underline{\mu}}$, and in actual fact some considerable care has to be taken in proving this. Thus, the actual proof that there are infinitely many ${}_n\mathsf{pe}$-stages will be a little more complicated (and, of course, more detailed) than what we have just described, but is essentially the same.

4.4. **The Construction.** The construction proceeds in a typical priority argument fashion. At stage s, unless the stage is ended prematurely via the instruction (C1), the stage visits nodes $\mu \in \mathcal{TS}$ (determined by the current outcomes) until a node is reached of length s. When a node $\mu \in \mathcal{TS}$ is visited, the associated Ξ-requirement acts as described and the associated Φ_ℓ-requirement (where $\ell := |\mu|$) dictates the outcome of μ.

4.5. **The Verification.** It will often be necessary to be precise about exactly when during some stage of the construction certain instructions are carried out. We therefore subdivide each stage of the construction into *steps*. By a step of the construction, we mean the part of a single stage during which the instructions for one strategy visited at that stage are carried out. So if n strategies are visited at stage s, then stage s consists of n steps.

Before we prove anything at all about the construction, we actually have to be careful to ensure that it is well-defined. Consider what happens when μ performs the instructions (C2) at some stage s. Suppose that we find strings σ, σ', τ such that $\sigma \in \mathrm{T}^{\underline{\mu^-}}$ and $\sigma' \in R^\mu$, σ is similar to σ', such that $\tau(0)$ is not a $\underline{\mu}$-term, and $\sigma^\frown\tau \in \mathcal{A}$ but $\sigma'^\frown\tau \notin \mathcal{A}$. The instructions tell us that we must enumerate $\sigma'^\frown\tau$ into $\mathcal{A}$ and add all the labels on $\sigma^\frown\tau$ to $\sigma'^\frown\tau$. We must rule out the possibility, however, that for some other σ'' the same conditions hold with respect to σ' and τ, but that the labels on $\sigma''^\frown\tau$ are different than those on $\sigma^\frown\tau$. In this case the instructions may be ambiguous and will certainly fail to achieve what is required of them. We shall shortly prove that this potentially problematic situation does not arise. For now, however, in order to ensure that the construction is well-defined one can simply imagine that, were this situation to arise, we choose the first σ for which the given conditions hold for each σ' and τ.

As an important initial step, we start by showing that a true path $\mathcal{TP}$ exists.

Lemma 4.4. *There is a (unique) infinite path through $\mathcal{TS}$ of nodes which are visited infinitely often. Indeed, every node visited infinitely often is on this path.*

Proof. We show by induction on n that there is a unique μ of length n which is visited infinitely often. Clearly this holds for $n = 0$. Suppose that the result holds for n, and let μ be the node of length n which is visited infinitely often. If there exists a greatest k such that μ has infinitely many ${}_k\texttt{pe}$-stages, then let s be a ${}_k\texttt{pe}$-stage for μ, which is not the first stage at which μ is visited and after which μ never has a ${}_{k'}\texttt{pe}$-stage for any $k' > k$. The node μ' of length $n+1$ visited at stage s is then the unique node of length $n+1$ visited infinitely often. If there exists no such k but μ has infinitely many expansionary stages, then let s be an $\texttt{se}$-stage for μ after which there are no ${}_{k'}\texttt{pe}$-stages for any k'. The node μ' of length $n+1$ visited at stage s is the unique node of length $n+1$ visited infinitely often. Finally, if there exists a stage s after which μ always has finitary outcome, then the node μ' of length $n+1$ visited at stage s is the unique node of length $n+1$ visited infinitely often. $\square$

Lemma 4.5. *Only acceptable sequences are enumerated into $\mathcal{A}$.*

Proof. Sequences are either enumerated via (C1) or (C2). The fact that sequences enumerated via (C1) are acceptable is immediate from the definition of R^μ. In order to deal with sequences enumerated via (C2), suppose that $\sigma \in R^\mu$, $\beta := \mathrm{rank}(\sigma)$, that t is a ν-term, that if β is a limit then $\beta' := \mathrm{rank}(t)$ is a successor less than β and that if β is not a limit then $\beta' = \mathrm{pred}(\beta)$. It suffices for us to show that if $\sigma^\frown t$ is not an acceptable sequence then $\nu \subseteq \mu$. In order to see this let $t' = \mathrm{last}(\sigma)$ and let $\mu' \subseteq \mu$ be such that t' is a μ'-term. The fact that $\sigma^\frown t$ is not an acceptable sequence means that ν is comparable with μ' and that either:

(i) β is a successor, t' is of the form (β, n, μ') and $t = (\mathrm{pred}(\beta), n, \nu)$, or;
(ii) β is a limit, t' is of the form $(\beta, n, (\beta', n'), \mu')$ and $t = (\beta', n', \nu)$.

Let i and j be such that $\beta = \alpha_i$ and $\beta' = \alpha_j$ (according to the enumeration fixed by Convention 3.7). If (i) holds then $i, n \leq |\mu'|$, and by Convention 3.7 this means that $j < |\mu'|$ and that, since t is a ν-term, $\nu \subseteq \mu'$. If (ii) holds then we have that $i, j, n, n' \leq |\mu'|$. The fact that t is a ν-term again gives $\nu \subseteq \mu'$, as required. $\square$

Next we work towards showing that P1 is satisfied. The following lemma shows that the potentially problematic situation concerning ambiguous instructions which we outlined at the start of the verification cannot arise.

Definition 4.6. At any stage, we say that $\mathcal{A}$ is *symmetric* with respect to μ if, for all $\rho \equiv_\mu \rho'$, we have $\rho \in \mathcal{A}$ iff $\rho' \in \mathcal{A}$ and that if either of these strings belong to $\mathcal{A}$ then they have the same labels.

Lemma 4.7. *Immediately after any step when μ is visited except the first, $\mathcal{A}$ is symmetric with respect to μ.*

Proof. We actually prove the following by induction on the step of the construction, simultaneously for all μ: Immediately after any step when μ is visited except the first, $\mathcal{A}$ is symmetric with respect to μ' for all $\mu' \subseteq \mu$, and after the first step at which μ is visited $\mathcal{A}$ is symmetric with respect to all $\mu' \subset \mu$. Note that since there is only one $\emptyset$-term, $\equiv_\emptyset$ implies equality. So the result is trivial for $\mu = \emptyset$. Now suppose $|\mu| > 0$ and that μ is visited at stage s.

Note that any μ' only enumerates sequences into $\mathcal{A}$ which extend elements of $R^{\mu'}$. It follows that no extensions of elements of R^μ can be enumerated into $\mathcal{A}$ by nodes to the left or right of μ. We wish to show that no $\mu' \subset \mu$ can enumerate sequences into $\mathcal{A}$ extending elements of R^μ. So suppose that there is a first step at which some $\mu' \subset \mu$ enumerates a sequence ρ into $\mathcal{A}$ which extends an element of R^μ. This must be because μ' finds σ, σ' which are similar and τ such that $\sigma \in \mathrm{T}^{\underline{\mu'}-}$, $\sigma' \in R^{\mu'}$, $\tau(0)$ is not a $\underline{\mu'}$-term, $\sigma^\frown\tau \in \mathcal{A}$ but $\rho = \sigma'^\frown\tau \notin \mathcal{A}$. Since ρ extends an element of R^μ, $\tau = \tau_0{}^\frown\tau_1$ with $\sigma'^\frown\tau_0 \in R^\mu$, and thus $\sigma^\frown\tau_0 \in R^\mu$. So $\sigma^\frown\tau$ extends an element of R^μ, and since it has already been enumerated into $\mathcal{A}$, by our choice of ρ it must have been enumerated by a node $\mu'' \supseteq \mu$.

