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EMBEDDINGS INTO THE TURING DEGREES.

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1. Introduction

The structure of the Turing degrees was introduced by Kleene and Post in 1954 [KP54]. Since then, its study has been central in the area of Computability Theory. One approach for analyzing the shape of this structure has been looking at the structures that can be embedded into it. In this paper we do a survey of this type of results.

The Turing degree structure is a very natural object; it was defined with the intention of abstracting the properties of the relation "computable from", which is the most important notion in computability theory introduced by Turing in [Tur39]. It is defined as follows. Consider $\mathcal{P}(\omega)$, the set of sets of natural numbers. Given $A, B \in \mathcal{P}(\omega)$, we say that A is *computable from* B, if there is a computer program which, on input $n \in \omega$, decides whether $n \in A$ or not using B as an *oracle*. That means that the program is allowed to ask questions to the oracle of the form "does m belong to B?" We write $A \leqslant_T B$ if A is computable from B. The relation $\leqslant_T$ is a quasi-ordering on $\mathcal{P}(\omega)$. This quasi-ordering induces an equivalence relation on $\mathcal{P}(\omega)$, given by

$$A \equiv_T B \quad \Leftrightarrow \quad A \leqslant_T B \ \& \ B \leqslant_T A,$$

and a partial ordering on the equivalence classes. The equivalence classes are called *Turing degrees*. (The concept of Turing degree was introduced by Post [Pos44].) We use $\langle \mathbf{D}, \leqslant_T \rangle$ to denote this partial ordering. One of the main goals of Computability Theory is to understand the structure of $\langle \mathbf{D}, \leqslant_T \rangle$.

There are two basic but important remarks to make here. First, when we talk about a computer program, we are fixing a programming language, say for example the language of Turing machines, or Java. The notion of computability is independent of the programming language chosen. Second, we note that we chose to work with subsets of ω because every finite object can be encoded by a single number (using, for instance, the binary representation of the number). For example, strings, graphs, trees, simplicial complexes, group presentations, etc., if they are finite, they can be effectively coded by a natural number. There would be no essential difference if we had chosen to work with subsets of $\mathbb{Z}$, $2^{<\omega}$ or $V(\omega)$, instead of ω.

Before we go into the embeddability results, we will start by mentioning basic facts about the structure of the Turing Degrees. The embeddability results are divided in four sections: embeddings of countable structures, initial segments embeddings, embeddings

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of larger structures, and embedding into the high/low hierarchy. Embeddability results are very closely related to decidability results, so we dedicate our last section to them.

No knowledge of Computability Theory is assumed. Basic references on the topic are [Ler83] and [Soa87]. Two nice surveys have been recently written. One is by Ambos-Spies and Fejer [ASF], where they describe the history of the Turing Degrees. The other one, by Shore [Sho06], describes the current situation of this research program, and also looks at its history and possible future directions. Our paper has something of both of those papers, but it concentrates just on embeddability results, and is mostly about the global structure. We will not mention results about other reducibilities, even though many have been considered and studied.

2. Background

2.1. First observations. Let us start by making the most basic observations about the structure of the Turing degrees.

There is a least Turing degree that we denote by $\mathbf{0}$. It is the degree whose members are the computable sets.

Every degree has at most countably many degrees below it. We call this property, the *countable predecessor property* or *c.p.p.* The reason is that there are only countably many programs one can write, so there are at most countably many sets that are computable from a fixed set. It also follows that each Turing degree contains at most countably many sets.

There are $2^{\aleph_0}$ many Turing degrees. Because there are $2^{\aleph_0}$ many subsets of ω and each equivalence class is countable.

The Turing degrees form an *upper semilattice*, or *usl*; that is, every pair of elements has a least upper bound. We denote the least upper bound of $\mathbf{a}$ and $\mathbf{b}$ by $\mathbf{a} \vee \mathbf{b}$, and we refer to it as the *join* of $\mathbf{a}$ and $\mathbf{b}$. Given $A, B \in \mathcal{P}(\omega)$, let

$$A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\}.$$

Is not hard to note that $A \leqslant_T A \oplus B$, $B \leqslant_T A \oplus B$, and that if both $A \leqslant_T C$ and $B \leqslant_T C$, then $A \oplus B \leqslant_T C$. We let $\mathbf{a} \vee \mathbf{b}$ be the degree of $A \oplus B$, where A and B are sets in $\mathbf{a}$ and $\mathbf{b}$ respectively.

2.2. Turing Jump. There is another naturally defined operation on the Turing degrees called the *Turing jump* (or just *jump*). The jump of a degree $\mathbf{a}$, denoted $\mathbf{a}'$, is given by the degree of the *Halting Problem* relativized to some set in $\mathbf{a}$. Given $A \subseteq \mathcal{P}(\omega)$, we define A', the *Halting Problem relative to A*, as follows.

A' is the set of codes for programs that, when run with oracle A, halt.

Note that a computer program is a finite sequence of characters and hence can be encoded by a natural number. It can be shown that the jump operation is strictly increasing and monotonic. That is, for every $\mathbf{a}, \mathbf{b} \in \mathbf{D}$,

(1) $\mathbf{a} <_T \mathbf{a}'$, and
(2) $\mathbf{a} \leqslant_T \mathbf{b} \Rightarrow \mathbf{a}' \leqslant_T \mathbf{b}'$.

The only non-trivial fact here is that $A' \not\leqslant_T A$, and it is proved the same way one proves that the Halting problem is not computable.

Definition 2.1. A *jump upper semilattice* is a structure

$$\mathcal{J} = \langle J, \leqslant_{\mathcal{J}}, \cup, \mathfrak{j} \rangle$$

such that

- $\langle J, \leqslant_{\mathcal{J}} \rangle$ is a partial ordering,
- for all $x, y \in J$, $x \cup y$ it is the least upper bound of x and y, and
- $\mathfrak{j}(\cdot)$ is a unary operation such that for all $x, y \in J$, $x <_{\mathcal{J}} \mathfrak{j}(x)$; and if $x \leqslant_{\mathcal{J}} y$, then $\mathfrak{j}(x) \leqslant_{\mathcal{J}} \mathfrak{j}(y)$.

