RICE SEQUENCES OF RELATIONS

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Abstract. We propose a framework to study the computational complexity of definable relations on a structure. Many of the notions we discuss are old, but the viewpoint is new. We believe all the pieces fit together smoothly under this new point of view. We also survey related results in the area.

More concretely, we study the space of sequences of relations over a given structure. On this space we develop notions of c.e.-ness, reducibility, join and jump. These notions are equivalent to other notions studied in other settings. We explain the equivalences and differences between these notions.

1. Introduction

The study of the complexity of definable relations over a given structure is a main theme in mathematical logic. In this paper we are interested in using computational ways of measuring the complexity of relations. The key notion of this paper is the one of *rice sequence of relations*, where rice stands for *relatively intrinsically computably enumerable*. A rice relation on a structure $\mathcal{A}$ is one that is always computably enumerable relative to any given presentation of $\mathcal{A}$ (Definition 3.1). Rather than looking at relations (i.e., subsets of A^n for some n), we will look at sequences of relations which can be used, for instance, to code subsets of $A^{<\omega}$, and even subsets of $A^{<\omega} \times \mathbb{N}$. This idea of going beyond subsets of A^n to develop a better theory of computability is not new, and it appears, for instance, in the work on hereditarily finite extensions or Moschovakis extensions that we will mention later. Once we have a notion of c.e.-ness (namely rice) on the space of sequences of relations over a fixed structure $\mathcal{A}$, we can define a notion of relative computability, of join and of jump. This notion of jump of a relation, or of a sequence of relations, can then be extended to the notion of the jump of a structure. Different notions of jump for structures have been developed in the recent years by researchers in the computability groups of Novosibirsk and Sofia (see Sections 5 and 6), and by the author.

In this paper we survey some of this recent work in the context of the study of sequences of relations. We believe that all the pieces fit together nicely under this new viewpoint, which is even slightly different from the one used be the author in the last few years [Mon09, Mona, Mon10]. Much of the notation introduced in those papers is revisited here. For instance, we remark that what used to be called a jump of $\mathcal{A}$ in [Mon09, Mona, Mon10], is now called *a structural jump of* $\mathcal{A}$ defined in Section 6, and different from *the* jump of $\mathcal{A}$ as defined in Section 5.

The reason why we like the viewpoint developed here is that it is closer, in style, to the notions used by many of the people already working on computable structure theory, particularly in the west, which makes it more approachable. Of course, other researches might disagree.

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This paper also contains historic information explaining what the other known similar notions are, how they were developed, and how they connect to the notions here.

The author's original motivation in this topic [Mon09, Mona] was to study relations on a given structure $\mathcal{A}$ that contain all the structural Σ_n^c information about $\mathcal{A}$. In many cases one can find a nice, small set of relations which, alone, give you everything you need to know about all other Σ_n^c relations. Finding such relations can, of course, be useful for other applications. We will talk about this in the last two sections of this paper.

Most of the results in this paper are not new, at least not in essence. This is except for the work in Section 7, where we study finite complete sets of rice relations. The work in that section was done during a visit to Sofia the week before CiE'11, by Knight, R. Miller, Soskov, A. Soskova, M. Soskova, VanDenDriessche, Vatev, and the author. (The author would like to thank them for allowing him to publish these results here.)

2. Background

We only consider relational languages, since functions can be coded as relations without changing the computational complexity of the objects we are interested in. Our languages are always computable. That is, they are countable and we can effectively list all their symbols and their arities. We will always use $\mathcal{L}$ to denote a language, where $\mathcal{L} = \{P_0, P_1, ...\}$ is finite or infinite, and where P_i has arity p_i. Since $\mathcal{L}$ is computable, the function $i \mapsto p_i$ is computable. By an *$\mathcal{L}$-structure* we mean a tuple $\mathcal{A} = (A; P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, ...)$ where $P_i^{\mathcal{A}} \subseteq A^{p_i}$ for all i. We only allow countable structures, as we will not deal with larger structures in this paper. By a *copy of* $\mathcal{A}$, or by a *presentation of* $\mathcal{A}$, we mean another structure $\mathcal{B} = (B; P_0^{\mathcal{B}}, P_1^{\mathcal{B}}, ...)$ which is isomorphic to $\mathcal{A}$ and where $B \subseteq \mathbb{N}$. Since all our structures are countable, it does not hurt to assume that the domains are always subsets of $\mathbb{N}$. However, the words "copy" or "presentation" emphasize that we are talking about this particular representation of $\mathcal{A}$, and not about the isomorphism type of $\mathcal{A}$.

Given a structure $\mathcal{A}$ with $A \subseteq \mathbb{N}$, we let $D(\mathcal{A})$ be the atomic diagram of $\mathcal{A}$. More concretely: Let $a_0, a_1, ...$ be constant symbols naming the elements of $\mathcal{A}$ where the number $i \in \mathbb{N}$ is named by a_i. (If $i \notin A$, then a_i does not name anybody.) Let $\varphi_0^{at}, \varphi_1^{at}, ...$ be an effective listing of all the atomic $(\mathcal{L} \cup \{a_0, a_1, ...\})$-formulas. Finally, let $D(\mathcal{A}) \in 2^\omega$ be defined by

$$D(\mathcal{A})(i) = \begin{cases} 1 & \text{if } \mathcal{A} \models \varphi_i^{at}, \\ 0 & \text{if } \mathcal{A} \not\models \varphi_i^{at}. \end{cases}$$

In particular, we let $D(\mathcal{A})(i) = 0$ if φ_i^{at} uses some constant a_j for which $j \notin A$. Notice that $D(\mathcal{A})$ computes A by looking at the formulas $a_i = a_i$.

When we say that a set X is c.e. (computable) in a structure $\mathcal{A}$, we mean that X is c.e. (computable) in $D(\mathcal{A})$, which of course depends on the given presentation of $\mathcal{A}$. The *spectrum* of a structure $\mathcal{A}$ is

$$Sp(\mathcal{A}) = \{\mathbf{x} \in \mathcal{D} : \mathbf{x} \text{ computes a copy of } \mathcal{A}\},$$

where $\mathcal{D}$ is the set of Turing degrees. (The spectrum of $\mathcal{A}$ is often as the set of degrees of all copies of $\mathcal{A}$. The two definitions are equivalent for non-trivial structures, as proved by Knight [Kni86].)

All throughout we will use the letters $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ to denote structures with domains A, B and C, and we will use the letters X, Y and Z for sets of natural numbers. Unless we specify otherwise, $\mathcal{A}$ is always an $\mathcal{L}$-structure where $\mathcal{L}$ is as above.

We will use Σ_n^c and Π_n^c to denote the set of *computable infinitary* Σ_n and Π_n formulas. These are first-order $\mathcal{L}$-formulas where we allow infinitary disjunctions and infinitary conjunctions so long as they are taken over a computable list of formulas, and so long as there are only finitely many different free variables over all the formulas in the list. When we count alternation of quantifiers, infinitary disjunctions count as $\exists$ and infinitary conjunctions count as $\forall$. See [AK00, Chapter 7] for more background on these formulas.

Given a tuple $\bar{a}$ in $\mathcal{A}$, we let Σ_1-$tp_{\mathcal{A}}(\bar{a})$, the Σ_1-*type of* $\bar{a}$ *in* $\mathcal{A}$, be the set of Gödel numbers of all Σ_1 formulas true about $\bar{a}$ in $\mathcal{A}$.

We use $\mathcal{P}_{fin}(X)$ to denote the set of finite subsets of X.

3. Rice relations

The following well-known notion is central to this paper.

Definition 3.1. A relation $R \subseteq A^n$ is *rice (relatively intrinsically computably enumerable)* if for every copy $(\mathcal{B}, R^{\mathcal{B}})$ of $(\mathcal{A}, R)$, we have that $R^{\mathcal{B}}$ is c.e. in $D(\mathcal{B})$.

Notice that the notion of being rice is independent of the presentation of $\mathcal{A}$, and depends only on its isomorphism type.

Example 3.2. Let $\mathcal{A}$ be a linear ordering. We say that x and $y \in A$ are *adjacent*, and write $Adj(x, y)$, if there is no element in between them. The relation $\neg Adj(\cdot, \cdot)$ is rice in $\mathcal{A}$.

We shall call a relation whose complement is rice, *co-rice*.

Example 3.3. Let $\mathcal{G}$ be a graph that consists of infinitely many disjoint cycles, one of each size n for $n \geq 3$. Let R be the set of vertices x in $\mathcal{G}$ such that x belongs to a cycle of size n, for some $n \in 0'$ (i.e., with $\{n\}(n)\downarrow$). Then R is rice in $\mathcal{G}$.