If $\sigma^\frown\tau \in R^{\mu''}$, then $\sigma'^\frown\tau \in R^{\mu''}$. If instead $\sigma^\frown\tau$ was enumerated by μ'' via (C2), then there are some σ'', τ_2, τ_3 with $\tau_1 = \tau_2{}^\frown\tau_3$, $\sigma'' \in \mathrm{T}^{\underline{\mu''^-}}$, $\sigma^\frown\tau_0{}^\frown\tau_2 \in \mathrm{T}^{\mu''}$, σ'' similar to $\sigma^\frown\tau_0{}^\frown\tau_2$ and $\tau_2(0)$ not a $\underline{\mu''}$-term. But then $\sigma'^\frown\tau_0{}^\frown\tau_2 \in \mathrm{T}^{\mu''}$ and by transitivity of similarity, σ'' is similar to $\sigma'^\frown\tau_0{}^\frown\tau_2$. In either case, we see that when μ'' enumerated $\sigma^\frown\tau$, it would also have enumerated $\sigma'^\frown\tau$ if this sequence had not already been enumerated into $\mathcal{A}$, a contradiction.

So suppose that μ is visited for the first time at stage s. By the induction hypothesis we know that, immediately prior to μ acting, $\mathcal{A}$ is symmetric with respect to all $\mu' \subset \mu$ (since any node terminates the construction for the stage after the first time it is visited, μ^- has been visited at least twice). Now if μ enumerates $\sigma^\frown\tau$ into $\mathcal{A}$, where $\sigma \in \mathrm{T}^{\underline{\mu'}}$ for some $\mu' \subset \mu$ and $\tau(0)$ is not a μ'-term, then it also enumerates all sequences $\sigma'^\frown\tau$ such that $\sigma' \in \mathrm{T}^{\underline{\mu'}}$ is similar to σ and gives them the same labels.

Suppose next that this is the second time that μ is visited, and that it was visited first at stage s'. At stage s', μ enumerated all elements of R^μ into $\mathcal{A}$, whereupon the construction for that stage was immediately terminated. So when μ is visited at stage s, we have by the induction hypothesis that $\mathcal{A}$ is symmetric with respect to all $\mu' \subset \mu$, and we also know that no proper extensions of elements of R^μ have been enumerated into $\mathcal{A}$. Thus at stage s, μ enumerates sequences into $\mathcal{A}$ and gives them labels sufficient to ensure that $\mathcal{A}$ is symmetric with respect to μ at the end of the step. We must also ensure that $\mathcal{A}$ remains symmetric with respect to all $\mu' \subset \mu$. Suppose that $\rho = \sigma^\frown\tau$ where $\sigma \in \mathrm{T}^{\underline{\mu'}}$ for some $\mu' \subset \mu$ and $\tau(0)$ is not a $\underline{\mu'}$-term. If μ enumerates ρ into $\mathcal{A}$ at stage s then it enumerates all $\sigma'^\frown\tau$ such that $\sigma' \in \mathrm{T}^{\underline{\mu'}}$ is similar to σ and gives them the same labels. If μ rejuvenates the labels for ρ then $\rho \in R^\mu$, but then so are all $\sigma'^\frown\tau$ such that $\sigma' \in \mathrm{T}^{\underline{\mu'}}$ is similar to σ, and all these sequences have their labels rejuvenated simultaneously.

In the case that μ is visited at stage s, and has been visited at least twice before, we may argue in a fairly similar way. Let s' be the greatest stage prior to s at which μ was visited. By the induction hypothesis we have that at the end of stage s', $\mathcal{A}$ was symmetric with respect to μ. By the induction hypothesis, we also know that before μ performs any action at stage s, $\mathcal{A}$ is symmetric with respect to all $\mu' \subset \mu$. Subsequent to stage s' and before μ acts at stage s, no extensions of elements of R^μ can have been enumerated into $\mathcal{A}$. Thus, before μ acts at stage s, if $\sigma \in R^\mu$ and $\sigma' \in \mathrm{T}^{\underline{\mu^-}}$ are similar and $\tau(0)$ is not a $\underline{\mu}$-term, then $\sigma^\frown\tau \in \mathcal{A}$ implies $\sigma'^\frown\tau \in \mathcal{A}$. If $\sigma'^\frown\tau \in \mathcal{A}$ but $\sigma^\frown\tau \notin \mathcal{A}$, then for all $\sigma'' \in R^\mu$ similar to σ, $\sigma''^\frown\tau \notin \mathcal{A}$ and μ will enumerate these sequences at stage s and give them all the same labels. If $\sigma^\frown\tau \in \mathcal{A}$ and $\sigma'^\frown\tau \in \mathcal{A}$ prior to μ acting at stage s, we must ensure that neither sequence has had their labels rejuvenated subsequent to stage s'. Since $\tau(0)$ is not a $\underline{\mu}$-term, this could only happen if $\sigma'^\frown\tau \in R^{\mu'}$ for some μ' to the right of μ, but then μ' must have been visited prior to stage s' and cannot be visited subsequent to stage s' since μ is to the left of μ' and is visited at stage s'. Finally, the fact that when μ acts at stage s, $\mathcal{A}$ remains symmetric with respect to all $\mu' \subset \mu$ is argued exactly as for the case that μ is visited for the second time at stage s. □

We isolated an important principle during the course of the proof of Lemma 4.7:

Observation 4.8. Only $\mu' \supseteq \mu$ can enumerate sequences extending elements of R^μ. In particular, this means that when μ enumerates $\sigma'^\frown\tau$ in $\mathcal{A}$ via (C2), where $\tau(0)$ is not a $\underline{\mu}$-term, $\sigma'^\frown\tau(0)$ does not belong to $R^{\mu'}$ for *any* μ'. Thus a simple inspection immediately tells us which node must be responsible for enumerating a sequence in $\mathcal{A}$. If $\rho \in R^\mu$ then it can only be enumerated by μ. Otherwise let σ be the longest initial segment of ρ which belongs to some R^μ. Then ρ can only be enumerated by μ such that $\sigma \in R^\mu$.

The following terminology will be useful.

Definition 4.9. We say that ρ is a *μ-boundary* sequence if $\rho \notin \mathrm{T}^{\underline{\mu}}$ but $\rho^- \in \mathrm{T}^{\underline{\mu}}$. We say that ρ is *pure* if it belongs to R^μ for some μ. We say that ρ is *broken* if it is not pure. If ρ is pure, let *origin*(ρ) be the μ such that $\rho \in R^\mu$. If ρ is broken, let *origin*(ρ) be the origin of the longest pure initial segment of ρ. We say that ρ_0 and ρ_1 are *siblings* if there exist ρ and $\mathcal{TS}$-terms $t \neq t'$ such that $\rho_0 = \rho^\frown t$ and $\rho_1 = \rho^\frown t'$.

So we have observed that ρ is always enumerated by origin(ρ). If ρ is pure then it is enumerated via (C1), and if is it broken then it is enumerated via (C2).

Lemma 4.10. *The structure $\mathcal{A}$ is symmetric with respect to $\mathrm{T}^{\mathcal{TP}}$.*

Proof. Property S1 follows since the action of a node $\mu \in \mathcal{TP}$ the first time it is visited ensures $R^\mu \subseteq \mathcal{A}$.

S2 then follows almost immediately from Lemma 4.7. If $\sigma, \sigma' \in \mathrm{T}^{\mathcal{TP}}$ are similar and $\sigma^\frown\tau(0) \notin \mathrm{T}^{\mathcal{TP}}$, then let μ be the shortest member of $\mathcal{TP}$ such that σ and σ' both belong to $\mathrm{T}^{\underline{\mu}}$. If $\sigma^\frown\tau \in \mathcal{A}$ then it is only given finitely many labels, so let s be a stage at which μ is visited and $\sigma^\frown\tau$ has already been enumerated into $\mathcal{A}$ and will not be given any more labels after μ acts at stage s. By Lemma 4.7, at every stage $\geq s$ at which μ is visited, $\sigma'^\frown\tau \in \mathcal{A}$ and has precisely the same finite set of labels. □

The following will complete the properties required for the analysis in Sections 3.3 and 3.4.