2.3. **The picture.** We have observed so far that $\mathcal{D} = \langle \mathbf{D}, \leqslant_T, \vee, ' \rangle$ is a jump upper semilattice of size $2^{\aleph_0}$, with a least element called $\mathbf{0}$, and with the countable predecessor property.

The next natural question is whether $\mathcal{D}$ is a lattice. The answer is no. Kleene and Post [KP54] proved that there exists degrees $\mathbf{a}$ and $\mathbf{b}$ with no greatest lower bound. There are aslo pairs of incomparable degrees which do have greatest lower bounds.

The only particular degree we have mentioned so far is $\mathbf{0}$. We have also mentioned the Halting problem, which has degree $\mathbf{0}'$. The structure of degrees below $\mathbf{0}'$, that we denote by $\mathcal{D}(\leqslant_T \mathbf{0}')$, is already very rich. For instance, all the computable enumerable sets are computable from $0'$. A set is *computable enumerable, or c.e.*, if there is a computer program that lists all its elements. The study of the structure of the c.e. degrees is also a topic where extensive research has been done.

$\mathbf{0}'$ is very low down inside the whole structure of the Turing degrees. We can start going up and construct a sequence of degrees $\mathbf{0} <_T \mathbf{0}' <_T \mathbf{0}'' <_T$ .... This way we get all the way up the arithmetic hierarchy: It is not hard to show that the sets that are Turing below $0^{(n)}$ for some $n \in \omega$ are exactly the *arithmetic* ones, that is, the ones that can be defined by a formula of first order arithmetic. (We use $X^{(n)}$ to denote the nth iteration of the Turing jump.) Then, we can take the uniform join of all these sets and get $0^{(\omega)} = \{\langle n, m \rangle : m \in 0^{(n)}\}$, which is Turing equivalent to the set of sentences true in first order arithmetic. We can then continue taking jumps and define $0^{(\omega+1)} = 0^{(\omega)\prime}$, and even define $0^{(\alpha)}$ for any countable computable ordinal α by taking uniform joins at limit levels. The situation when α is a non-computable ordinal is a bit more delicate. A *computable ordinal* is one which can be presented as a computable ordering of a computable subset of the natural numbers. We use ω_1^{CK} to denote the first ordinal which does not have a computable presentation. A set which is computable in $0^{(\alpha)}$ for some $\alpha < \omega_1^{CK}$ is said to be *hyperarithmetic*. These are exactly the Δ^1_1 sets (Kleene and Suslin [Kle55]). Higher up comes Kleene's O, the set of computable indices (i.e. programs) of computable well-orderings. Kleene's O is Π^1_1-complete and computes all the hyperarithmetic sets. We could then take Kleene's O relative to Kleene's O, and so on. Much higher up is the set of true sentences of second order arithmetic, and there are still many more degrees higher up. Whenever we have a countable set of degrees, there exists a degree that bounds them all.

So far, our picture looks thin and tall. But actually, $\mathcal{D}$ not taller than it is wide. Since $\mathcal{D}$ has the countable predecessor property, every chain in $\mathcal{D}$ can have size at most $\aleph_1$. However, it is known that there is an antichain that contains $2^{\aleph_0}$ minimal degrees (Lacombe [Lac54]). A degree $\mathbf{a} >_T \mathbf{0}$ is *minimal* if there is no degree $\mathbf{x}$, with $\mathbf{0} <_T \mathbf{x} <_T \mathbf{a}$. (The existence of minimal degrees is due to Spector [Spe56].)

3. Embeddings of countable structures

We now start analyzing the structures that embed into $\mathcal{D}$.

3.1. Upper semilattices. The first result in this direction was proved by Kleene and Post [KP54] in the same paper where they introduced the Turing degrees. They showed that there is an infinite independent set of degrees, that is, a set of degrees none of which can be computed from the other ones altogether. They prove it using the method of finite approximations. Today we would refer to such a construction as a forcing construction. The ideas in [KP54] can be easily extended to get the following result.

Theorem 3.1 (Kleene and Post)**.** *Every countable upper semilattice embeds into the Turing degrees.*

Proof sketch: It is enough to show that the countable atomless Boolean algebra embeds into $\mathcal{D}$ since every countable upper semilattice embeds into it. Let $G \subseteq \omega$ be sufficiently generic. In other words, G meets the countably many dense open sets considered for the proof. Via a computable bijection between ω and the set of rational numbers $\mathbb{Q}$, think of G as a subset of $\mathbb{Q}$. It is well know that the countable atomless Boolean algebra is isomorphic to $\mathrm{Int}(\mathbb{Q})$, the interval algebra of $\mathbb{Q}$, that is, the algebra whose elements are the finite unions of closed-open intervals of $\mathbb{Q}$. Now, define $h\colon \mathrm{Int}(\mathbb{Q}) \to \mathcal{P}(\mathbb{Q})$ by $h(I) = G \cap I$. The proof that h preserves $\leqslant_T$, $\vee$ and 0 does not use the genericity of G and is quite simple. The genericity of G is used to show that h preserves $\not\leqslant_T$. It can also be used to show that h preserves greatest lower bounds. □

It also follows from the proof above that every countable distributive lattice can be embedded into $\mathcal{D}$, even preserving greatest lower bounds, since they can be embedded into the atomless Boolean algebra.

The fact that every countable lattice can be embedded into $\mathcal{D}$ preserving greatest lower bounds follows from a much stronger result of Lachlan and Lebeuf (see Theorem 4.1 below).

3.2. Local structures. A local structure is one of the form $\mathbf{D}(\leqslant_T \mathbf{a}) = \{\mathbf{x} \in \mathbf{D} : \mathbf{x} \leqslant_T \mathbf{a}\}$. There has been a lot of research done on local structures, and we will just quickly refer to some of it.