The relations in the examples above have quite a different feel to them. The former contains structural information, while the latter codes "Turing-degree information," namely $0'$. We will say more about this later.

The following theorem characterizes rice relations in purely syntactical terms, as opposed to the definition which refers to computations whose oracles are the diagrams of the copies of the given structure.

Theorem 3.4 (Ash, Knight, Manasse, Slaman [AKMS89]; Chisholm [Chi90])**.** *Let $\mathcal{A}$ be a structure, and $R \subseteq A^n$ a relation on it. The following are equivalent:*

(1) *R is rice.*
(2) *R is definable by a Σ_1^{c} formula with parameters from $\mathcal{A}$.*

Recall that a Σ_1^{c} formula is nothing more than an infinitary disjunction of a computable list of finitary Σ_1 $\mathcal{L}$-formulas.

Once we have a notion of c.e.-ness among relations on $\mathcal{A}$, we can develop a notion of computability.

Definition 3.5. Let R and Q be relations on $\mathcal{A}$. We say that R is *relatively intrinsically computable in* Q, and we write

$$R \leq_T^{\mathcal{A}} Q,$$

if R is both rice and co-rice in $(\mathcal{A}, Q)$.

Historic Remark 3.6. *The notion of rice relation appeared already in* [AKMS89]*–see also* [AK00, Chapter 10]*. The equivalent notion of Σ-definable relation on $\mathbb{HF}_{\mathcal{A}}$ was used by Ershov as part of the study of admissibility over abstract structures, and is still used in Russia quite a bit. We will say more about Σ-definability in Section 4.1. Moschovakis* [Mos69] *defined an equivalent notion called* semi-search computable relation*, which is also defined on an extended domain (of the sort of $\mathbb{HF}_{\mathcal{A}}$), and appears often in the work of Soskov et.al. The equivalence between these notions is due to Gordon* [Gor70].

3.1. **Sequences of relations.** The space of relations on a structure $\mathcal{A}$ is not rich enough to develop a good theory of computability because, for instance, it does not always have a universal rice relation. There are various approaches to solve this issue. One is to consider relations defined on extensions of the structure like the hereditarily finite extension (see Section 4.1 below), or the Moschovakis extension (see [Mos69]). Here we take a different approach that is probably friendlier for the audience accustomed to the style of Ash and Knight's book [AK00].

Definition 3.7. Let $\mathrm{RSeq}(\mathcal{A})$ be the set of all sequences of relations $\vec{R} = (R_0, R_1, ...)$, where $R_i \subset A^{r_i}$ and the arity function $i \mapsto r_i$ is computable.

We say that $\vec{R}$ is *rice in* $\mathcal{A}$ if for every copy $(\mathcal{B}, \vec{R}^{\mathcal{B}})$ of $(\mathcal{A}, \vec{R})$, we have that $\vec{R}^{\mathcal{B}}$ is uniformly c.e. in $D(\mathcal{B})$, that is, the set $\bigoplus_{i\in\omega} R_i^{\mathcal{B}} = \{(i, \bar{b}) \subseteq \mathbb{N} \times B^{<\omega} : \bar{b} \in R_i^{\mathcal{B}}\}$ is c.e. in $D(\mathcal{B})$.

Given $\vec{R}$ and $\vec{Q} \in \mathrm{RSeq}(\mathcal{A})$, we say that $\vec{R}$ is *r.i. computable in* $\vec{Q}$, and write

$$\vec{R} \leq_T^{\mathcal{A}} \vec{Q},$$

if both $\vec{R}$ and $\neg\vec{R}$ are rice in $(\mathcal{A}, \vec{Q})$, where $\neg\vec{R}$ is the sequence of complements of the relations in $\vec{R}$, and $(\mathcal{A}, \vec{Q})$ is a new structure whose language is augmented with infinitely many new relations symbols Q_i, one for each $i \in \mathbb{N}$, interpreted in the obvious way according to $\vec{Q}$.

Example 3.8. Let $\mathcal{V}$ be a $\mathbb{Q}$-vector space. Then $\vec{LD} = (LD_2, LD_3, ...)$, given by $LD_i = \{(v_1, ..., v_i) \in \mathcal{V}^i : v_1, ..., v_i$ are linearly dependent$\}$, is rice in $\mathcal{V}$.

Example 3.9. Let $\mathcal{A}$ be a ring. Then $\vec{R} = (R_1, R_2, ...)$, given by $R_i = \{(a_0, ..., a_i) \in A^{i+1} : a_i x^i + ... + a_1 x + a_0$ is a reducible polynomial$\}$, is rice in $\mathcal{A}$.

Remark 3.10. Note that not only can we represent subsets of $A^{<\omega}$ as sequences of relations, but also subsets of $A^{<\omega} \times \mathbb{N}$, for instance, by considering sequences $\vec{R} = (R_{i,j} : i, j \in \mathbb{N})$ where $R_{i,j}$ has arity i. Furthermore, restricting ourselves to work just with subsets of $A^{<\omega} \times \mathbb{N}$ would be essentially equivalent to working with $\mathrm{RSeq}(\mathcal{A})$.

Historic Remark 3.11. *An equivalent notion of computability on subsets of $A^n \times \mathbb{N}^k$, for the structure $\mathcal{E} = (A;\ )$ on an empty language, was already considered by Soskov and Baleva* [Bal06].

3.1.1. *Information sequences.* We can also use sequences of relations to code subsets of $\mathbb{N}$ in a natural way. We will allow ourselves to consider relations $R \subseteq A^r$ where $r = 0$. Recall that $A^0 = \{\langle\rangle\}$, where $\langle\rangle$ is the empty tuple, and hence either $R = \varnothing$ or $R = \{\langle\rangle\}$. In the former case we say that $R = \bot$, and that $R = \top$ in the latter. (The reader that is uncomfortable with 0-ary relations, can work with 1-ary relations R instead, where either $R = \varnothing$ or $R = A$.)

Definition 3.12. If $\vec{R}$ is a sequence of relations, all of arity 0, we say that $\vec{R}$ is an *information sequence*. Given $X \subseteq \mathbb{N}$, let $\vec{X} \in \mathrm{RSeq}(\mathcal{A})$ be the information sequence $\vec{X} = (X_0, X_1, ...)$ where

$$X_i = \begin{cases} \top & \text{if } i \in X, \\ \bot & \text{if } i \notin X. \end{cases}$$

We observe that $\vec{X}$ is c.e. in an oracle Z if and only if X is c.e. in Z. Thus, $\vec{X}$ is rice in $\mathcal{A}$ if and only if X is c.e. in the diagrams of all the copies of $\mathcal{A}$. In particular, for every c.e. set X, $\vec{X}$ is rice in $\mathcal{A}$. The set of all $X \subseteq \mathbb{N}$ such that $\vec{X}$ is rice in $\mathcal{A}$, called the *co-spectrum of* $\mathcal{A}$ (see Soskov [Sos04]), forms an ideal in the enumeration degrees. This ideal is characterized by a theorem of Knight (Corollary 3.16) below.

Also note that, for $X, Y \subseteq \mathbb{N}$, we have

$$X \leq_T Y \Rightarrow \vec{X} \leq_T^{\mathcal{A}} \vec{Y}.$$

3.1.2. *Join of sequences.* We also have a least-upper-bound operation on $\mathrm{RSeq}(\mathcal{A})$.

Definition 3.13. Given $\vec{R} = (R_0, R_1, ...)$ and $\vec{Q} = (Q_0, Q_1, ...)$ in $\mathrm{RSeq}(\mathcal{A})$, let

$$\vec{R} \oplus \vec{Q} = (R_0, Q_0, R_1, Q_1, ...).$$

It is not hard to see that $\vec{R} \oplus \vec{Q}$ is the least upper bound of $\vec{R}$ and $\vec{Q}$ in the $\leq_T^{\mathcal{A}}$-ordering.

We will sometimes abuse notation and write $R \oplus \vec{Q}$, or $R_1 \oplus R_2$, even when R, R_1, R_2 are relations, rather than sequences of relations, interpreting a single relation R by the sequence $(R, \varnothing, \varnothing, \varnothing, ...)$.

3.1.3. *The Ash–Knight–Manasse–Slaman–Chisholm theorem, revisited.* This well-known theorem extends from relations to sequences of relations in a straightforward way. We include the proof here for completeness. The original proofs, which proved the result for r.i. Σ^0_α-relations, used forcing, but the rice case can be proved in a much simpler way. An interesting fact about this extended version is that it also extends two other well-known theorems, one of Knight's and one of Selman's. We will see how they follow as particular cases (Corollaries 3.16 and 3.17 below).