Lemma 4.11. *The structure $\mathcal{A}$ has properties P2 and P3.*

Proof. For P2, fix $\sigma \in \mathrm{T}^{\mathcal{TP}}$, and let $\mu \subset \mathcal{TP}$ be such that $\sigma \in R^\mu$. Suppose σ receives temporary label L_m at stage s_0 via (2) of Label Rejuvenation. Let $s_1 < s_2$ be the next two stages after s_0 at which μ is visited. Then at stage s_2, L_m is enumerated into the bag.

Conversely, suppose L_n is a label which enters the bag at stage s. Let $s_0 < s_1 < s_2$ be the next three stages after s at which μ is visited. Then at stage s_0, σ is labeled with L_m for some $m > n$. At stage s_2, L_m is enumerated into the bag, and L_n is made to hold of σ. So the temporary labels on σ are precisely the labels in the bag.

For P3, fix $\sigma \in \mathcal{A} \setminus \mathrm{T}^{\mathcal{TP}}$. If σ is broken, then it never gains any labels beyond the finitely many it had when it entered $\mathcal{A}$. If σ is pure, then let $\mu \not\subset \mathcal{TP}$ be such that $\sigma \in R^\mu$. Then the labels on σ are precisely the finitely many labels it had at the end of the final stage at which μ is visited. □

Computable Categoricity: To demonstrate that $\mathcal{A}$ is computably categorical when $\alpha \in \mathcal{O}$, we verify that $\mathcal{A}$ is computably isomorphic to $\mathcal{B}_\ell$ if the two are isomorphic. Thus, for the remainder of this section, we assume $\alpha \in \mathcal{O}$ (and is a limit) and $\mathcal{A} \cong \mathcal{B}_\ell$. We let μ be the strategy on the true path $\mathcal{TP}$ dedicated to building the computable isomorphism, i.e., we let $\mu := \mathcal{TP} \upharpoonright \ell$. We also let g be an isomorphism from $\mathcal{A}$ to $\mathcal{B}_\ell$. Recall that we say the ρ-tag placed on π is *correctly placed* if $\pi = g(\rho')$ for some $\rho' \equiv_\mu \rho$.

Lemma 4.12. *If there are infinitely many μ-expansionary stages, and the ρ-tag is not moved from some point on, then, letting f_ρ take its final value, $\mathcal{A}_\rho$ embeds into $\mathcal{B}_{\ell f_\rho}$ via a computable embedding which maps ρ to f_ρ.*

Proof. This follows directly from the fact that, given the conditions described in the statement of the lemma, for all $\sigma \in \mathcal{A}_\rho$, f_σ is defined at some point after the ρ-tag takes its final value (and therefore does not move from that point on). Then if $\mathsf{E}(\sigma, \sigma')$ for $\sigma, \sigma' \in \mathcal{A}_\rho$, $\mathsf{E}(f_\sigma, f_{\sigma'})$ holds in $\mathcal{B}_\ell$ because we do not place a σ'-tag on $\pi \in \mathcal{B}_\ell$ unless π^- has a σ-tag placed on it. When we place any σ-tag, we require that the element we place it on has the same rank and marker labels.

Suppose that a temporary label is given to $\sigma \in \mathcal{A}_\rho$ at some stage s. Let s' be a stage after which the σ-tag is placed and does not move. At the next expansionary stage after $\max\{s, s'\}$, we must have that f_σ also has this temporary label. It is not particularly important, but in observing that the next expansionary stage suffices here, rather than the one after that, note that sequences which μ is looking to place tags for in order to have an expansionary stage do not have their labels rejuvenated while we wait for the next expansionary stage. Therefore it makes no difference whether the instructions require f_σ to have *all* labels given to σ before we have a matching, or all labels given to σ by the end of the most recent expansionary stage. $\square$

Next we need a number of small technical lemmas, all of which lead up to the proof of Lemma 4.19, that there exist infinitely many μ-expansionary stages. The basic idea behind the proof of Lemma 4.19 is as follows. Suppose there is a last expansionary stage s for μ. We focus initially on what happens to the tags for μ-boundary sequences $\rho \in \mathcal{A}[s]$. These ρ do not have their labels rejuvenated subsequent to stage s. Firstly, this means that none of the tags can be placed in $g(\mathrm{T}^{\underline{\mu}})$ subsequent to stage s, because the two largest temporary labels given to each of these ρ are never added to the bag, and are therefore never given to any element of $\mathrm{T}^{\underline{\mu}}$. Secondly, this means that each value f_ρ must eventually be defined and come to a limit, because eventually the correct positions in $\mathcal{B}_\ell$ will have all the necessary labels and will be available to place these tags on.

Once we know that f_ρ comes to a limit somewhere, we then have to consider what this limit might be. We argue by induction on ρ that this limit is a correct position for ρ. By the inductive hypothesis and a pigeon hole argument, every $g(\sigma)$ for σ a μ-boundary sequence strictly shorter than ρ must have a tag placed on it, so this limit must be the image of some μ-boundary sequence. So the limit is either a correct position or a sibling of a correct position. However, siblings have distinct terms, so will have distinct largest temporary labels. So f_ρ settles at a correct position.

Once we know that labels for μ-boundary sequences settle at the correct locations, we can argue that the remaining labels settle and do so at the correct location: If ρ is a μ-boundary sequence, and $\rho^\frown\sigma$ is acceptable, then we argue by induction along $|\sigma|$, $f_{\rho^\frown\sigma}$ is eventually placed at the correct location (and never moves). By the inductive hypothesis, the only concern is that $f_{\rho^\frown\sigma}$ might be placed at a sibling of the correct location, but since siblings have distinct largest temporary labels, this cannot be.

So we will eventually have another matching, resulting in another expansionary stage.

Lemma 4.13. *If $\rho \in \mathcal{A} \setminus \mathrm{T}^{\mathcal{TP}}$ and $\rho^- \in \mathrm{T}^{\underline{\mu}}$, then the two largest temporary labels given to ρ are never added to the bag.*

Proof. If ρ is pure then this is clear. If it is broken (and thus ρ is a μ-boundary sequence), then the result follows from the following sublemma.

Sublemma 4.14. *Suppose ρ is a broken μ'-boundary sequence and s is a μ'-stage such that ρ is already in $\mathcal{A}$ by the end of the μ'-step at stage s. From this point on, until such a stage as the path is to the left of μ', the two largest labels on ρ are not added to the bag of labels.*

Proof. Let $\rho = \sigma^\frown t \in \mathcal{A}$, so that $\sigma \in \mathrm{T}^{\underline{\mu'}}$ but $\rho \notin R^{\mu''}$ for any μ''. We know that ρ must be enumerated by $\mathrm{origin}(\rho) \subseteq \mu'$ via (C2). When this enumeration is made, let μ_0 be the shortest initial segment of $\mathrm{origin}(\rho)$ such that there exists $\sigma' \in R^{\mu_0}$ which is similar to σ, and $\sigma'^\frown t$ has been enumerated into $\mathcal{A}$ (so ρ is enumerated by copying $\sigma'^\frown t$). If $\sigma'^\frown t$ is broken, then μ_0 enumerated it by copying some other string $\sigma''^\frown t$ already in $\mathcal{A}$. By our action at (C2), it must be that $\sigma'' \in \mathrm{T}^{\underline{\mu_0^-}}$ and σ'' similar to σ'. So $\mathrm{origin}(\sigma'') \subset \mu_0$, and by transitivity of similarity, σ'' is similar to σ, so $\mathrm{origin}(\sigma'')$ contradicts our choice of μ_0 to be shortest. So $\sigma'^\frown t \in R^{\mu_1}$ for some $\mu_1 \supseteq \mu_0$. If μ_1 were comparable with $\mathrm{origin}(\rho)$, it would contradict our assumption that ρ is broken. So $\mu_1 \supset \mu_0$ must be incomparable with origin(ρ). Also, μ_1 must be visited prior to the point at which ρ enters $\mathcal{A}$, but cannot be visited subsequent to this point until such a stage as the path is to the left of origin(ρ) (and thus μ'). Let s_1 be the last stage at which μ_1 is visited prior to stage s. At the end of stage s_1, $\sigma'^\frown t$ has two temporary labels which are not added to the bag of labels at any stage prior to one at which the path is to the left of μ'. When origin(ρ) enumerates ρ into $\mathcal{A}$, it gives it these labels. □