The first approach here is usually of the following sort. Theorem 3.1 says that every usl can be embedded into $\mathcal{D}$. Of course, the construction of this embedding cannot be computable, although, if we had an oracle smart enough, we could produce this embedding computably in the oracle. The question is how complex this oracle has to be. In the case of Theorem 3.1, a good answer is $0'$. A better answer is 1-generic. A 1-generic set is like a Cohen generic set, but it only needs to meet a small class of dense open sets: An infinite binary sequence $G \in 2^\omega$, is *1-generic* if for every Σ^0_1 set $S \subseteq 2^{<\omega}$ there exists a string $\sigma \subseteq G$ such that either $\sigma \in S$ or $\forall \tau \in 2^{<\omega}$ $(\tau \supseteq \sigma \Rightarrow \tau \notin S)$. (We are abusing notation and identifying 2^ω and $\mathcal{P}(\omega)$.) It is not hard to show that $\mathbf{0}'$ is able to compute a 1-generic set. (Moreover, any computable enumerable set or non-GL_2 set can compute a 1-generic.) So we get that every countable usl, and also every distributive lattice, embeds in $\mathbf{D}(\leqslant_T \mathbf{0}')$. Relativizing, one can get the whole embedding between $\mathbf{a}$ and $\mathbf{a}'$ for any $\mathbf{a} \in \mathbf{D}$.

For lattices in general it is not possible to get such a result. The reason is that there are $2^{\aleph_0}$ many lattices with finitely many generators [Sho82], but there are only countably many possibilities for those generators below $\mathbf{0}'$. However, Lerman [Ler83] proved that every computable presentable lattice embeds in $\mathbf{D}(\leqslant_T \mathbf{0}')$. Actually he proved this for $0''$-computable lattices and embedded them even as initial segments below $\mathbf{0}'$. Moreover, if $\mathbf{a}$ bounds a 1-generic degree, then every computable lattice embeds in $\mathbf{D}(\leqslant_T \mathbf{a})$ (Shore [Sho82]). This is not true if we also want to preserve top element. This follows from Kumabe's [Kum00] construction of a strong minimal cover of a 1-generic. However, it is true for $\mathbf{a} = \mathbf{0}'$, as it was proved by Fejer [Fej89]. Moreover, it is known that every computable lattice embeds in $\mathbf{D}(\leqslant_T \mathbf{a})$ preserving top element if $\mathbf{a}$ is non-GL_2 [Fej89], even array non-recursive (Downey, Jockusch and Stob [DJS96]) or if it is 1-generic (Greenberg and Montalbán [GM03]).

Since we are here, we should mention that the question of which lattices embed into the structure of the c.e. degrees is open, and a lot of effort has been put into it. (For a survey on this topic see Lempp, Lerman and Solomon [LLS06].)

3.3. **Jump Partial orderings.** If we forget about joins but add jump to the language, we get the the following type of structure.

Definition 3.2. A *jump partial ordering*, or *jpo*, is a structure

$$\mathcal{P} = \langle P, \leqslant_P, \mathfrak{j} \rangle$$

such that

- $\langle P, \leqslant_P \rangle$ is a partial ordering, and
- $\mathfrak{j}$ is a unary operation such that for all $x, y \in P$, $x <_P \mathfrak{j}(x)$; and if $x \leqslant_P y$, then $\mathfrak{j}(x) \leqslant_P \mathfrak{j}(y)$.

A *jump partial ordering with 0*, or *jpo w/0*, is a structure $\mathcal{P} = \langle P, \leqslant_P, \mathfrak{j}, 0 \rangle$, where $\mathcal{P} = \langle P, \leqslant_P, \mathfrak{j} \rangle$ is a jump partial ordering and 0 is the least element.

As we mentioned in Section 2.3, if $\langle P, \leqslant_P \rangle$ is a well ordering and the jump function corresponds to the successor function on $\langle P, \leqslant_P \rangle$, then $\mathcal{P}$ can be embedded into $\mathcal{D}$. Such an embedding is called a *jump hierarchy*. Even if $\langle P, \leqslant_P \rangle \cong \omega_1^{CK}(1 + \eta)$, we get the embeddability result, where ω_1^{CK} is the least non-computable ordinal and η is the order type of the rationals. Such an embedding is a *Harrison pseudo-hierarchy* [Har68].

If we let $\langle P, \leqslant_P \rangle \cong \mathbb{Z}$, the ordering of the integers, and we let $\mathfrak{j}(n) = n + 1$, the fact that $\mathcal{P}$ embeds into $\mathcal{D}$ follows from Harrison's pseudo-hierarchy theorem [Har68] and Friedberg's jump inversion theorem [Fri57]. Such an embedding has to be high up in $\mathcal{D}$; it can be proved that every degree in the image of such an embedding has to compute all the hyperarithmetic sets (Enderton and Putnam [EP70]). A curiosity, is that if we want to get an embedding $h \colon \mathbb{Z} \to \mathcal{P}(\omega)$ such that $h(n)' = h(n+1)$, (where equality here is as sets, not only as Turing degrees,) we cannot (Steel [Ste75]).

The most general theorem in this setting is the following one.

Theorem 3.3 (Hinman and Slaman [HS91])**.** *Every countable jump partial ordering can be embedded into the Turing Degrees (of course, preserving order and jump).*

The proof is via a complicated forcing construction. Much more that 1-genericity is needed in this case. One needs to consider sets that are arithmetically generic over a Harrison pseudo-hierarchy.

3.4. **Jump Partial orderings with 0.** If we add 0 to the language the problem becomes much more complicated, and very different techniques are required. The reason is that the constructions before used sets which are very generic and very far from arithmetically definable. But now, if for example we have that $x \leqslant_P \mathfrak{j}^n(0)$, then we need to map x to a degree below $\mathbf{0}^{(n)}$, and hence to a set which is arithmetically definable, with no more than $n+1$ quantifiers.