Theorem 3.14 (Ash, Knight, Manasse, Slaman [AKMS89]; Chisholm [Chi90])**.** *Let $\vec{R} = (R_0, R_1, ...)$ be a sequence of relations in $\mathcal{A}$. The following are equivalent:*

(A1) *$\vec{R}$ is rice.*
(A2) *There is a tuple $\bar{p} \in A^{<\omega}$ and a computable list $\phi_0, \phi_1, ...$ of $\Sigma^{\mathtt{c}}_1$-formulas such that, for all $i \in \mathbb{N}$ and all $\bar{a} \in A^{r_i}$ (where r_i is the arity of R_i),*

$$\bar{a} \in R_i \iff \mathcal{A} \models \phi_i(\bar{p}, \bar{a}).$$

Proof. It is easy to see that (A2) implies (A1) because deciding $\Sigma^{\mathtt{c}}_1$ formulas about $\mathcal{A}$ is c.e. in $D(\mathcal{A})$. We prove the other direction. We will attempt to build a copy $\mathcal{B}$ of $\mathcal{A}$ where $\vec{R}^{\mathcal{B}}$ is not uniformly c.e. in $D(\mathcal{B})$. By (A1), this attempt is bound to fail, and we will use this failure to find the list of formulas $\phi_0, \phi_1, ...$ that we need.

Let $A^\star$ be the set of finite tuples from A all whose entries are different. At stage s we will define $\bar{p}_s \in A^\star$ such that $\bar{p}_{s-1} \subseteq \bar{p}_s$ (where inclusion here is as strings). At the end of stages we will obtain $G = \bigcup_{s\in\mathbb{N}} \bar{p}_s \colon \mathbb{N} \to A$. Along the way we will make sure that every element of A is in some $\bar{p}_s$, and hence that G is a bijection between $\mathbb{N}$ and A. We can then let $\mathcal{B}$ be the pull-back of $\mathcal{A}$ via G. That is, $\mathcal{B}$ has domain $\mathbb{N}$, and if P is a relation symbol of $\mathcal{L}$, $P^{\mathcal{B}}(\bar{x})$ holds if and only if $P^{\mathcal{A}}(G(\bar{x}))$ holds. By (A1), we know that for some index e,

$$\bigoplus_{n\in\mathbb{N}} R_n^{\mathcal{B}} = W_e^{D(\mathcal{B})}.$$

Given $\bar{q} \in A^\star$, we let $D(\bar{q})$ be the initial segment of $D(\mathcal{B})$ of length $|\bar{q}|$ which is determined by $\bar{q}$ assuming we have that $\bar{q} \subseteq G$. More formally, let $\{b_0, b_1, ...\}$ be a list of constant symbols where b_i is interpreted as i in $\mathcal{B}$, and let $\{\varphi_i^{at} : i \in \mathbb{N}\}$ be a list of all atomic $\mathcal{L} \cup \{b_0, ...\}$-sentences, and assume that φ_i^{at} only uses constants b_j for $j \leq i$. Given $\bar{q} = (q_0, ..., q_{k-1}) \in A^\star$, let $D(\bar{q}) \in 2^k$ be such that $D(\bar{q})(i) = 1$ if and only if $\mathcal{A} \models \varphi_i^{at}[b_j \mapsto q_j : j < k]$. This way we have that

$$D(\mathcal{B}) = \bigcup_{s\in\mathbb{N}} D(\bar{p}_s).$$

For $\sigma \in 2^{<\omega}$ we let W_e^σ be the step $|\sigma|$ approximation to W_e^S for any $S \supseteq \sigma$, noticing that W_e^S can not read the oracle S beyond position $|\sigma|$ in less than $|\sigma|$ steps. So we have that

$W_e^{D(\mathcal{B})} = \bigcup_{s\in\mathbb{N}} W_e^{D(\bar{p}_s)}$. We also notice that for $\sigma \in 2^k$ there is an atomic formula ψ_σ such that $D(\bar{q}) = \sigma \iff \mathcal{A} \models \psi_\sigma(\bar{q})$.

Now that we have got the background notation out of the way, here is the construction. Let $\bar{p}_0$ be the empty sequence. At stage $s+1 = 2e$, we try to force $\bigoplus_{n\in\mathbb{N}} R_n^{\mathcal{B}} \neq W_e^{D(\mathcal{B})}$ as follows. Ask if there exist $n \in \mathbb{N}$, $\bar{\jmath} = (j_1, ..., j_{r_n}) \in \mathbb{N}^{r_n}$ and $\bar{q} \supseteq \bar{p}_s$ such that $(n, \bar{\jmath}) \in W_e^{D(\bar{q})}$ but $\mathcal{A} \not\models R_n(q_{\bar{\jmath}})$ (where $q_{\bar{\jmath}} = (q_{j_1}, ..., q_{j_{r_n}})$). If so, let $\bar{p}_{s+1}$ be such $\bar{q}$, and otherwise let $\bar{p}_{s+1} = \bar{p}_s$. Notice that in the former case we have succeeded making $\bigoplus_{n\in\mathbb{N}} R_n^{\mathcal{B}} \neq W_e^{D(\mathcal{B})}$ because we are forcing $(n, \bar{\jmath}) \in W_e^{D(\mathcal{B})}$ and $b_{\bar{\jmath}} \notin R_n^{\mathcal{B}}$. At stage $s+1 = 2e+1$, we take care of making G onto: If the eth element of $\mathcal{A}$ is not already in $\bar{p}_s$, add it to the range of $\bar{p}_{s+1}$, otherwise let $\bar{p}_{s+1} = \bar{p}_s$.

Since we are assuming (A1), for some e we do get $\bigoplus_{n\in\mathbb{N}} R_n^{\mathcal{B}} = W_e^{D(\mathcal{B})}$. Thus, at stage $s+1 = 2e$, there were no n, $\bar{\jmath}$, and $\bar{q}$ as wanted, as otherwise we would have acted. Let $\bar{p} = \bar{p}_s$. We now claim that for all $n \in \mathbb{N}$ and all $\bar{a} \in A^{r_n}$, we have that

$$\begin{aligned}&\mathcal{A} \models R_n(\bar{a}) \text{ if and only if for some } \bar{q} \in A^\star\text{, with } \bar{q} \supseteq \bar{p} \text{ and with } \bar{q}(\langle j_1, ..., j_{r_n}\rangle) = \bar{a}\\ &\text{for some } j_1, ..., j_{r_n} < |\bar{q}|\text{, we have that } (n, \langle j_1, ..., j_{r_n}\rangle) \in W_e^{D(\bar{q})}.\end{aligned}$$

Note that this can be written as the disjunction over all $\sigma \in 2^{<\omega}$ and all $\bar{\jmath}$ with $(n, \bar{\jmath}) \in W_e^\sigma$, of the formulas that say that there exists $\bar{y} \in A^{<\omega}$ such that if we let $\bar{q} = \bar{p}\bar{y}$, we get $\bar{q} \in A^\star$, $\bar{q}(\bar{\jmath}) = \bar{a}$ and $D(\bar{q}) = \sigma$. So, the claim gives us a $\Sigma_1^{\mathtt{c}}$ definition $\phi_n(\bar{x})$ of $R_n^{\mathcal{A}}$, with parameters $\bar{p}$ and which is uniform on n. To prove the claim, notice that the left-to-right direction follows from $\bigoplus_{n\in\mathbb{N}} R_n^{\mathcal{B}} = W_e^{D(\mathcal{B})}$ by taking $\bar{q}$ to be a long enough segment of G. For the other direction, notice that if such $\bar{\jmath}$ and $\bar{q}$ existed, but $\mathcal{A} \not\models R_n(\bar{a})$, we would have chosen them and acted at step $s+1$. Contradicting the fact that we did not act. □

For the next corollaries we recall the definition of *enumeration reducibility.*

Definition 3.15. A set $X \subseteq \mathbb{N}$ is *e-reducible to* $Y \subseteq \mathbb{N}$ if there exists a c.e. set $\Phi \subseteq \mathbb{N} \times \mathcal{P}_{fin}(\mathbb{N})$ (called an *enumeration operator*) such that for all $n \in \mathbb{N}$, $n \in X$ if and only if there exists $D \in \mathcal{P}_{fin}(\mathbb{N})$ such that $(n, D) \in \Phi$ and $D \subseteq Y$.

Corollary 3.16 (Knight, see [AK00, Theorem 10.17])**.** *Let $X \subseteq \mathbb{N}$. The following are equivalent:*

(B1) *X is c.e. in every copy of $\mathcal{A}$.*
(B2) *X is e-reducible to Σ_1-$tp_{\mathcal{A}}(\bar{p})$ for some $\bar{p} \in A^{<\omega}$.*

Proof. It is not hard to show that (B2) implies (B1) using that all Σ_1-types are c.e. in every copy of $\mathcal{A}$. We prove the other direction.