Since μ is on the true path by assumption, and origin$(\rho) = \mu$, then the path is never to the left of μ after the stage at which ρ is enumerated into A. The lemma follows. □

Lemma 4.15. *If ρ is broken, then no extension of ρ is enumerated after the step at which ρ is enumerated.*

Proof. The proof is by induction on the step of the construction. So suppose that the result holds prior to a step z at which μ' is visited. If μ' enumerates sequences via (C1), then these do not extend sequences enumerated via (C2), so suppose that it enumerates sequences via (C2). If any sequence which μ' enumerates is to extend a sequence previously enumerated via (C2), then this previous sequence must also have been enumerated by μ'. So suppose that μ' previously enumerated $\rho = \sigma'^\frown\tau$ at some step z' such that $\sigma' \in R^{\mu'}$ and $\tau(0)$ is not a $\underline{\mu'}$-term, and that at step z it enumerates $\sigma'^\frown\tau^\frown\tau'$. This must be because there exists $\sigma \in \mathrm{T}^{\underline{\mu'^-}}$ which is similar to σ' with $\sigma^\frown\tau^\frown\tau' \in \mathcal{A}$. By Lemma 4.7, by the end of the step z', $\sigma^\frown\tau$ had already been enumerated. We have that $\sigma^\frown\tau^\frown\tau'$ cannot already have been enumerated, otherwise $\sigma'^\frown\tau^\frown\tau'$ would be enumerated at step z'. It must therefore be enumerated at a later point, but strictly prior to step z, from which it follows by the induction hypothesis that $\sigma^\frown\tau \in R^{\mu''}$ for some μ'' which must be to the left or right of μ'. This means that μ'' cannot have been visited between steps z' and z, meaning that no extensions of sequences in $R^{\mu''}$ can have been enumerated between steps z' and z, a contradiction. □

Lemma 4.16. *Suppose that there is a last μ-expansionary stage s. Then at no point subsequent to stage s is any tag on an element $\pi \in g(\mathrm{T}^{\underline{\mu}})$.*

Proof. Recall that if ρ is not a μ-boundary sequence, then we do not place f_ρ until we have placed f_{ρ^-}. So if there is such a tag f_ρ, we may assume ρ is a μ-boundary sequence. Further, by construction we only seek to place f_ρ if $\rho \in \mathcal{A}[s]$, and thus $\rho \in \mathcal{A} \setminus \mathrm{T}^{\mathcal{TP}}$. By Lemma 4.13 the two largest labels given to ρ are never added to the bag, and are therefore never given to elements of $\mathrm{T}^{\underline{\mu}}$. Now if the ρ-tag was already placed on π at stage s, then π has the all the labels which had been given to ρ by the start of stage s; ρ may have been given a new largest label at stage s, but whether or not this happened, ρ's second largest label after stage s was one of its labels at the start of stage s. If f_ρ is placed on π subsequent to stage s, then π has the largest label given to ρ. In either case we see that π has one of the two largest labels given to ρ, which is a contradiction. □

It is easily observed, by induction on the point at which sequences are enumerated, that at the end of the step at which any sequence ρ is enumerated into $\mathcal{A}$, all initial segments (other than $\emptyset$) have also been enumerated.

Lemma 4.17. *At any point, if siblings ρ_0 and ρ_1 have both been enumerated into $\mathcal{A}$, then the two largest temporary labels given to ρ_0 have not been given to ρ_1.*

Proof. The proof is by induction on the point of the construction. Let $\rho_0 = \rho^\frown t$ and $\rho_1 = \rho^\frown t'$. First of all, consider the case that ρ is enumerated by some μ' via (C2). By Lemma 4.15, this means that $\rho^\frown t$ and $\rho^\frown t'$ must also be enumerated via (C2) at this step. Suppose that σ' is the longest initial segment of ρ in $R^{\mu'}$ and that $\rho = \sigma'^\frown\tau$. Then μ' enumerates ρ because it sees $\sigma \in \mathrm{T}^{\underline{\mu'-}}$ which is similar to σ' with $\sigma^\frown\tau \in \mathcal{A}$. By Lemma 4.7, by the end of this step, $\sigma^\frown\tau^\frown t$ and $\sigma^\frown\tau^\frown t'$ have both been enumerated into $\mathcal{A}$, and by Lemma 4.15 again, must therefore have already been in $\mathcal{A}$. The induction hypothesis therefore holds for these strings. By Lemma 4.7, μ' gives $\rho^\frown t$ precisely the same labels as $\sigma^\frown\tau^\frown t$ and gives $\rho^\frown t'$ precisely the same labels as $\sigma^\frown\tau^\frown t'$.

We are left to consider the case that ρ is pure and is enumerated by μ', say. If $\rho^\frown t$ and $\rho^\frown t'$ are both pure then the result follows immediately, since at any point the two most recent labels given to either of these sequences have not been added to the bag. Suppose that both of these sequences are broken. Then they must both be enumerated by μ'. In this case the result follows from Sublemma 4.14 by considering the last stage at which μ' enumerates one of the sequences (or both together). We are therefore left to consider the case that one of the sequences, $\rho^\frown t$, say, is broken, while the other is pure. Then at any point, the two most recent labels given to $\rho^\frown t'$ have not been added to the bag, and so have not been given to $\rho^\frown t$. On the other hand, at the stage at which $\rho^\frown t$ is enumerated, or at any subsequent stage at which $\rho^\frown t'$ has its temporary labels rejuvenated, μ' must be visited and the result then follows again from Sublemma 4.14. $\square$

Lemma 4.18. *If ρ and ρ' are similar, pure, have the same origin and last(ρ) =last(ρ'), then at every point s of the construction, $\mathcal{A}_\rho[s] = \mathcal{A}_{\rho'}[s]$.*

Proof. This follows immediately by induction on the point of the construction at which sequences are enumerated, or labels rejuvenated. Let μ' be the common origin of ρ and ρ'. If any sequence $\rho^\frown\tau$ is enumerated into $\mathcal{A}$ by some μ'' via (C1), then μ'' also enumerates into $\rho'^\frown\tau$ at this step and gives it the same labels. Similarly, if μ'' rejuvenates the temporary labels for one of these sequences, then it does so for the other in the same way. If ν enumerates $\rho^\frown\tau$ via (C2) then $\nu \supseteq \mu'$. Let σ be the longest initial segment of τ such that $\rho^\frown\sigma \in R^\nu$ and let $\tau = \sigma^\frown\tau'$. Since $\rho'^\frown\sigma$ is similar to $\rho^\frown\sigma$ and is in R^ν, it follows from Lemma 4.7 that at the step at which $\rho^\frown\tau$ is enumerated, $\rho'^\frown\tau$ is also enumerated and is given the same labels. $\square$

Lemma 4.19. *There are infinitely many μ-expansionary stages.*

Proof. Suppose towards a contradiction, that there is a last μ-expansionary stage s. We concentrate initially on what happens to tags for μ-boundary sequences $\rho \in \mathcal{A}[s]$. We wish to show first that these tags reach a limit value.