Hinman and Slaman [HS91] started to look at the quantifier-free 1-types of jump partial orderings with 0 realizable in $\mathcal{D}$. Note that realizing a quantifier-free n-type is equivalent to embedding a jpo w/0 and with n many generators. They got some partial results, that were rounded off later by Hinman in [Hin99]. He showed that every quantifier-free 1-type $p(x)$ of jump partial orderings with 0, and with a formula of the form $x \leqslant_P \mathfrak{j}^m(0)$, is realizable in $\mathcal{D}$. Then, Montalbán [Mon03], showed the same for 1-types $p(x)$ with no formula of the form $x \leqslant_P \mathfrak{j}^m(0)$. Putting these results together we get the following one.

Theorem 3.4 (Hinman [Hin99], Montalbán [Mon03])**.** *Every quantifier-free 1-type of jump partial orderings with 0, is realizable in $\mathcal{D}$.*

The following question remainds open.

Question 3.5. Which quantifier-free n-type of jpo w/0 can be realized in $\mathcal{D}$?

Here is one of the difficulties to solve this question. The main tool used in Hinman's result about 1-types is the Shoenfield [Sho59] jump inversion theorem: If A is c.e.a. B', there there is a set C, c.e.a. B such that $C' \equiv_T A$. (By Y c.e.a. X we mean that Y computable enumerable in X and is Turing above X.) For n-types, we do not have such an inversion theorem. Worst than that, we have Shore's non-inversion theorem [Sho88]: There are sets B_0, B_1 and B_2 c.e. over $0'$ with $0' <_T B_0, B_1 \leqslant_T B_2$, for which there are no sets $A_0, A_1 \leqslant_T A_2 \leqslant_T 0'$ with $A_i' \equiv_T B_i$.

On the positive side, Montalbán [Mon03] showed that if every quantifier free n-type $p(x_1, ..., x_n)$ of jpo w/0, which contains a formula $x_1 \leqslant \mathfrak{j}^m(0)$ & ... & $x_n \leqslant \mathfrak{j}^m(0)$ for some m, is realized in $\mathcal{D}$, then every quantifier free n-type of jpo w/0 is realized in $\mathcal{D}$.

A very nice result is the following one. In [LL96], Lempp and Lerman used their method of Iterated Trees of Strategies and showed that every formula $\varphi(x_1, ..., x_n)$ consistent with the axioms jpo w/0 plus $x_1 \leqslant \mathfrak{j}(0)$ & ... & $x_n \leqslant \mathfrak{j}(0)$ is realizable in $\mathcal{D}$, getting some interesting decidability results as corollaries.

3.5. **Jump upper semilattices.** The reader might be wondering by now what happens if we have both join and jump. We get the following extension of Hinman and Slaman's theorem.

Theorem 3.6 (Montalbán [Mon03])**.** *Every countable jump upper semilattice can be embedded into the Turing Degrees (of course, preserving jump and join).*

The proof uses ideas from Hinman and Slaman [HS91], but it also needs a array of new ideas.

Outline of the proof: The proof has two main steps. First, we introduce the notion of h-embeddable jusl. We say that a jusl $\mathcal{J}$ is *h-embeddable* ('h' for hierarchy) if there is a map $H\colon J \to \mathcal{P}(\omega)$ such that for all $x, y \in J$,

- if $x <_{\mathcal{J}} y$ then $H(x)' \leqslant_T H(y)$,
- $\mathcal{J} \leqslant_T H(y)$, and $\bigoplus_{x \leqslant_{\mathcal{J}} y} H(x) \leqslant_T H(y)$.

We call such a map H, a *jump hierarchy.*

Via a forcing construction, we get that for every h-embeddable jusl $\mathcal{J}$, there is an embedding $f\colon \mathcal{J} \to \mathcal{D}$. Essentially, the forcing notion has to make sure that $x \leqslant_{\mathcal{J}} y \Rightarrow f(x) \leqslant_T f(y)$; $f(x \vee y) \equiv_T f(x) \vee f(y)$ and that $f(\mathfrak{j}(x)) \leqslant_T (f(x))'$. Genericity is used to ensure that $x \nleqslant_{\mathcal{J}} y \Rightarrow f(x) \nleqslant_T f(y)$ and that $(f(x))' \leqslant_T f(\mathfrak{j}(x))$. The jump hierarchy is used for this last reduction, $(f(x))' \leqslant_T f(\mathfrak{j}(x))$. The point is to have that for every x, $f(x) \geqslant_T H(x)$, and use $H(\mathfrak{j}(x))$ to decode $(f(x))'$ from $f(\mathfrak{j}(x))$. There are many subtleties one has to worry about here.

Now we can embed a big family of jusl's. For instance, every well founded jusl is h-embeddable: If $\mathcal{J}$ and the rank function on it are computable in X, take $H(x) = X^{\mathrm{rk}(x)}$. However, there is no reason to believe that every jusl is h-embeddable.

The second step is to prove that every jusl embeds into an h-embeddable one. This part of the proof is more algebraic and uses Harrison linear orderings, Fraïssé limits and well-quasi-orderings. □

As we mentioned right after Definition 3.2, even for simple jpo's such as $\mathbb{Z}$, these embeddings cannot be done inside the hyperarithmetic degrees. For the proof above, again one needs to consider a Harrison pseudo-hierarchy and a set arithmetically generic over it. These sets can be found below Kleene's O, and even hyperarithmetically-low. So we get that every computable jusl embeds in $\mathcal{D}(\leqslant_T \text{Kleene's } O)$.

3.6. **Jump upper semilattices with 0.** The situation when we add 0 to the language is again very different. In this case we get a negative answer right away.

Theorem 3.7 (Montalbán [Mon03])**.** *Not every countable jusl w/0 can be embedded into $\mathcal{D}$. Indeed, there is a jusl w/0 and with only one generator (other than 0) which cannot be embedded in $\mathcal{D}$.*

Idea of the proof: The reason is that there are $2^{\aleph_0}$ many jusl w/0 with one generator x satisfying $x \leqslant_{\mathcal{J}} \mathfrak{j}^2(0)$. But there are only countably many degrees $\mathbf{x} \leqslant_T \mathbf{0}''$. □

Question 3.8. Is there a simple (say computable) jusl w/0 and with one generator that cannot be embedded into $\mathcal{D}$?