As we mentioned before, X is c.e. in every copy of $\mathcal{A}$ if and only if $\vec{X}$ is rice in $\mathcal{A}$. So, we have that (A1), and hence (A2), of Theorem 3.14 hold for $\vec{R} = \vec{X}$. Let $\{\phi_n : n \in \mathbb{N}\}$ be a computable sequence of $\mathcal{L}$-sentences with parameters $\bar{p}$ witnessing (A2). Each ϕ_n is of the form $\bigvee_{j\in\mathbb{N}} \exists\bar{y}\varphi^\Sigma_{i_{n,j}}(\bar{p}, \bar{y})$, where φ^Σ_i is the ith Σ_1-$\mathcal{L}$-formula. Let $\Phi = \{(n, \{i_{n,j}\}) : n, j \in \mathbb{N}\}$, so that $n \in \Phi^{\Sigma_1\text{-}tp_{\mathcal{A}}(\bar{p})}$ if and only if, for some $j \in \mathbb{N}$, $i_{n,j} \in \Sigma_1$-$tp_{\mathcal{A}}(\bar{p})$, which happens if and only if ϕ_n holds. So, $X = \Phi^{\Sigma_1\text{-}tp_{\mathcal{A}}(\bar{p})}$. □

Corollary 3.17 ([Sel71])**.** *Let $X, Y \subseteq \mathbb{N}$. The following are equivalent:*

(C1) *Every enumeration of Y computes an enumeration of X.*
(C2) *X is e-reducible to Y.*

(By enumeration of Y *we mean a function $f\colon \mathbb{N} \to \mathbb{N}$ with range Y.)*

Proof. It is not hard to show that (C2) implies (C1). We prove the other direction.

Assume Y is infinite, otherwise both statements are trivially equivalent to X being c.e. Consider a language with constants $c_0, c_1, ...$ and a binary relation Q. Let $\mathcal{A}$ be the structure with domain $\mathbb{N}$, where c_i is interpreted as $2i$, and no constants are assigned to the odd numbers, and where Q is a bijection between the odd numbers and the set $\{c_i : i \in Y\}$.

Now, it is clear that every presentation of $\mathcal{A}$ computes an enumeration of the set Y. Hence, (C1) implies that X is c.e. in every presentation of $\mathcal{A}$, and thus that $\vec{X}$ is rice in $\mathcal{A}$. So, we have that (B1), and hence (B2), of the previous corollary hold, that is, that X is e-reducible to Σ_1-$\mathrm{tp}_{\mathcal{A}}(\bar{p})$ for some $\bar{p} \in A^{<\omega}$. Now, every Σ_1-type in $\mathcal{A}$ is e-reducible to the set Y: It is not hard to show, using disjunctive normal forms in a standard way, that every Σ_1 formula about $\mathcal{A}$ is equivalent to a finite disjunction of formulas of the form $\exists x\, Q(x, c_i)$ (which holds if and only if $i \in Y$). So we have that X is e-reducible to Y. □

3.2. **The jump of a sequence of relations.** So far, on $\mathrm{RSeq}(\mathcal{A})$ we have defined a notion of c.e.-ness, of computability and of join. Now, as the central notion of this paper, we define a notion of jump. We start by defining *universal rice sequence of relations.*

Let $\varphi_0^{\Sigma^c}, \varphi_1^{\Sigma^c}, \ldots$ be an effective listing of all Σ_1^c $\mathcal{L}$-formulas, where $\varphi_i^{\Sigma^c}$ has arity k_i.

Definition 3.18. Let $\vec{K}^{\mathcal{A}} = (K_0^{\mathcal{A}}, K_1^{\mathcal{A}}, \ldots)$ be defined by

$$K_i^{\mathcal{A}} = \{\bar{a} \in A^{k_i} : \mathcal{A} \models \varphi_i^{\Sigma^c}(\bar{a})\}.$$

$\vec{K}^{\mathcal{A}}$ is nothing more than the Σ_1^c-diagram of $\mathcal{A}$.

It should be clear that $\vec{K}^{\mathcal{A}}$ is rice.

Observation 3.19. $\vec{K}^{\mathcal{A}}$ is *universal among rice sequences of relations* in $\mathcal{A}$ in the following sense. If $\vec{Q}$ is rice, there is $\bar{p} \in A^{<\omega}$ and a computable $f\colon \mathbb{N} \to \mathbb{N}$ such that

$$\forall i \in \mathbb{N} \forall \bar{a} \in A^{q_i}\ (\bar{a} \in Q_i \iff (\bar{p}, \bar{a}) \in K_{f(i)}^{\mathcal{A}}),$$

where q_i is the arity of Q_i, and the arity of $K_{f(i)}^{\mathcal{A}}$ is $|\bar{p}| + q_i$. To prove this, we use the extended Ash–Knight–Manasse–Slaman–Chisholm theorem 3.14: Just let $\bar{p}$ and $\{\phi_i : i \in \mathbb{N}\}$ be as given by the theorem, and let f be the computable function such that ϕ_i is $\varphi_{f(i)}^{\Sigma^c}$.

Definition 3.20. Given $\vec{Q} \in \mathrm{RSeq}(\mathcal{A})$, let $(\mathcal{A}, \vec{Q})$ be the structure $\mathcal{A}$ augmented with infinitely many new relations interpreting Q_i for $i \in \mathbb{N}$. Let *the jump of* $\vec{Q}$ *in* $\mathcal{A}$ be $\vec{K}^{(\mathcal{A}, \vec{Q})}$. We denote it by $\vec{Q}'$.

We can also define $\vec{Q}''$ as $\vec{K}^{(\mathcal{A}, \vec{Q}')}$, etc.

Remark 3.21. Let us use $\vec{\varnothing}_{\mathcal{A}}$ to denote the sequence of empty (unary) relations $(\varnothing, \varnothing, \ldots) \in \mathrm{RSeq}(\mathcal{A})$. Let us emphasize the difference between $\vec{\varnothing}'_{\mathcal{A}}$ and $\overrightarrow{0'}$. The former is $\vec{K}^{\mathcal{A}}$ as in Definition 3.18 where the relations in the sequence have all possible arities, each arity appearing infinitely often. The latter is the information sequence coding $0'$, so it consists only of 0-ary relations and contains no structural information about $\mathcal{A}$. We have that $\overrightarrow{0'} \leq_T^{\mathcal{A}} \vec{\varnothing}'_{\mathcal{A}}$ always holds just because $\overrightarrow{0'}$ is rice. However, in most cases, $\vec{\varnothing}'_{\mathcal{A}}$ has structural information about $\mathcal{A}$ that $\overrightarrow{0'}$ alone does not. The last two sections of this paper are dedicated to studying this structural information.

Example 3.22. Let $\mathcal{A}$ be a *linear ordering.* Then

$$\vec{\varnothing}'_{\mathcal{A}} \equiv_T^{\mathcal{A}} Adj(\cdot, \cdot) \oplus \overrightarrow{0'}.$$

This proof is given in [Mon09, Theorem 2.1] using different notation.

Example 3.23. Let $\mathcal{V}$ be a $\mathbb{Q}$-vector space. Then

$$\vec{\varnothing}_{\mathcal{V}}^{(n)} \equiv_T^{\mathcal{V}} \vec{LD} \oplus \overrightarrow{0^{(n)}}.$$

This is because we can use $\vec{LD}$ to compute an isomorphism between $\mathcal{V}$ and the standard computable presentation of $\mathbb{Q}^d$, where $d = dim_{\mathbb{Q}}(\mathcal{V})$, and then we can use $0^{(n)}$ to decide Σ_n^{c}-relations on $\mathbb{Q}^d$.

3.2.1. *Diagonalization.* We now prove that, on the space of sequences of relations, the jump is actually a jump in the sense that is strictly increasing.

Theorem 3.24 (Vatev [Vat11], Stukachev)**.** *For every $\vec{Q} \in \mathrm{RSeq}(\mathcal{A})$, $\vec{Q} <_T^{\mathcal{A}} \vec{Q}'$.*

Proof. (Montalbán) It is easy to see that $\vec{Q} \leq_T^{\mathcal{A}} \vec{Q}'$ because the Σ_1^{c} diagram of $(\mathcal{A}, \vec{Q})$ clearly computes the atomic diagram of $(\mathcal{A}, \vec{Q})$. We now show that $\vec{Q}'$ is not r.i. computable in $\vec{Q}$. It is enough to show that $\vec{K}^{\mathcal{A}}$ is not r.i. computable in $\mathcal{A}$ for any given $\mathcal{A}$.