Showing that the tags reach a limit value. We have to prove that for each μ-boundary sequence $\rho \in \mathcal{A}[s]$, there is a stage after which f_ρ is always defined and takes the same value π. By Lemma 4.13, the two largest labels given to ρ are never added to the bag of labels, so that subsequent to stage s the tag for ρ can only ever be on π which is $g(\rho')$ for ρ' with $\mathrm{last}(\rho) = \mathrm{last}(\rho')$ (by the argument in the proof of Lemma 4.16, it does not matter that this tag may have been placed on π before stage s); by definition of how we place tags, ρ' must be similar to ρ. Now consider all of the π on which the tag for ρ would be correctly placed, i.e., $\pi \in \mathcal{B}_\ell$ such that $\pi = g(\tau)$ for some $\tau \equiv_\mu \rho$. By Lemma 4.16, if a tag for any ρ' is placed on such π then ρ' must be a μ-boundary sequence. Thus by the above argument, reversing ρ and ρ', we see that $\mathrm{last}(\rho') = \mathrm{last}(\rho)$, and so $\rho' \equiv_\mu \rho$. There are only finitely many $\rho' \equiv_\mu \rho$, k-many, say. This means that there are k-many positions in $\mathcal{B}_\ell$ which are correct for these k-many tags and which, subsequent to stage s, can only be tagged with these k-many tags. The sequences $\rho' \equiv_\mu \rho$ are only given a finite number of labels during the construction, and so there is a stage after which the g-images of all of these sequences together with their labels have appeared in $\mathcal{B}_\ell$. Therefore, after a certain stage, there is always some position $g(\rho')$ with $\rho' \equiv_\mu \rho$ which, by Lemma 4.10, has all the labels given to ρ, and on which we may place the tag for ρ. This suffices to show that there is not a stage after which f_ρ is always undefined. Now suppose that f_ρ becomes undefined infinitely many times. Note that if the tag for ρ is placed on π' and then becomes undefined, it can never be placed on π' again, and that if the tag is correctly placed then, by Lemma 4.16, f_ρ will never subsequently become undefined. Therefore, after some point, a correct position for the tag will be the Gödel least available option, and we will therefore place the tag for ρ correctly. This gives the required contradiction.

Showing that the tags reach an appropriate limit value. So, for each μ-boundary sequence $\rho \in \mathcal{A}[s]$, there is a stage after which f_ρ is always defined and takes the same value π. Furthermore, $\pi = g(\rho')$ for some ρ' with $\mathrm{last}(\rho') = \mathrm{last}(\rho)$ and ρ' similar to ρ. We show, by induction on $|\rho|$, that $\rho' \equiv_\mu \rho$.

Suppose first that ρ' is not a μ-boundary sequence. By Lemma 4.16, $\rho' \notin g(\mathrm{T}^{\underline{\mu}})$, so there exists a unique $\tau \subset \rho'$ which is a μ-boundary sequence. By the inductive hypothesis, for every $\tau' \equiv_\mu \tau$, the limit value of $f_{\tau'}$ is $g(\tau'')$ for some $\tau'' \equiv_\mu \tau$. Since there are only finitely many of these, by the pigeon hole principle there must be some $\tau' \equiv_\mu \tau$ for which the limit value of $f_{\tau'}$ is $g(\tau)$. But when this tag is placed on $g(\tau)$, it will prove that the f_ρ tag is wrongly placed on π, contrary to our assumption that π is the limit value of f_ρ. So it must be that ρ' is a μ-boundary sequence. Since ρ' is similar to ρ and $\mathrm{last}(\rho') = \mathrm{last}(\rho)$, it follows that $\rho' \equiv_\mu \rho$.

Since being an acceptable sequence is a local property, if $\rho^\frown\tau$ is acceptable and $\rho \equiv_\mu \rho'$, then $\rho'^\frown\tau$ is acceptable, and in fact $\rho^\frown\tau \equiv_\mu \rho'^\frown\tau$. By Lemma 4.7, it follows that for any τ, if $\rho^\frown\tau \in \mathcal{A}$, then $\rho'^\frown\tau \in \mathcal{A}$ and has the same labels.

Dealing with non-boundary sequences. So far we have been able to observe that for μ-boundary sequences $\rho \in \mathcal{A}[s]$, f_ρ takes a final value $g(\rho')$ such that $\rho' \equiv_\mu \rho$. Now we wish to show that after stage s, for each ρ of this kind and each τ such that $\rho^\frown\tau \in \mathcal{A}$, subsequent to the point at which $f_\rho = g(\rho')$, the tag for $\rho^\frown\tau$ is only ever placed on $g(\rho'^\frown\tau)$ and is eventually placed here. The proof is by induction on $|\tau|$. Suppose that $|\tau| > 0$ and the result holds for τ^-. We have shown that $\rho'^\frown\tau \in \mathcal{A}$ and has the same labels as $\rho^\frown\tau$. Note that the labels on $\rho^\frown\tau$ are not updated subsequent to stage s. Let $t = \mathrm{last}(\tau)$. By Lemma 4.17 the two largest labels given to $\rho^\frown\tau$ cannot ever be given to any $\rho'^\frown\tau^{-\frown}t'$ for $t' \neq t$. Thus, after stage s, for any t' (with t' possibly equal to t), the tag for $\rho^\frown\tau^{-\frown}t'$ can only be placed on $g(\rho'^\frown\tau^{-\frown}t')$ once the tag for $\rho^\frown\tau^-$ has taken its final position, and will eventually be placed here.

The reader may worry that the tag for $\rho^\frown\tau$ might have been placed by stage s, but the same argument as in the proof of Lemma 4.16 shows that this is not a concern: If the tag for $\rho^\frown\tau$ were placed on $\rho'^\frown\tau'$ by stage s, then since s was expansionary, all labels which $\rho^\frown\tau$ had at stage s are present on $\rho'^\frown\tau'$. At least one of $\rho^\frown\tau$'s two largest labels was present at stage s, so by Lemma 4.17, it must be that $\tau' = \tau$.

All of this suffices to show that we do eventually get a matching, which gives the required contradiction. □

Let m be the longest length of any $\rho \in \mathrm{T}^{\underline{\mu}}$. We outlined at the end of Section 4.3 the basic idea behind the proof that, for each k with $1 \leq k \leq m$, there are infinitely many ${}_k\mathtt{pe}$-stages and that all tags are eventually correctly placed. What follows is essentially the same proof, but modified to deal with the fact that as we do the induction on k, we also have to prove that tags are not placed in $g(\mathrm{T}^{\underline{\mu}})$.

Lemma 4.20. *Let m be the longest length of any $\rho \in \mathrm{T}^{\underline{\mu}}$. For each k with $1 \leq k \leq m$:*

(1) *There are infinitely many ${}_k\mathtt{pe}$-stages.*
(2) *No tag is ever placed on an element of $g(\mathrm{T}^{\underline{\mu}} \restriction k)$.*
(3) *For all μ-boundary sequences ρ with $|\rho| \leq k+1$, the ρ-tag is eventually permanently correctly placed and is never placed in $g(\mathrm{T}^{\underline{\mu}})$, unless possibly* $\mathrm{last}(\rho)$ *is of the form $(\beta, n, (\beta', n'), \mu')$ and the ρ-tag is placed in $g(\mathrm{T}^{\underline{\mu}})$.*

Thus, for all μ-boundary sequences ρ, the ρ-tag is eventually correctly placed (and the possibility allowed in (3) cannot actually occur).

Proof. The proof is by induction on k.