If we do not require the jump operation to be total, it makes sense to talk about finite jusl's. The problem of whether every finite jusl w/0 can be embedded into $\mathcal{D}$ is still open. It is believed that a positive answer could be achieved using Lempp and Lerman's method of Iterated Trees of Strategies (see for instance [LL96]). This method gives a general framework to do $0^{(n)}$-priority arguments and is very complicated. [1]

[1] A positive answer to this question has been recently claimed by Lerman. He is now in the process of circulating a 39 page manuscript called *The Existential Theory of the Uppersemilattice of Turing Degrees with Least Element and Jump is Decidable.*

4. Initial segment embeddings

A completely different family of embeddability results are initial segment embeddings.

There is a long history of results in this area. We mention only some of them. Hugill [Hug69] showed that every countable linear ordering embeds into $\mathcal{D}$ as an initial segment. In [Lac68], Lachlan proved that every countable distributive lattice is isomorphic to an initial segment of $\mathcal{D}$. Then, in [Ler71], Lerman showed the same for every finite usl. A complete characterization of the countable initial segments of $\mathcal{D}$ was later given Lachlan and Lebeuf.

Theorem 4.1 (Lachlan and Lebeuf [LL76])**.** *Every countable upper semilattice with least element is isomorphic to an initial segment of* $\mathcal{D}$

These embeddings can be done quite locally, as long as the usl is not too complex. Lerman [Ler83, XII] showed that every countable usl w/0 that is computable in $0''$ is isomorphic to an initial segment of $\mathcal{D}$ below $\mathbf{0}'$. This result was later extended by Kjos-Hanssen [KH03], who showed that a countable usl w/0 is isomorphic to an initial segment of $\mathcal{D}$ below $\mathbf{0}'$ if and only if it has a presentation c.e. in $0''$.

The methods used for this kind of results are forcing with computable perfect trees and lattice tables. Forcing with computable perfect trees, or Sacks forcing, was already used in the first construction of a minimal degree by Spector [Spe56], as noticed by Sacks [Sac71]. A more complex class of trees is necessary to get other initial segments results. Lerman's book [Ler83] contains all these embeddability results.

5. Embeddings of larger structures

Now we look at uncountable structures. Recall that $\mathcal{D}$ has the countable predecessor property (c.p.p.), and hence any subordering of it has to have it too.

5.1. **Partial Orderings.** The first result of this sort is due to Sacks and the key step of his proof is the following extensions-of-embeddings lemma. The finite version of this lemma is due to Kleene and Post [KP54].

Lemma 5.1 (Sacks [Sac61])**.** *Let $\mathcal{P} \subseteq \mathcal{Q}$ be two countable partial orderings such that $\mathcal{P}$ is downward closed in $\mathcal{Q}$ and for every $q \in Q$ we have that every two elements of P below q have an upper bound in P also below q. Then any embedding of $\mathcal{P}$ into $\mathcal{D}$ extends to an embedding of $\mathcal{Q}$ into $\mathcal{D}$.*

Sacks actually proved a slightly stronger lemma where $|\mathcal{P}| < 2^{\aleph_0}$, $|\mathcal{Q} \smallsetminus \mathcal{P}| \leqslant \aleph_0$, and $\mathcal{Q}$ has the c.p.p.

Theorem 5.2 (Sacks [Sac61])**.** *Every partial ordering of size $\aleph_1$ with the c.p.p. can be embedded into $\mathcal{D}$.*

Proof: First extend the partial ordering to an usl, also with the c.p.p., and then decompose it as an increasing union of countable partial orderings so that we can apply the lemma above. □

He also showed that there is a maximal independent set of degrees size $2^{\aleph_0}$. That is, a set $\{\mathbf{x}_\xi : \xi < 2^{\aleph_0}\}$ such that for every $\xi_0, ..., \xi_k \in 2^{\aleph_0}$, $\mathbf{x}_{\xi_0} \not\leqslant_T (\mathbf{x}_{\xi_1} \vee ... \vee \mathbf{x}_{\xi_k})$ (unless, of course, $\xi_0 = \xi_i$ for some $i = 1, .., k$). It followed that every partial ordering with the *finite predecessor property* and size $2_0^{\aleph}$ embeds into $\mathcal{D}$.

He made the following conjecture which is still unsolved.

Conjecture 5.3 (Sacks [Sac63])**.** Every partial ordering of size $2^{\aleph_0}$ with the c.p.p. embeds into $\mathcal{D}$.

Of course, the affirmative answer is consistent with ZFC, as it is implied by the theorem above if $2^{\aleph_0} = \aleph_1$. However, the lemma used to show Theorem 5.2 cannot be extended to higher cardinalities in ZFC. Groszek and Slaman [GS83] showed that it is consistent with ZFC that $2^{\aleph_0} \geqslant \aleph_2$ and there is an independent set of size $\aleph_2$ which cannot be extended to a larger independent set. In contrast, Simpson [Sim77] pointed out that if Martin's Axiom holds at κ, then there is no maximal independent set of size κ. (See, for instance, [Kun80] for information on Martin's Axiom.)

5.2. **Upper semilattices.** Sacks theorem was improved by Abraham and Shore the following way.

Theorem 5.4 (Abraham and Shore [AS86])**.** *Every usl of size $\aleph_1$ with the c.p.p. and with 0 is isomorphic to an initial segment of $\mathcal{D}$.*

With respect to usl embeddings, this is as far we can go in ZFC. Slaman and Groszek [GS83] show that there is a model of ZFC where $2^{\aleph_0} \geqslant \aleph_2$, and there is an usl of size $\aleph_2$ with the c.p.p. which does not embed into $\mathcal{D}$ preserving joins.