We start by re-indexing $\vec{K}^{\mathcal{A}}$ so that the arity of each relation is reflected in the index. Let $K_{i,j}^{\mathcal{A}}(\bar{x}) \equiv \varphi_{i,j}^{\Sigma^{\mathsf{c}}_1}(\bar{x})$ where $\varphi_{i,j}^{\Sigma^{\mathsf{c}}_1}$ is the ith Σ_1^{c} formula with arity j. Suppose, toward a contradiction, that $\vec{K}^{\mathcal{A}}$ is co-rice. For each $e, j \in \mathbb{N}$, let

$$R_{e,j}(\bar{x}) = \begin{cases} \neg K_{\{e\}(e,j),2j}^{\mathcal{A}}(\bar{x}, \bar{x}) & \text{if } \{e\}(e,j)\downarrow, \\ \varnothing & \text{otherwise,} \end{cases}$$

where $\{e\}$ is the eth Turing functional. Note that under the assumption that $\vec{K}^{\mathcal{A}}$ is co-rice, $\vec{R}$ is rice. By the universality of $\vec{K}^{\mathcal{A}}$ (Observation 3.19), there is an $n \in \mathbb{N}$, an $\bar{a} \in A^n$, and an index k for a total computable function $\{k\}$ such that

$$R_{e,j}(\bar{x}) \iff K_{\{k\}(e,j),n+j}^{\mathcal{A}}(\bar{a}, \bar{x}).$$

We then get the following contradiction.

$$R_{k,n}(\bar{a}) \iff K_{\{k\}(k,n),n+n}^{\mathcal{A}}(\bar{a}, \bar{a}) \iff \neg R_{k,n}(\bar{a}).$$ □

Historic Remark 3.25. *The proof given above is new, although it is clearly similar to the standard proof of the incomputability of the Halting problem. Theorem 3.24 had been previously proved for a different, yet equivalent, notion of jump (notion J2 in page 11) by Vatev in [Vat11]. Vatev's proof, restated in our terms, goes by showing that if $\mathcal{B}$ is a generic copy of $\mathcal{A}$, then $\vec{K}^{\mathcal{B}}$ has degree $D(\mathcal{B})'$ (which, of course, is not computable in $D(\mathcal{B})$), and hence $\vec{K}^{\mathcal{A}}$ is not r.i. computable in $\mathcal{A}$. In a personal communication, Stukachev has told me he has another proof which has not been translated to english yet.*

4. Superstructures

We mentioned in the introduction, the notion of rice sequences of relations is equivalent to other notions that were known many decades ago. In this section we briefly sketch two of these other notions: the study of Σ-definable subsets of the hereditarily finite superstructures, and the study of semi-search computable subsets of the Moschovakis superstructure.

The reader can skip this section without affecting the understanding of the rest of the paper.

4.1. **The hereditarily finite superstructure.** As we mentioned before, another approach to the study of rice relations is using Σ-definability on admissible structures. We will not use admissible structures in general but just the hereditarily finite extension of an abstract structure $\mathcal{A}$, which we define below. We will see how this is essentially equivalent to studying rice sequences of relations. For more background see Barwise's book [Bar75, Chapter II] or Stukachev's survey paper [Stu].

Definition 4.1. Let $\mathcal{P}_{fin}(X)$ denote the collection of finite subsets of X. Given a set A, we define:

(1) $HF_A(0) = \varnothing$,
(2) $HF_A(n+1) = \mathcal{P}_{fin}(A \cup HF_A(n))$, and
(3) $\mathbb{HF}_A = \bigcup_{n\in\mathbb{N}} HF_A(n)$.

Now, given an $\mathcal{L}$-structure $\mathcal{A}$ we define the $\mathcal{L} \cup \{\in, D\}$-structure $\mathbb{HF}_{\mathcal{A}}$ whose domain has two sorts, A and $\mathbb{HF}_A$, and where the symbols of $\mathcal{L}$ are interpreted in the A-sort as in $\mathcal{A}$, '$\in$' is interpreted in the obvious way, and D is a unary relation coding the atomic diagram of $\mathcal{A}$, as we explain below.

A quantifier of the form $\forall x \in y$... and $\exists x \in y$.... is called a *bounded quantifier*. A Σ-*formula* is one that is build out of atomic and negation of atomic formulas using disjunction, conjunction, bounded quantifiers and existential unbounded quantifiers. A subset of $\mathbb{HF}_{\mathcal{A}}$ is Δ-*definable* if it and its complement are Σ-definable.

Clearly, on $\mathbb{HF}_{\mathcal{A}}$ we have the usual pairing function $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$, and, of course, we can also encode n-tuples, strings etc. Notice, also, that $\mathbb{HF}_{\mathcal{A}}$ includes the finite ordinals (denoted by $\underline{n}$, where $\underline{0} = \varnothing$ and $\underline{n+1} = \{\underline{n}\} \cup \underline{n}$). We use ω to denote the Δ-definable set of finite ordinals of $\mathbb{HF}_{\mathcal{A}}$. Well-known arguments in admissibility theory show that every c.e. subset of ω is Σ-definable in $\mathbb{HF}_{\mathcal{A}}$, and every computable function is Δ-definable (see, for instance, [Bar75, Theorem II.2.3]).

We let $D(\mathcal{A})$ be the *satisfaction relation for atomic formulas*, that is $D(\mathcal{A}) = \{\langle i, \bar{a}\rangle : \mathcal{A} \models \varphi_i^{at}(\bar{a})\} \subseteq \mathbb{HF}_{\mathcal{A}}$, where $\{\varphi_0^{at}, \varphi_1^{at}, ...\}$ is an effective enumeration of all the atomic formulas of $\mathcal{A}$. Notice that if the language of $\mathcal{A}$ is finite, this is a finite list. So, when the language of $\mathcal{A}$ is finite, $D(\mathcal{A})$ is Δ-definable in $\mathbb{HF}_{\mathcal{A}}$, without using $D(\mathcal{A})$ of course, and hence it does not need to be added to the definition of $\mathbb{HF}_{\mathcal{A}}$.

Now, given any $\vec{R} \in \mathrm{RSeq}(\mathcal{A})$, we can encode it by

$$h(\vec{R}) = \{\langle \underline{n}, \bar{a}\rangle : n \in \mathbb{N}, \bar{a} \in R_n\} \subseteq \mathbb{HF}_A.$$

Another set we will use is the *satisfaction relation for* Σ_1 *formulas*, that is

$$hK(\mathcal{A}) = \{\langle i, \bar{a}\rangle : \mathbb{HF}_{\mathcal{A}} \models \varphi_i^{\Sigma}(\bar{a})\} \subseteq \mathbb{HF}_A,$$

where $\{\varphi_0^{\Sigma}, \varphi_1^{\Sigma}, ...\}$ is an enumeration of all the Σ_1 formulas of $\mathbb{HF}_{\mathcal{A}}$. Using recursion on the size of formulas, it is not hard to prove that $hK(\mathcal{A})$ is Σ-definable in $\mathbb{HF}_{\mathcal{A}}$.

Theorem 4.2. *Let $\vec{R} \in \mathrm{RSeq}(\mathcal{A})$. The following are equivalent:*

(1) *$\vec{R}$ is rice in $\mathcal{A}$.*
(2) *$h(\vec{R})$ is Σ-definable in $\mathbb{HF}_{\mathcal{A}}$ with parameters.*

Historic Remark 4.3. *This theorem is credited to Vaĭtsenavichyus* [Vaĭ89] *in* [Stu] *and appears in some form in* [BT79].

Sketch of the Proof. We start by proving that $h(\vec{K}^{\mathcal{A}})$ is Σ-definable in $\mathbb{HF}_A$, where $\vec{K}^{\mathcal{A}}$ is as in Definition 3.18. Let hK_0 be the satisfaction relation for finitary Σ_1 formulas in $\mathcal{A}$, that is

$$hK_0(\mathcal{A}) = \{\langle i, \bar{a}\rangle : A \models \varphi_i^{\Sigma}(\bar{a})\} \subseteq \mathbb{HF}_A,$$

where $\{\varphi_0^{\Sigma}, \varphi_1^{\Sigma}, ...\}$ is an enumeration of all the Σ_1 formulas of $\mathcal{A}$. As with $hK(\mathcal{A})$, it is not hard to prove that $hK_0(\mathcal{A})$ is Σ-definable in $\mathbb{HF}_{\mathcal{A}}$. Each $\Sigma_1^{\mathtt{c}}$ formula is a disjunction over some c.e. set W_e of formulas φ_i^{Σ} for $i \in W_e$. Using the Σ-definitions of $\{(e, n) \in \mathbb{N}^2 : n \in W_e\}$ and of $hK_0(\mathcal{A})$, we get a Σ-definition of $h(\vec{K}^{\mathcal{A}})$.