The base case. First we deal with the case $k = 1$. It is clear that no tag is ever placed on the single element of $\mathcal{B}_\ell$ of length 1, since we do not look to place tags for ρ of this length. When any tag for ρ of length 2 is placed it cannot subsequently be moved. So each f_ρ for ρ of length 2 reaches a final value, and there are infinitely many ${}_1\texttt{pe}$-stages. Let the final value f_ρ be $g(\rho')$. First suppose that $\rho \in \mathrm{T}^{\mathcal{TP}}$. By Lemma 4.12, $\mathcal{A}_\rho$ embeds into $\mathcal{A}_{\rho'}$ which means that $\rho' \in \mathrm{T}^{\mathcal{TP}}$, since otherwise ρ' would have only finitely many labels. Now since $\mathcal{A}_\rho$ embeds into $\mathcal{A}_{\rho'}$, $\mathrm{rank}(\rho)$ is a successor and both of ρ and ρ' are in $\mathrm{T}^{\mathcal{TP}}$, it follows from Lemma 3.13 that $\mathrm{term}(\rho) = \mathrm{term}(\rho')$. Next suppose that $\rho \notin \mathrm{T}^{\mathcal{TP}}$. In this case ρ only receives finitely many labels, and so it follows from Lemma 4.17 and the fact that there are infinitely many expansionary stages that $\rho = \rho'$.

Understanding the situation given by the induction hypothesis. Suppose the result holds for all $j < k$ (and that $1 < k \leq m$). So no tag is ever placed in $g(\mathrm{T}^{\underline{\mu}} \upharpoonright j)$ for $j < k$, and all tags for ρ with $\rho^- \in \mathrm{T}^{\underline{\mu}} \upharpoonright j$ such that $j < k$ are eventually placed correctly, except perhaps when $|\rho| = k$, $\mathrm{last}(\rho)$ is of the form $(\beta, n, (\beta', n'), \mu'')$ and the ρ-tag is placed in $g(\mathrm{T}^{\underline{\mu}})$. Suppose the latter possibility occurs. Note that the ρ-tag is not moved once it is placed in $g(\mathrm{T}^{\underline{\mu}})$. Then $\rho \in \mathrm{T}^{\mathcal{TP}}$ since otherwise, by Lemma 4.13, the last two temporary labels given to ρ are never added to the bag of labels, and so are never given to any element of $\mathrm{T}^{\underline{\mu}}$ (again contradicting the fact that there are infinitely many expansionary stages). By Lemmas 4.12 and 3.13 the only possibility is that the tag for ρ is placed on $\pi = g(\sigma)$, for some $\sigma \in \mathrm{T}^{\underline{\mu}}$ which is similar to ρ with $\mathrm{term}(\sigma) = (\beta, n)$.

Suppose $|\rho| = k$ is a boundary sequence, and let $\Lambda = \{\rho' : \rho' \equiv_\mu \rho\}$. Since $|\Lambda|$ is finite, for every $\rho' \in \Lambda$ with the ρ'-tag not (permanently) correctly placed, there is a $\rho'' \in \Lambda$ such that no tag is (permanently) correctly placed on $g(\rho'')$. By the inductive hypothesis, it must be that $f_{\rho'} \in g(\mathrm{T}^{\underline{\mu}})$ and $g(\rho'')$ is not tagged at all (since $g(\rho'') \notin g(\mathrm{T}^{\underline{\mu}})$, the inductive hypothesis tells us that any permanent tag on $g(\rho'')$ is correctly placed). We can therefore fix an injective function h such that for $\sigma \in \mathrm{T}^{\underline{\mu}}$, $h(\sigma) = \sigma$ unless a tag for some ρ is placed on $g(\sigma)$, in which case $h(\sigma) \equiv_\mu \rho$ and no tag is ever placed on $gh(\sigma)$. For any $\sigma \in \mathrm{T}^{\underline{\mu}}$, by the inductive hypothesis no proper initial segment of $gh(\sigma)$ is ever tagged. Then $gh(\mathrm{T}^{\underline{\mu}} \upharpoonright k)$ is the set of all elements of $\mathcal{B}_\ell$ of length k which do not receive a tag on any initial segment. It follows that there will be no difficulty in satisfying clause (iib) in the conditions for a ${}_k\texttt{pe}$-stage. Note also that for any $\sigma \in \mathrm{T}^{\underline{\mu}}$, $h(\sigma) \in \mathrm{T}^{\mathcal{TP}}$: If $h(\sigma) = \sigma$, this is immediate; if $h(\sigma) \neq \sigma$, $h(\sigma) \equiv_\mu \rho$ for some ρ with the ρ-tag misplaced, and by the earlier argument, such ρ are in $\mathrm{T}^{\mathcal{TP}}$. The point of all of this is to establish a clear picture of the situation we are working with as we prove there are infinitely many ${}_k\texttt{pe}$-stages.

Proving (1) for the induction step. Suppose towards a contradiction, that there is a last stage s_0 which is a ${}_j\texttt{pe}$-stage for some $j \geq k$ (note that for $m \geq j \geq k$ the existence of infinitely many ${}_j\texttt{pe}$-stages implies the existence of infinitely many ${}_k\texttt{pe}$-stages). We observed that for each $1 \leq j \leq k$, all the elements in $\mathcal{B}_\ell$ of length j, except for $|\mathrm{T}^{\underline{\mu}} \upharpoonright j|$-many, eventually have a tag permanently placed on some initial segment. Given the induction hypothesis, it therefore suffices to show that for each μ-boundary sequence $\rho \in \mathcal{A}[s_0]$ with $|\rho| = k+1$, the value f_ρ reaches a limit. Suppose that ρ is of this form, and consider the set of all $\tau \equiv_\mu \rho$. We extend h, as given above, to these sequences by defining $h(\tau) = h(\tau^-)^\frown \mathrm{last}(\tau)$. Note that $h(\tau^-)$ is necessarily similar to τ^-. Let Λ be the set of all values $h(\tau)$ such that $\tau \equiv_\mu \rho$. By Lemma 4.10, all sequences $h(\tau)$ belong to $\mathcal{A}$ and have the same finite set of labels as ρ. For no element of $g(\Lambda)$ is any tag ever placed on a proper initial segment. Now if any tag is ever on an element of $g(\Lambda)$ subsequent to stage s_0, then by Lemma 4.17 this must be a tag for some $\tau \equiv_\mu \rho$. Once a stage is reached at which all elements of $g(\Lambda)$ have appeared in $\mathcal{B}_\ell$ together with their finite set of labels, the positions in $g(\Lambda)$ will be available as positions for placing the tags for $\tau \equiv_\mu \rho$. If $\tau \equiv_\mu \rho$ and f_τ does not reach a limit, then it will eventually be placed in $g(\Lambda)$ and never subsequently moved, a contradiction. This proves (1) for the induction step.

Proving (2) for the induction step. This is the fiddliest case. In order to help with readability we shall conform to a certain convention as regards the use of variables as we prove this case. We shall use the variable σ (by which we mean also σ' and σ'') for sequences of length k, and we shall use the variables τ and ρ for sequences of length $k+1$. Note that it suffices to prove this for μ-boundary sequences.

Now suppose σ' is a μ-boundary sequence with $|\sigma'| = k$ and that the σ'-tag is placed on $\pi \in g(\mathrm{T}^{\underline{\mu}})$. As we have discussed previously, it must be the case that $\sigma' \in \mathrm{T}^{\mathcal{TP}}$ and $\mathrm{last}(\sigma')$ is of the form $(\beta, n, (\beta', n'), \mu'')$. Further, π must be $g(\sigma)$ for some $\sigma \in \mathrm{T}^{\underline{\mu}}$ similar to σ' with $\mathrm{term}(\sigma) = (\beta, n)$. Since σ' is a μ-boundary sequence, $\mu'' \supset \mu$. Since $\sigma \in \mathrm{T}^{\underline{\mu}}$, it must be that (β', n', μ'') is a μ''-term. Let ρ be $\sigma^\frown(\beta', n', \mu'')$. We partition the set of all $\rho' \equiv_\mu \rho$ into three parts, depending on which tag (if any) is placed on $g(\rho'^-)$.