5.3. **Jump partial orderings.** If we add jump to the language, we get the following negative result.

Theorem 5.5 (Montalbán [Mon03])**.** *There is a jpo of size $2^{\aleph_0}$ and with the c.p.p. which cannot be embedded into $\mathcal{D}$.*

Montalbán [Mon03] also showed that if Martin's Axiom holds at κ, then every jusl with the c.p.p. and size $\leqslant \kappa$ can be embedded into $\mathcal{D}$. As a corollary we get that whether every jpo (or jusl) with the c.p.p. and size $\aleph_1$ is embeddable into $\mathcal{D}$ or not is independent of ZFC. It is false if $\aleph_1 = 2^{\aleph_0}$ and true if Martin's Axiom holds at $\aleph_1$.

6. GH-embeddings

There are other very meaningful predicates on the Turing degrees that are defined in terms of $\leqslant_T$, $\vee$ and $'$. To understand these predicates better, we now look a embeddings of structures which preserve them.

6.1. **The Low/High hierarchy.** The degrees below $\mathbf{0}'$ are classified depending on how close they are to being computable or how close they are to being complete (i.e. to compute $\mathbf{0}'$) via the Low/High hierarchy. This classification has been extremely useful in the study of the degrees below $\mathbf{0}'$. We say that a degree $\mathbf{a} \leqslant_T \mathbf{0}'$ is *low*, if its jump is as low as it could be, that is, if $\mathbf{a}' \equiv_T \mathbf{0}'$. We say that $\mathbf{a} \leqslant_T \mathbf{0}'$ is *high*, if its jump is as high as it could be, that is, if $\mathbf{a}' \equiv_T \mathbf{0}''$. More generally:

Definition 6.1 (Soare [Soa74], Cooper [Coo74])**.** A Turing degree $\mathbf{a} \leqslant_T \mathbf{0}'$ is

- low$_n$ (L_n) if $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$.

- high$_n$ (H_n) if $\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}$.
- intermediate (I) if $\forall n\ (\mathbf{0}^{(n)} <_T \mathbf{a}^{(n)} <_T \mathbf{0}^{(n+1)})$.

Note that for each n, $L_n \subseteq L_{n+1}$, $H_n \subseteq H_{n+1}$, and L_n, H_n and I are disjoint. These classes induce a partition, $\mathcal{C}^*$, of the degrees $\leqslant \mathbf{0}'$.

$$\mathcal{C}^* = \{L_1^*, L_2^*, ...\} \cup \{I^*\} \cup \{H_1^*, H_2^*, ...\},$$

where $L_1^* = L_1$, $H_1^* = H_1$, $I^* = I$ and for $n > 1$, $L_n^* = L_n \smallsetminus L_{n-1}$, and $H_n^* = H_n \smallsetminus H_{n-1}$. We define an ordering, $\prec$, on $\mathcal{C}^*$ as follows:

$$L_1^* \prec L_2^* \prec \cdots \prec I^* \prec \cdots \prec H_2^* \prec H_1^*.$$

It follows from the monotonicity of the jump that if $\mathbf{x} \leqslant_T \mathbf{y}$, $\mathbf{x} \in X \in \mathcal{C}^*$ and $\mathbf{y} \in Y \in \mathcal{C}^*$, then $X \preceq Y$. The following theorem of Lerman's helps us to understand how the degrees in the different classes of the hierarchy are located.

Definition 6.2. An *H-poset* is a structure $\mathcal{P} = \langle P, \leqslant, 0, 1, f(\cdot)\rangle$, where $\langle P, \leqslant\rangle$ is a partial ordering, 0 and 1 are the least and greatest elements respectively, and f is a labeling function from P to $\mathcal{C}^*$ such that for every $x, y \in P$,

$$x \leqslant y \Rightarrow f(x) \preceq f(y),$$

$f(0) = L_1^*$ and $f(1) = H_1^*$.

Theorem 6.3 (Lerman [Ler85])**.** *Every finite H-poset can be embedded into $\mathcal{D}$ (of course, preserving labels).*

6.2. **The Generalized Low/High hierarchy.** As a generalization of this hierarchy to all the Turing degrees we get the *generalized high/low hierarchy.* In [JP78], Jockusch and Posner defined the *generalized high/low hierarchy* with the intention of classifying all the Turing degrees depending on how close a degree is to being computable, and on how close it is to computing the Halting Problem. This classification coincides with the High/Low hierarchy on the degrees below $\mathbf{0}'$.

Definition 6.4. For $n \geqslant 1$ we say that a degree $\mathbf{x}$ is *generalized low$_n$*, or GL_n, if $\mathbf{x}^{(n)} = (\mathbf{x} \vee 0')^{(n-1)}$. We say that a degree $\mathbf{x}$ is a *generalized high$_n$ degree*, or GH_n, if $\mathbf{x}^{(n)} = (\mathbf{x} \vee 0')^{(n)}$, and it is *generalized intermediate*, or GI, if $\forall n\big((\mathbf{x} \vee 0')^{(n-1)} <_T \mathbf{x}^{(n)} <_T (\mathbf{x} \vee 0')^{(n)}\big)$.