Assume now that $\vec{R}$ is rice. Thus, there is $\bar{p} \in A^{<\omega}$ and a computable function $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $i \in \mathbb{N}$ and all $\bar{a} \in A^{<\omega}$, $R_i(\bar{a})$ holds in $\mathcal{A}$ if and only if $K^{\mathcal{A}}_{f(i)}(\bar{p}, \bar{a})$ holds. Using the Σ-definition of $h(\vec{K}^{\mathcal{A}})$, we get a Σ-definition of $h(\vec{R})$ with parameters $\bar{p}$.

Suppose now that $h(\vec{R})$ is Σ-definable in $\mathbb{HF}_{\mathcal{A}}$ with parameters; we want to prove that $\vec{R}$ is rice. Let $\mathcal{B}$ be a copy of $\mathcal{A}$. Computably in $D(\mathcal{B})$ build a copy of $\mathbb{HF}_{\mathcal{B}}$ and then use the Σ-definition of $h(\vec{R})$ to enumerate $h(\vec{R})^{\mathbb{HF}_{\mathcal{B}}}$. We end up with a computable enumeration of $\vec{R}^{\mathcal{B}}$ relative to $D(\mathcal{B})$. □

There is also a natural way of going the other way around: from relations in $\mathbb{HF}_{\mathcal{A}}$ to sequences of relations on $\mathcal{A}$. Let $X = \{x_0, x_1, ...\}$ where the x_i's are variable symbols. Every $t \in \mathbb{HF}_X$ is essentially a term, and we write $t(\bar{x})$ to show the variables that appear in t. Observe that $\mathbb{HF}_A = \{t(\bar{a}) : t(\bar{a}) \in \mathbb{HF}_X, \bar{a} \in A^{|\bar{x}|}\}$. Let $\{t_i : i \in \mathbb{N}\}$ be an effective enumeration of $\mathbb{HF}_X \cup X$, and let q_i be number of different variables in t_i. Now, given $Q \subseteq \mathbb{HF}_{\mathcal{A}}$, we define

$$s(Q) = (Q_0, Q_1, ...) \in \mathrm{RSeq}(\mathcal{A}) \quad \text{ where } \quad Q_i = \{\bar{a} \in A^{q_i} : t_i(\bar{a}) \in Q\}.$$

With a bit of effort one can show that the relation $\{\langle a, \underline{n}, \bar{a}\rangle : a \in A, n \in \mathbb{N}, \bar{a} \in A^{q_n} \;\&\; a = t_{\underline{n}}(\bar{a})\}$ is Δ-definable in $\mathbb{HF}_{\mathcal{A}}$. This can be used to prove the following theorem.

Theorem 4.4. *Given $Q \subseteq \mathbb{HF}_A$, the following are equivalent:*
1. *$s(Q)$ is rice in $\mathcal{A}$.*
2. *Q is Σ-definable in $\mathbb{HF}_{\mathcal{A}}$ with parameters.*

Proof. Assume $s(Q)$ is rice in $\mathcal{A}$. Then $h(s(Q))$ is Σ-definable. Now, $a \in Q$ if and only if there exist $n \in \mathbb{N}$ and $\bar{a} \in A^{q_n}$ such that $a = t_n(\bar{a})$ and $\langle \underline{n}, \bar{a}\rangle \in h(s(Q))$. This gives a Σ-definition of Q from the one of $h(s(Q))$.

Suppose now that Q is Σ-definable. As in the proof of the previous theorem, it is not hard to show that for every copy $\mathcal{B}$ of $\mathcal{A}$, $s(Q)^{\mathcal{B}}$ is c.e. in $D(\mathcal{B})$. □

4.2. **The Moschovakis enrichment.** The Moschovakis extension $\mathcal{A}^*$ of a structure $\mathcal{A}$ is not too far from $\mathbb{HF}_{\mathcal{A}}$.

Definition 4.5. [Mos69] Let 0 be a new constant symbol. Given a set A, we define $A^0 = A \cup \{0\}$, and we let A^* be the closure of A^0 under a pairing operation $x, y \mapsto (x, y)$.

Moschovakis [Mos69] then defines a class of partial multi-valued functions from $(A^*)^n$ to A^* which he calls *search computable functions*. This class is defined as the least class closed under certain primitive operations, much in the style of Kleene's definition of primitive recursive and partial recursive functions, where instead of the Kleene's least-element operator μ, we have a multivalued search operator ν. A subset of A^* is *search computable* if its characteristic function is, and it is *semi-search computable* if it has a definition of the form $\exists y\ (f(x, y) = 1)$, where f is search computable.

The definition of search computable allows us to add a list of new primitive functions to our starting list (so long as they are given in an effective list, with computable arities), obtaining a sort of *relativized version* of search computability. If we have a structure $\mathcal{A}$, we would add to the list of primitive functions the characteristic functions of the relations in $\mathcal{A}$ to obtain a notion of partial, multi-valued, *search computable function in* $\mathcal{A}$.

Much in the same way as we did for $\mathbb{HF}_{\mathcal{A}}$ above, we have a natural way of encoding sequences $\vec{R} \in \mathrm{RSeq}(\mathcal{A})$ by subsets of $\mathcal{A}^*$, and vice-versa. Maybe even more directly, one can go from subsets of A^* to subsets of $\mathbb{HF}_{\mathcal{A}}$ and back. Gordon [Gor70] proved that the notions of search computable in $\mathcal{A}$ and semi-search computable in $\mathcal{A}$ for subsets of A^* coincide with the notions of Δ-definable and Σ-definable for subsets of $\mathbb{HF}_{\mathcal{A}}$. And hence, when you add parameters, they also coincide with the notions of r.i. computable and rice for sequences of relations in $\mathrm{RSeq}(\mathcal{A})$.

5. The jump of a structure

In this section we look at the jump as a map from abstract structures to abstract structures, rather than from $\mathrm{RSeq}(\mathcal{A})$ to $\mathrm{RSeq}(\mathcal{A})$.

Definition 5.1. Given an $\mathcal{L}$-structure $\mathcal{A}$, we let

$$\mathcal{A}' = (\mathcal{A}, \vec{K}^{\mathcal{A}}),$$

where $\vec{K}^{\mathcal{A}}$ is as in Definition 3.18, and $\mathcal{A}'$ is now an $\mathcal{L}'$-structure, where $\mathcal{L}' = \mathcal{L} \cup \{K_i : i \in \mathbb{N}\}$.

Notice that the isomorphism type of $\mathcal{A}'$ does not depend on the presentation of $\mathcal{A}$. However it does depend, in a non-essential way, on the Gödel numbering of the $\Sigma^{\mathtt{c}}_1$ $\mathcal{L}$-formulas, in the same way as the Turing jump of a set depends on the numbering of the partial computable functions. Also notice that the extended language $\mathcal{L}'$ is still a computable relational language. $\mathcal{L}'$ is now infinite, even if $\mathcal{L}$ was finite, and this can be viewed as a disadvantage of this definition. That is the price to pay if we want $\mathcal{A}$ and $\mathcal{A}'$ to have the same domain.

We note that the definition above also works for uncountable structures. However, we will still assume all our structures are countable throughout this paper.