Let Λ_0 be the set of $\rho' \equiv_\mu \rho$ such that $g(\rho'^-)$ is tagged with $f_{\sigma''}$ for some σ'' with $\mathrm{last}(\sigma'') = (\beta, n, (\beta', n'), \mu'')$. Note that $\rho \in \Lambda_0$. Let Λ_1 be the set of $\rho' \equiv_\mu \rho$ such that $g(\rho'^-)$ is tagged with $f_{\sigma''}$ for some σ'' with $\mathrm{last}(\sigma'') \neq (\beta, n, (\beta', n'), \mu'')$. Let Λ_2 be the set of $\rho' \equiv_\mu \rho$ which do not have any tag placed on $g(\rho'^-)$. Finally, let Λ_0^- be the set of all ρ'^- for $\rho' \in \Lambda_0$, and let Λ^- be the set of all ρ'^- for $\rho' \equiv_\mu \rho$. We extend h to $\rho' \equiv_\mu \rho$ as before: $h(\rho') = h(\rho'^-)^\frown\mathrm{last}(\rho')$.

Now let s_0 be a ${}_k\mathtt{pe}$-stage large enough that all elements of $gh(\mathrm{T}^{\underline{\mu}} \upharpoonright k)$ (and so also their initial segments) have appeared in $\mathcal{B}_\ell$, and ρ has already been enumerated into $\mathcal{A}$ by stage s_0. We claim that at any ${}_k\mathtt{pe}$-stage $s_1 > s_0$, every ρ'-tag such that $\rho' \equiv_\mu \rho$ must be placed in $gh(\Lambda_1)$ or $gh(\Lambda_2)$. Since $|\Lambda_0| > 0$ and two tags cannot be placed on the same element of $\mathcal{B}_\ell$ at the same time, this gives the required contradiction.

In order to see the claim, first let $s_2 < s_1$ be as in the definition of s_1 being eligible as a ${}_k\mathtt{pe}$-stage. Then with at most $|\mathrm{T}^{\underline{\mu}} \upharpoonright k|$-many exceptions, for any ζ of length k with $g(\zeta) \in \mathcal{B}_\ell[s_2]$, $g(\zeta)$ has a tag placed on an initial segment by stage s_1. Since g and h are injective, and the elements of $gh(\mathrm{T}^{\underline{\mu}} \upharpoonright k)$ never receive such tags, it follows that $g(\zeta)$ has a tag placed on an initial segment by stage s_1 unless $\zeta \in h(\mathrm{T}^{\underline{\mu}} \upharpoonright k)$. Since the elements of $gh(\mathrm{T}^{\underline{\mu}} \upharpoonright k)$ never receive tags on initial segments, if $g(\zeta) \in \mathcal{B}_\ell[s]$ of length k does not have a tag placed on an initial segment at stage s_1, it will never have a tag placed on an initial segment.

Next, suppose that the tag for $\rho' \equiv_\mu \rho$ is placed on $g(\tau)$ at stage s_1. By definition of s_1 being eligible as a ${}_k\mathtt{pe}$-stage, $\tau \in \mathcal{B}_\ell[s_2]$, and thus $\tau^- \in \mathcal{B}_\ell[s_2]$. Since ρ' is a μ-boundary sequence there cannot be a tag placed on an initial segment of $g(\tau^-)$ at stage s_1. It follows that $\tau^- \in h(\mathrm{T}^{\underline{\mu}} \upharpoonright k)$ and $g(\tau)$ never has a tag placed on a proper initial segment, and thus the tag for ρ' will never be moved. Let $\tau^- = h(\sigma''')$. Since h preserves backbones, $\mathrm{bb}(\sigma''') = \mathrm{bb}(\tau^-) = \mathrm{bb}(\rho'^-) = \mathrm{bb}(\sigma)$. Let $\rho'' = \sigma'''^\frown(\beta', n', \mu'')$. Then $\rho'' \equiv_\mu \rho$ and ρ'' demonstrates that $\tau^- \in h(\Lambda^-)$.

Since the ρ' tag is never moved, by Lemmas 3.13 and 4.12 we have $\mathrm{last}(\tau) = \mathrm{last}(\rho')$. Now suppose $\sigma''' \in \Lambda_0^-$. Then by definition of h, $\mathrm{last}(h(\sigma''')) = (\beta, n, (\beta', n'), \mu'')$. But then $\tau = \tau^{--\frown}(\beta, n, (\beta', n'), \mu'')^\frown(\beta', n', \mu'')$, which is not an acceptable sequence. So it must be that $\sigma'' \notin \Lambda_0^-$, and thus $\tau \notin h(\Lambda_0)$, as claimed.

Proving (3) for the induction step. Let s_0 be large enough that $g(\mathrm{T}^{\underline{\mu}}) \subseteq \mathcal{B}_\ell[s_0]$. Suppose that ρ is a μ-boundary sequence with $|\rho| = k+1$. Let $s_1 > s_0$ be a ${}_k\mathtt{pe}$-stage at which ρ has already been enumerated into $\mathcal{A}$. If $\mathrm{last}(\rho)$ is of the form $(\beta, n, (\beta', n'), \mu')$ then let μ'' be the unique initial segment of μ' such that there is a μ''-term (β, n, μ''), let $\rho' = \rho^{-\frown}(\beta, n, \mu'')$ and if $\rho' \notin \mathrm{T}^{\underline{\mu}}$ then suppose we have already shown the result for all $\tau \equiv_\mu \rho'$ and let $s_2 > s_1$ be large enough such that all tags for $\tau \equiv_\mu \rho'$ have been correctly placed by stage s_2. Otherwise let $s_2 = s_1$.

By the same argument as in (2), if f_ρ is placed on $g(\sigma)$, then $\sigma^- \in gh(\mathrm{T}^{\underline{\mu}} \restriction k)$. By (2), $h(\mathrm{T}^{\underline{\mu}} \restriction k) = \mathrm{T}^{\underline{\mu}} \restriction k$, so the tag for ρ must be placed on a one-element extension of an element of $g(\mathrm{T}^{\underline{\mu}})$. If $\rho \notin \mathrm{T}^{\mathcal{TP}}$ then it follows from Lemma 4.17 that it is correctly placed, so suppose $\rho \in \mathrm{T}^{\mathcal{TP}}$. If $\mathrm{last}(\rho)$ is not of the form $(\beta, n, (\beta', n'), \mu')$ then it follows from the fact that the tag will never be moved, and from Lemmas 3.13 and 4.12, that it is correctly placed. If $\mathrm{last}(\rho)$ is of the form $(\beta, n, (\beta', n'), \mu')$ then it follows in the same way that the tag is either correctly placed, or placed on $\pi := g(\tau)$ for some τ with $\tau^- \in \mathrm{T}^{\underline{\mu}}$ and $\mathrm{last}(\tau) = (\beta, n, \mu'')$. Unless $\tau \in \mathrm{T}^{\underline{\mu}}$, however, the latter possibility cannot hold since tags for all $\tau \equiv_\mu \rho'$ have already been placed correctly, so that one of these tags is placed on π. □

Remark 4.21. In the previous Lemma, the proof of the base case relied on an application of Lemma 4.12, which we can only apply once we know there are infinitely many μ-expansionary stages. This is why it is important to have a μ-expansionary outcome of lower priority than the ${}_1$pe-outcomes, i.e. the se-outcomes.

Also, when proving (1) for the induction step it was important that for all $\rho \in \mathcal{A}[s_0]$, the labels for ρ are not rejuvenated before a ${}_k$pe-stage. This is why it is important to different μ-outcomes for ${}_k$pe-stages and ${}_{k-1}$pe-stages.