This classification has also been very useful in the study of $\mathcal{D}$. Many order-theoretic properties of $\mathbf{0}'$ have been proven to hold for the members in the higher classes of this hierarchy. For instance, every non-GL_2 cups to every degree above it [JP78]; every GH_1 degree bounds a minimal degree [Joc77], but not every GH_2 does [Ler86]; and every GH_1 degree has the complementation property [GMS04]. Also, degrees that should not contain much information appear in the lower classes: every 1-generic set is GL_1 (see [Ler83, IV.2]); every 2-random real is GL_1 [Kau91]; every minimal degree is GL_2 [JP78]. So, one could argue that degrees in the upper classes of this hierarchy are more complex than the ones in the lower classes. One would think that generalized high degrees should be above generalized low degrees, or at least not below. However, there are generalized low degrees which compute generalized high degrees. (Take an H_1 degree $\mathbf{x} <_T \mathbf{0}'$. By the Posner and Robinson join theorem relative to $\mathbf{x}$ [PR81], there exists $\mathbf{y} >_T \mathbf{x}$ with $\mathbf{y} \vee \mathbf{0}' = \mathbf{y}' = \mathbf{x}' = \mathbf{0}''$. So we get that $\mathbf{y} >_T \mathbf{x}$, $\mathbf{x}$ is GH_1 and $\mathbf{y}$ is GL_1.) Moreover, we have the worst situation possible in this respect:

Let

$$\mathcal{G}^* = \{GL_1^*, GL_2^*, ...\} \cup \{GI^*\} \cup \{GH_1^*, GH_2^*, ...\},$$

where $GL_1^* = GL_1$, $GH_1^* = GH_1$, $GI^* = GI$ and for $n > 1$, $GL_n^* = GL_n \setminus GL_{n-1}$, and $GH_n^* = GH_n \setminus GH_{n-1}$.

A *GH-poset* is a structure $\mathcal{P} = \langle P, \leqslant, 0, f(\cdot)\rangle$, where $\langle P, \leqslant\rangle$ is a partial ordering, 0 is the least element and f is a function from P to $\mathcal{C}^*$ such that $f(0) = GL_1^*$. Note that no condition at all is imposed on the labels of a GH-poset except for $f(0) = GL_1^*$.

Theorem 6.5 (Montalbán [Mon06])**.** *Every finite GH-poset can be embedded into $\mathcal{D}$.*

7. Decidability

It is impossible to talk about embeddings, extensions of embeddings and initial segment results without mentioning decidability results. For instance, since every finite distributive lattice embeds into $\mathcal{D}$ as an initial segment [Lac68], we can reduce the theory of distributive lattices to the theory of $\langle \mathbf{D}, \leqslant_T\rangle$ (by quantifying over all the top elements of initial segments of $\mathcal{D}$ which are distributive). Since the theory of distributive lattices is undecidable (Ervsov and Tauĭclin [ET63]), we get the following theorem:

Theorem 7.1 (Lachlan [Lac68])**.** *The theory of $\langle \mathbf{D}, \leqslant_T\rangle$ is undecidable.*

However, if we restrict ourself to certain classes of formulas, many decidability results have been proved. The next question is what fragments of the theory of $\langle \mathbf{D}, \leqslant_T\rangle$ are decidable.

7.1. Existential Theories.

The decidability of existential theories is closely related to embeddability results.

Lemma 7.2. *Let $\mathcal{L}$ be a finite relational language, and let $\mathcal{F}$ be an $\mathcal{L}$-structure. Then the following are equivalent*

1. *The $\exists$-theory of $\mathcal{F}$ in the language $\mathcal{L}$ is decidable;*
2. *There is an algorithm that decides which finite $\mathcal{L}$-structures can be embedded into $\mathcal{F}$.*

Proof: For the implication 1⇒2 note that for each finite $\mathcal{L}$-structure there is an existential formula in $\mathcal{L}$ which holds in $\mathcal{F}$ if and only if $\mathcal{P}$ embeds in $\mathcal{F}$.

For the other direction consider a existential sentence φ of $\mathcal{L}$. We can write φ as a disjunction of formulas of the form $\psi_j = \exists x_1, ..., x_k\ (\varphi_{j,1}\ \&\ ..., \&\ \varphi_{j,n})$, where each $\varphi_{j,i}$ is a literal (either an atomic formula or a negation of one). Note that $\mathcal{F} \models \psi_j$ if and only if there is an $\mathcal{L}$-structure of k elements which satisfies ψ_j and embeds into $\mathcal{F}$. All we have to do now is check all the $\mathcal{L}$-structures of size k. □

We can now apply the embeddability results of Section 3 to get decidability results.

Corollary 7.3 (Kleene and Post [KP54])**.** *The $\exists$-theory of $\langle \mathbf{D}(\leqslant_T \mathbf{0}'), \leqslant_T, \vee\rangle$ is decidable.*

Proof: Think of $\vee$ as a 3-ary relation and let $\mathcal{L}$ be the language with $\leqslant_T$ and $\vee$. From [KP54] it follows that a finite $\mathcal{L}$-structure embeds into $\mathcal{D}(\leqslant_T \mathbf{0}')$ if and only if it is a partial upper semilattice. □

Corollary 7.4 (Montalbán [Mon03])**.** *The $\exists$-theory of $\mathcal{D} = \langle \mathbf{D}, \leqslant_T, \vee, '\rangle$ is decidable.*

The results in Section 6 imply the following decidability results.

Theorem 7.5. *(Lerman [Ler85]) The* $\exists$*-theory of* $\langle \mathbf{D}, \leqslant_T, \mathbf{0}, \mathbf{0}', \mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{I}, ..., \mathcal{H}_2, \mathcal{H}_1 \rangle$ *is decidable.*
(Montalbán [Mon06]) The $\exists$ *theory of* $\langle \mathbf{D}, \leqslant_T, \mathbf{0}, GL_1, GL_2, ..., GI, ..., GH_2, GH_1 \rangle$ *is decidable.*

7.2. **Two quantifier theories and extensions of embeddings.** When we look at $\forall\exists$-theories, more than embeddability results, we need extension of embedding results.

Lemma 7.6. *Let* $\mathcal{L}$ *be a finite relational language, and let* $\mathcal{F}$ *be an* $\mathcal{L}$*-structure. Then, the following are equivalent*

1. *The* $\forall\exists$*-theory of* $\mathcal{F}$ *in the language* $\mathcal{L}$ *is decidable;*
2. *There is an algorithm such that given finite* $\mathcal{L}$*-structures* $\mathcal{P}, \mathcal{Q}_1, ..., \mathcal{Q}_l$*, with* $\mathcal{P} \subseteq \mathcal{Q}_i$ *for each* i*, it decides whether every embedding of* $\mathcal{P}$ *into* $\mathcal{F}$ *extends to an embedding of* $\mathcal{Q}_i$ *for some* $i \leqslant l$*.*

We leave the proof to the reader.