Historic Remark 5.2. *The jump of structures has been independently defined various times in the last few years. We have four different definitions of maps from abstract structures to abstract structures that we call jump. All these definitions are equivalent up to Σ-equivalence (see Definition 5.5 below).*

- (J1) *The first appearance in print of such a definition for all countable abstract structures is due to Vessela Baleva [Bal06], as part of her Ph.D. thesis under the supervision of Ivan Soskov. That definition uses, as domain for the jump of $\mathcal{A}$, the Moschovakis extension $(A \times \mathbb{N})^*$ of $A \times \mathbb{N}$ (see Section 4.2). Baleva's jump of $\mathcal{A}$ adds to $(A \times \mathbb{N})^*$ a universal $\Sigma^{\mathtt{c}}_1$ set, very similar to the set $h(\vec{K}^{\mathcal{A}})$ mentioned in the Section 4.1. Baleva sets up her definition in the semi-lattice of countable structures over a fixed domain ordered by* **SC***-reducibility. This reducibility is essentially $\leq^{\mathcal{E}}_T$ as defined in 3.7 where $\mathcal{E}$ is the countable structure on the empty language, and all other structures are viewed as having the same domain as $\mathcal{E}$.*
- (J2) *Alexandra Soskova and Ivan Soskov [Sos07, SS09] worked with another map from structures to structures, which they did not call the jump of the structure until later papers. There, the domain of $\mathcal{A}'$ is the Moschovakis extension A^* of A, and the added relation is one that codes (the negation of) the forcing relation on Π_1 formulas, which is essentially $\{\langle \bar{p}, e, x\rangle \in A^\star \times \mathbb{N}^2 : \exists \bar{q} \in A^\star\ (\bar{q} \supseteq \bar{p}\ \&\ x \in W_e^{D(\bar{q})})\}$ following the notation of the proof of Theorem 3.14.*
- (J3) *Alexey Stukachev [Stu09, Stu10], who was working on the semi-lattice of structures of any size below an arbitrary cardinal, ordered under Σ-reducibility, provided a new definition of the jump of a structure in that setting. Stukachev was the first one to work with uncountable structures of any size. For him, the domain of $\mathcal{A}'$ is $\mathbb{HF}_{\mathcal{A}}$, and the added relation is the satisfaction relation for Σ_1-formulas, namely $hK(\mathcal{A})$. The importance of the role of Σ-reducibility in this context was previously shown by Stukachev in [Stu07, Stu08].*

  Another independent definition of the jump is due to Puzarenko [Puz09]. However, he only defines the jump of an admissible structure, and does not deal with structures in general. The definition is almost the same as that of Stukachev when one considers the jump of the admissible structure $\mathbb{HF}_{\mathcal{A}}$; although they only realized this afterwards. Puzarenko's definition generalized an earlier definition of jump by Morozov [Mor04], that only works for recursively listed admissible structures, *which are the admissible structures where there is a Σ-definable bijection between the whole domain and the class of ordinals.*
- (J4) *Montalbán [Mon09], independently of all this previous work, introduced yet another definition. The original definition in [Mon09] is not completely equivalent to the definitions above; it is what we call the* structural jump *in Section 6. In that definition, the domain of the jump of*

a structure $\mathcal{A}$ is the same as the domain of $\mathcal{A}$, and the language is extended with finitely or infinitely many new relations that, altogether, form a complete set of Π^c_1-relations.

5.1. **Comparing structures.** Working with structures as objects, we need to have a way of comparing the computational complexity of different structures. There is more than one way to do this. A measurement that is widely accepted is to say that a structure is more complicated than another if it is harder to compute, in the sense that it takes smarter oracles to produce a presentation of it. This reduction is more commonly used implicitly than explicitly as in the following definition.

Definition 5.3. Given two structures $\mathcal{A}$ and $\mathcal{B}$, we say that $\mathcal{A}$ is *Muchnik reducible* to $\mathcal{B}$, and write $\mathcal{A} \leq_w \mathcal{B}$, if

$$\forall X \subseteq \mathbb{N}, \quad X \text{ computes a copy of } \mathcal{B} \quad \Rightarrow \quad X \text{ computes a copy of } \mathcal{A},$$

or equivalently, if $Sp(\mathcal{B}) \subseteq Sp(\mathcal{A})$. The "$w$" stands for "weak," where the strong reduction is the Medvedev reduction which asks for a uniform way of computing a copy of $\mathcal{A}$ from a copy of $\mathcal{B}$.

This reduction defines a pre-ordering on the class of all countable structures, and hence an equivalence, $\equiv_w$, as usual.

In some cases this equivalence may be not appropiate, as it does not look too deeply into the model theoretic aspects of a structure. For example, any two computable structures are equivalent under $\equiv_w$, even if the structures are model-theoretically very different. The following reduction is nothing more than an effective version of the notion of interpretability used in model theory.

Definition 5.4. Let $\mathcal{A}$ be an $\mathcal{L}$-structure, and $\mathcal{B}$ be any structure, where $\mathcal{L} = \{P_0, P_1, ...\}$ and P_i has arity p_i.

We say that $\mathcal{A}$ is *effectively interpretable* in $\mathcal{B}$, and write $\mathcal{A} \leq_I \mathcal{B}$, if, for some $n \in \mathbb{N}$, in $\mathcal{B}$, there is a rice set $D \subseteq B^n$, a r.i. computable subset $\eta \subseteq B^n \times B^n$ which is an equivalence relation on D, and a r.i. computable sequence of sets $R_i \subseteq B^{n \cdot p_i}$, closed under the equivalence η within D, such that

$$(A; P_0^{\mathcal{A}}, P_1^{\mathcal{A}}, ...) \cong (D/\eta; R_0, R_1, ...).$$

The sets R_i do not need to be subsets of D^{p_i}, and, when we refer to the structure $(D/\eta; R_0, R_1, ...)$ we, of course, mean $(D/\eta; (R_0 \cap D^{p_0})/\eta, (R_1 \cap D^{p_1})/\eta, ...)$.

Being $\equiv_I$ is a very strong notion of equivalence, which reflects both effective and model-theoretic aspects of the structure. However, in some cases, the restriction of D to being a subset of B^n can be too strong. We already argued that, when studying rice relations, one needs to go beyond the study of subsets $D \subseteq B^n$. The following notion fixes this problem.

Definition 5.5. Consider structures $\mathcal{A}$ and $\mathcal{B}$ as above. We say that $\mathcal{A}$ is Σ-*reducible to* $\mathcal{B}$ if there is an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$ as in the previous definition, but allowing D to be a rice subset of $A^{<\omega} \times \mathbb{N}$, and η and $\vec{R}$ be r.i. computable relations on $A^{<\omega} \times \mathbb{N}$, of course viewed as elements of $\mathrm{RSeq}(\mathcal{A})$.

Observation 5.6. Consider structures $\mathcal{A}$ and $\mathcal{B}$ as a above. Then

$$A \leq_\Sigma \mathcal{B} \iff \mathcal{A} \leq_I \mathbb{HF}_{\mathcal{B}},$$

Theorem 5.7. *For any structures $\mathcal{A}$ and $\mathcal{B}$,*

(1) *$\mathcal{A} \leq_I \mathcal{B}$ implies $\mathcal{A} \leq_\Sigma \mathcal{B}$, and*
(2) *$\mathcal{A} \leq_\Sigma \mathcal{B}$ implies $\mathcal{A} \leq_w \mathcal{B}$.*

None of the implications above reverses.

Proof. None of the implications is hard to prove. To see that the first implication does not reverse consider $\mathcal{A} = (\omega; Adj)$, and $\mathcal{B} = \mathcal{E} = (B; \ )$ the countable infinite structure over the empty language. Then $\mathcal{A} \leq_\Sigma \mathcal{E}$ because $\mathcal{A}$ can be effectively interpreted in arithmetic, and hence $\mathcal{A}$ is $\leq_\Sigma$-reducible to any structure. On the other hand, it is not hard to prove that $\mathcal{A} \not\leq_I \mathcal{E}$, because the only rice subsets of B^n in $\mathcal{E}$ are either finite or co-finite.

There are various proofs that $\mathcal{A} \leq_w \mathcal{B}$ does not imply $\mathcal{A} \leq_\Sigma \mathcal{B}$. See for instance [Kal09, KP09, Stu07]. For one such example: Let $\mathcal{A}$ be the structure coding the family of the graphs of all computable functions, i.e., it is a family of sets of pairs, each set being the graph of a computable function. Let $\mathcal{B}$ be the family of all c.e. sets. Then, it is proved in [KP09] that $\mathcal{A} \leq_w \mathcal{B}$ but $\mathcal{A} \not\leq_\Sigma \mathcal{B}$. □

Historic Remark 5.8. *The notion of* Σ*-reducibility between abstract structures was introduced independently by Khisamiev* [Khi04] *and Stukachev in* [Stu07], *and it is based on the notion of* Σ*-definability of a structure inside an admissible set by Ershov. Stukachev* [Stu07, Stu08] *and Puzarenko* [Puz09, KP09] *showed the importance of this reducibility and compared it to various other reducibilities between structures (including* $\leq_w$*). The definitions of* [Khi04] *and* [Stu07] *used, of course, the notions of* Σ*-definability rather than rice. The equivalence with the definition given here follows from Theorems* 4.2 *and* 4.4*.*

5.2. **The three main theorems about the jump.**

We now state the three main results about the jump of structures.

5.2.1. *The first jump inversion theorem.* This theorem is a generalization of the Friedberg jump inversion theorem to the semi-lattice of structures under $\leq_\Sigma$-reducibility.