Lemma 4.22. *The presentations $\mathcal{A}$ and $\mathcal{B}_\ell$ are computably isomorphic.*

Proof. Note first that, for $\rho \in \mathcal{A} \setminus \mathrm{T}^{\underline{\mu}}$, if $\rho \equiv_\mu \rho'$ then $\mathcal{A}_\rho$ is isomorphic to $\mathcal{A}_{\rho'}$. If $\rho \notin \mathrm{T}^{\mathcal{TP}}$ then this follows from Lemma 4.10, and otherwise it follows from Lemma 4.18. The computable isomorphism is constructed as follows. We start with the finite amount of nonuniform information which is g restricted to $\mathrm{T}^{\underline{\mu}}$. For each $\rho \in \mathcal{A} \setminus \mathrm{T}^{\underline{\mu}}$ with $\rho^- \in \mathrm{T}^{\underline{\mu}}$, let $t := \mathrm{last}(\rho)$. Run the construction and wait for a stage at which the ρ-tag is placed on π with $\pi^- \in g(\mathrm{T}^{\underline{\mu}})$, $\pi^- = g(\sigma)$ say. Then map $\sigma^\frown t$ to π, and whenever a tag for $\rho^\frown\tau$ is placed on π' subsequent to this, map $\sigma^\frown t^\frown \tau$ to π'. □

5. Π^1_1-Completeness

It is not hard to see that the index set I_{cc} of the computable categorical structures is Π^1_1: A computable structure $\mathcal{M}_e$ is computably categorical if and only if for all $i \in \omega$ and for all $f \colon \omega \to \omega$, if f is an isomorphism $\mathcal{M}_e \to \mathcal{M}_i$, then there exists a computable isomorphism $\mathcal{M}_e \to \mathcal{M}_i$. Notice that the existence of a computable isomorphism is just Σ^0_3.

In this section, we show that the index set is in fact Π^1_1-hard.

Lemma 5.1. *Fix $\alpha \in \mathcal{O}^* \setminus \mathcal{O}$. Then $\mathcal{A}_\alpha$ is not computably categorical.*

Proof. This is merely a repetition of the proof sketched in Section 3.4. Fix $\mathcal{TP}$-terms t_0 and t_1 with $\mathrm{term}(t_0) = (\beta, n)$ and $\mathrm{term}(t_1) = (\beta, m)$ for a successor $\beta \in \mathcal{O}^* \setminus \mathcal{O}$ and $n \neq m$. Let $\sigma_0 = \langle(\alpha, 0, \emptyset), t_0)\rangle$ and $\sigma_1 = \langle(\alpha, 0, \emptyset), t_1)\rangle$. Let $\mathcal{A}[\sigma_0/\sigma_1] = \big((\mathcal{A} \setminus \mathcal{A}_{\sigma_0}) \times \{0\}\big) \cup \big(\mathcal{A}_{\sigma_1} \times \{1\}\big)$, with the induced relations and also $\big((\alpha, 0, \emptyset), 0\big) E(\sigma_1, 1)$ — so $\mathcal{A}[\sigma_0/\sigma_1]$ is made from $\mathcal{A}$ by replacing $\mathcal{A}_{\sigma_0}$ with a second copy of $\mathcal{A}_{\sigma_1}$. By Lemma 3.13, $\mathcal{A}_{\sigma_0} \cong \mathcal{A}_{\sigma_1}$, so $\mathcal{A} \cong \mathcal{A}[\sigma_0/\sigma_1]$.

However, suppose $f : \mathcal{A} \to \mathcal{A}[\sigma_0/\sigma_1]$ were an isomorphism, and consider $f(\sigma_0)$. Since f must preserve the edge relation, height labels and temporary labels, either $f(\sigma_0) = (\sigma_1, 1)$ or $f(\sigma_0) = (\sigma', 0)$ for some $\sigma' \in \mathrm{T}^{\mathcal{TP}}$ with $|\sigma'| = 2$ and $\mathrm{term}(\sigma') = (\beta, n')$ for some $n' \neq n$. So f would compute an isomorphism between $\mathcal{A}_{\sigma_0}$ and either $\mathcal{A}_{\sigma_1}$ or $\mathcal{A}_{\sigma'}$, as appropriate. By Lemma 3.13, f cannot be computable (indeed, cannot be hyperarithmetical). □

Theorem 5.2. *The index set I_{cc} is Π^1_1-complete.*

Proof. Fixing a Π^1_1-set S, we need to build a computable sequence of structures $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ such that $\mathcal{C}_n$ is computably categorical if and only if $n \in S$. By [FS62], there is a computable sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of limit elements of $\mathcal{O}^*$ such that $\alpha_n \in \mathcal{O}$ if and only if $n \in S$. Let $\mathcal{C}_n := \mathcal{A}_{\alpha_n}$. Since our construction was uniform in α, $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ is a computable sequence.

By Lemmas 4.22 and 5.1, $\mathcal{C}_n$ is computably categorical if and only if $n \in S$, so this sequence suffices. □

6. Remarks and Open Questions

Despite having established Theorem 1 and Theorem 2, many important questions remain. For example, the latter requires a different computably categorical structure $\mathcal{A}_\alpha$ for varying α. It is natural to ask whether there is a computably categorical structure that is not relatively Δ^0_α-categorical for any computable ordinal α. This is known to be equivalent to the failure of relative hyperarithmetic categoricity.

Question 6.1. Is there a computably categorical structure that is not relatively hyperarithmetically categorical?

We note that our structure $\mathcal{A}_\alpha$ is easily seen to be relatively Δ^0_α-categorical.

Soskov [Sos96] showed that an external relation R on a computable structure $\mathcal{S}$ is relatively intrinsically hyperarithmetical (i.e. $R^{\mathcal{A}}$ is hyperarithmetic on the diagram of $\mathcal{A}$ for every copy $\mathcal{A}$ of $\mathcal{S}$) if and only if it is intrinsically hyperarithmetical (i.e. $R^{\mathcal{A}}$ is hyperarithmetic for every computable copy $\mathcal{A}$ of $\mathcal{S}$). Our work yields and interesting corollary that is in stark contrast with Soskov's result.

Corollary 6.2. *For every computable ordinal α, there is a intrinsically computable relation R on a computable structure $\mathcal{S}$ that is not relatively intrinsically Δ^0_α.*

Proof. Fixing α, let $\mathcal{S}$ be the $\mathcal{L}_\alpha \cup \{\mathsf{C}_1, \mathsf{C}_2\}$-structure consisting of two disjoint copies of $\mathcal{A}_\alpha$, where the unary relation C_1 holds of one disjoint copy, and the unary relation C_2 holds of the other disjoint copy. Let R be the binary relation holding of the graph of the (unique) isomorphism between one disjoint copy and the other disjoint copy.

Then R is intrinsically computable as $\mathcal{S}$ was computably categorical. As $\mathcal{S}$ was not relatively Δ^0_α-categorical, there is a presentation $\mathcal{B}$ of $\mathcal{S}$ not isomorphic to an (arbitrary) fixed computable presentation $\mathcal{A}$ of $\mathcal{S}$ via any $\Delta^0_\alpha(\mathcal{B})$-computable isomorphism. Then the relation R is not $\Delta^0_\alpha(\mathcal{B})$-computable for the presentation of $\mathcal{S}$ consisting of the disjoint copies $\mathcal{A}$ and $\mathcal{B}$. Hence, the relation R is not relatively intrinsically Δ^0_α-computable. □

We end with the following

Question 6.3. If any two hyperarithmetic presentations $\mathcal{A}$ and $\mathcal{B}$ of a computable structure $\mathcal{S}$ are hyperarithmetically isomorphic, is $\mathcal{S}$ relatively hyperarithmetically categorical?

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