To get such an algorithm to solve the finite-extensions-of-embeddings problem in $\langle \mathbf{D}, \leqslant_T \rangle$, the two main ingredients are: Kleene and Post [KP54] finite version of Lemma 5.1; and Lerman's theorem [Ler71] that every finite usl $\mathcal{P}$ embeds into $\mathcal{D}$ as an initial segment. This was used independently by Shore [Sho78] and Lerman [Ler83, VII.4] to get that the $\forall\exists$-theory of $\langle \mathbf{D}, \leqslant_T \rangle$ is decidable. But Kleene and Post's [KP54] finite-extensions-of-embeddings result is not sufficient to get the $\forall\exists$-theory of $\langle \mathbf{D}, \leqslant_T, \vee \rangle$. Jockusch and Slaman [JS83] used a different forcing technique to prove that if $\mathcal{P}$ and $\mathcal{Q}$ are countable usl's w/0 and $\mathcal{P}$ is downward closed in $\mathcal{Q}$, then every usl-embedding of $\mathcal{P}$ into $\mathcal{D}$ extends to an embedding of $\mathcal{Q}$ into $\mathcal{D}$. The finite version of this result, together with Lerman [Ler71] initial segment theorem, gives us the algorithm needed in 7.6.2

Theorem 7.7 (Jockusch and Slaman [JS83])**.** *The* $\forall\exists$*-theory of* $\langle \mathbf{D}, \leqslant_T, \vee \rangle$ *is decidable.*

The situation below $\mathbf{0}'$ is more complicated. Lerman and Shore [LS88] showed that the $\forall\exists$-theory of $\langle \mathbf{D}(\leqslant_T \mathbf{0}'), \leqslant_T \rangle$ is decidable. However, the following question is still open.

Question 7.8. Is the $\forall\exists$-theory of $\langle \mathbf{D}(\leqslant_T \mathbf{0}'), \leqslant_T, \vee \rangle$ decidable?

Montalbán (2003, unpublished) made the following observation, which shows that, to solve the question above, it will be necessary to have more than an extensions-of-embeddings result like the one used for $\langle \mathbf{D}, \leqslant_T, \vee \rangle$ where only one $\mathcal{Q}$ is considered. For every $\mathbf{x}_1$ and $\mathbf{x}_2$ with $0 <_T \mathbf{x}_1 <_T \mathbf{x}_2 <_T \mathbf{0}'$, either there exists $\mathbf{y}$ such that $\mathbf{0} <_T \mathbf{y} <_T \mathbf{x}_1$, or there exists $\mathbf{y}$ such that $\mathbf{x}_1 <_T \mathbf{y} <_T \mathbf{0}'$ and $\mathbf{y} \vee \mathbf{x}_2 \equiv_T \mathbf{0}'$, but neither disjunct holds for every such $\mathbf{x}_1, \mathbf{x}_2$. (If $\mathbf{x}_1$ is a minimal degree, $\mathbf{0}'$ is high relative to it, and the existence of $\mathbf{y}$ follows from Posner and Robinson's join theorem [PR81]. To get $\mathbf{x}_1$ and $\mathbf{x}_2$ for which a $\mathbf{y}$ of the second type does not exists consider and c.e. operator which constructs a c.e. degree without the join property and then use Jockusch and Shore's pseudo-jump inversion theorem [JS83].)

As in a side, we should mention that it is also unknown whether the $\forall\exists$-theory of $\langle \mathcal{R} \leqslant_T, \vee \rangle$ is decidable, where $\mathcal{R}$ is the set of degrees of c.e. sets.

In larger languages, we do get to the boundary of decidability at the two quantifier level. For the theory of $\langle \mathbf{D}, \leqslant_T, \vee, ' \rangle$, Montalbán's result on its decidability is as far we can get.

Theorem 7.9 (Slaman and Shore [SS06])**.** *The $\forall\exists$-theory of $\langle \mathbf{D}, \leqslant_T, \vee, ' \rangle$ is undecidable.*

We also get undecidability if we add greatest lower bounds instead of jump.

Theorem 7.10 (Miller, Nies and Shore [MNS04])**.** *The $\forall\exists$-theory of $\langle \mathbf{D}, \leqslant_T, \vee, \wedge \rangle$ is undecidable, where $\wedge$ is any total extension of the infimum relation.*

The following seems to be a difficult open question.

Question 7.11. Is the $\forall\exists$-theory of $\langle \mathbf{D}, \leqslant_T, ' \rangle$ decidable?

A positive answer would give a decidability procedure for $\exists$-theory of $\langle \mathbf{D}, \leqslant_T, ', \mathbf{0} \rangle$.

7.3. **Other results.** Three quantifiers is the end of the story in term of decidability results. Schmerl (see [Ler83, VII.4.6]; the proof there needs a small correction) extended Lachlan's Theorem 7.1, and showed that the $\forall\exists\forall$-theory of $\langle \mathbf{D}, \leqslant_T \rangle$ is undecidable.

A quite interesting result about the theory of $\langle \mathbf{D}, \leqslant_T \rangle$ is that it is Turing (actually one-to-one) equivalent to true second order arithmetic (Simpson [Sim77]). Shore [Sho81], then proved that the theory of $\langle \mathbf{D}(\leqslant_T \mathbf{0}'), \leqslant_T \rangle$ is Turing (actually one-to-one) equivalent to true first order arithmetic. Moreover, also in [Sho81], he proved this result for $\langle \mathbf{D}(\leqslant_T \mathbf{a}), \leqslant_T \rangle$ where $\mathbf{a}$ is arithmetic and above $\mathbf{0}'$, computable enumerable, or high. Greenberg and Montalbán [GM03] extended this result to $\mathbf{a}$ n-CEA, 1-generic and below $\mathbf{0}'$, 2-generic and arithmetic, or arithmetically generic.

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