Theorem 5.9. *For every structure* $\mathcal{A}$ *which codes* $0'$ *(i.e.,* $\overrightarrow{0'}$ *is r.i. computable in* $\mathcal{A}$*), there is a structure* $\mathcal{C}$ *such that* $\mathcal{C}' \equiv_\Sigma \mathcal{A}$*.*

Historic Remark 5.10. *For the case of Muchnik equivalence, this theorem was proved independently in two occasions. One is due to Goncharov, Harizanov, Knight, McCoy, R. Miller and Solomon in 2005* [GHK+05, Lemma 5.5 for $\alpha = 2$]. *They do not state their result in terms of the jump of a structure, and they only prove the theorem for graphs, but any degree spectrum can be realized as the degree spectrum of a graph. They prove inversion, not only for a single jump, but also for the* α*th jump, for every* $\alpha < \omega_1^{CK}$*. They proved this result as a tool to get get other results about* Δ^0_α *categoricity and intrinsic* Σ^0_α *relations. Their* α*-jump inversion theorem was used in other occasions, as, for instance, by Greenberg, Montalbán and Slaman* [GMS] *to build a structure whose spectrum is exactly the non-hyperarithmetic degrees.*

The other proof of Theorem 5.9 *for* $\equiv_w$ *is due to Alexandra Soskova* [Sos07, SS09]. *Her construction is quite different and uses Marker extensions. The inspiration to use Marker's extension in this context came from Goncharov and Khoussainov* [GK04]. *Furthermore, she proved a relativized version of the theorem, where the relativization is to structures. That is, she proved that if* $\mathcal{A} \geq_w \mathcal{B}'$*, then there is a structure* $\mathcal{C} \geq_w \mathcal{B}$ *such that* $\mathcal{A} \equiv_w \mathcal{C}'$*.*

Stukachev [Stu10, Stu] *then proved this theorem for the stronger notion of* Σ*-equivalence, as in the statement of the theorem above, and for structures of arbitrary cardinality. Stukachev used the same structure Soskova used, although the proof that it works for* Σ*-equivalence is completely different and more model theoretic.*

5.2.2. *The second jump inversion theorem.* This second jump inversion theorem is not a generalization of the usual jump inversion theorem to a more general class of degrees, but a generalization in the sense that given $X \subseteq \mathbb{N}$ it gives $Y \subseteq \mathbb{N}$ with $Y' \equiv_T X$ and some extra properties.

Theorem 5.11. *If* X *computes a copy of* $\mathcal{A}'$*, then there there is a set* Y*, with* $Y' \equiv_T X$*, which computes a copy of* $\mathcal{A}$*. Equivalently, this says that*

$$Sp(\mathcal{A}') = \{\mathbf{y}' : \mathbf{y} \in Sp(\mathcal{A})\}.$$

This theorem is proved by showing that any copy of $\mathcal{A}'$ can compute a 1-generic copy $\mathcal{B}$ of $\mathcal{A}$, and that, for such a copy, $D(\mathcal{B})' \leq_T D(\mathcal{A}')$.

The equivalent formulation in the second sentence of the theorem can be viewed as a *correctness* statement: it reaffirms that our definition of the jump is an analog of the usual definition of the Turing jump on sets of numbers.

Corollary 5.12 ([Mon09, Theorem 3.5]). *The following are equivalent:*

(1) *$\mathcal{A}$ has the* low property*: If $X' \equiv_T Y'$ and X computes a copy of $\mathcal{A}$, then so does Y.*
(2) *$\mathcal{A}$* admits strong jump inversion*: If X' computes a copy of $\mathcal{A}'$, then X computes a copy of $\mathcal{A}$.*

Historic Remark 5.13. *Theorem 5.11 was first introduced by Soskov at a talk in the LC'02 in Munster; a full proof then appeared in* [SS09]. *That proof just shows the existence of a structure $\mathcal{B}$ with $Sp(\mathcal{B}) = \{\mathbf{y}' : \mathbf{y} \in Sp(\mathcal{A})\}$; this structure $\mathcal{B}$ was only later called the jump of $\mathcal{A}$, and it is notion J2 in page 11. Theorem 5.11 was first stated in print in Baleva* [Bal06, Theorem 5], *but the proof there is not complete (Lemma 4 is not correct).*

Another proof of the theorem then appeared independently in Montalbán [Mon09]. *Both proofs are essentially the same, although the great differences in the underlying setting make them look quite different.*

5.2.3. *The fixed point theorem.* The third theorem says that the jump operation on structures does not always jump.

Theorem 5.14 (Puzarenko [Puz11], Montalbán [Monb]). *There is a structure $\mathcal{A}$ such that $\mathcal{A} \equiv_I \mathcal{A}'$.*

Historic Remark 5.15. *Puzarenko's and Montalbán's proofs were found independently . Montalbán uses the existence of $0^\sharp$, and is a paragraph long once the definition of $0^\sharp$ is understood. Puzarenko's proof works inside ZFC, but is much more complicated. Both proofs work by building an ill-founded ω-model $\mathcal{A}$ of $V = L$ where for some ordinal α of the model, $(L_\alpha)^{\mathcal{A}} \cong \mathcal{A}$.*

More surprising than the theorem itself, is the complexity necessary for its proof.

Theorem 5.16 (Montalbán [Monb]). *Higher-order arithmetic cannot prove that there exists a structure $\mathcal{A}$ with $\mathcal{A} \equiv_w \mathcal{A}'$ (i.e. $Sp(\mathcal{A}) = Sp(\mathcal{A}')$). Higher-order arithmetic refers to the union of nth-order arithmetic for all $n \in \mathbb{N}$.*

One of the main steps to prove this theorem is to show that if $\mathcal{A} \equiv_w \mathcal{A}'$, then the co-spectrum of $\mathcal{A}$ (i.e., $\{X \subseteq \mathbb{N} : \vec{X} \text{ is rice in } \mathcal{A}\}$) is the second-order part of an ω-model of full second-order arithmetic. Generalizing this to higher orders, the author proves that the ω-jump of any presentation of $\mathcal{A}$ computes a countably-coded ω-model of higher-order arithmetic.

Puzarenko's proof of Theorem 5.14 uses KP plus $\omega_1^{CK}+1$ iterations of the Power-set axiom. So there is still a gap as to what is actually needed to prove the fixed point theorem.

6. The structural jump

The author's original definition of the jump of a structure [Mon09], which is different than the one we gave here, uses complete set of $\Pi_1^{\mathtt{c}}$-relations. Complete set of $\Pi_1^{\mathtt{c}}$-relations, or by taking complements, of rice relations, provide the link between the formal definition of the jump of a relation and its applications on concrete natural structures.

6.1. **Complete sets of rice relations.** The author's original definition [Mon09, Definition 1.1] is given as part (2) of Lemma 6.3 below. The following definition is equivalent:

Definition 6.1. A sequence of relations $\vec{R}$ is *structurally rice complete* if it is rice and
$$\vec{\varnothing}'_{\mathcal{A}} \equiv_T^{\mathcal{A}} \vec{R} \oplus \overrightarrow{0'}.$$
More generally, $\vec{R}$ is *structurally* Σ^{c}_n *(*Π^{c}_n*) complete* if it is Σ^{c}_n (Π^{c}_n) and
$$\vec{\varnothing}^{(n)}_{\mathcal{A}} \equiv_T^{\mathcal{A}} \vec{R} \oplus \overrightarrow{0^{(n)}}.$$

In [Mon09, Definition 1.1], instead of saying that $\vec{R}$ is structurally rice complete, we said that $\vec{R}$ is a complete sets of rice relations. The new notation is more accurate, but, as an abuse of notation, we will still use the old notation sometimes. The word "structurally" reflects that $\vec{R}$ is complete in the sense that it has all the structural information of $\vec{\varnothing}'_{\mathcal{A}}$, but it may need to borrow other information from $\overrightarrow{0'}$ to be actually equivalent to $\vec{\varnothing}'_{\mathcal{A}}$. Also, it is important that $\vec{R}$ is a sequence and not just a set, because the $\equiv_T^{\mathcal{A}}$-equivalence asks for uniform reductions between sequences.

Example 6.2. From Examples 3.22 and 3.23, we get that $\neg Adj$ alone, and $\vec{LD}$ are structurally rice complete for linear orderings and vector spaces respectively. More examples are given in Section 7.

The following lemma shows how having a simple complete set of rice relations on a structure $\mathcal{A}$ can be very useful: first because it gives a simple characterization of all the r.i. Σ^0_2 relations in $\mathcal{A}$, and second because it can be used to build copies of $\mathcal{A}$.

Lemma 6.3. *Let $\vec{R}$ be a finite or infinite Σ^{c}_n sequence of relation on $\mathcal{A}$. The following are equivalent:*

(1) *$\vec{R}$ is structurally Σ^{c}_n complete.*
(2) *Every $\Sigma^{\mathsf{c}}_{n+1}$ formula $\psi(\bar{x})$ about $\mathcal{A}$ is equivalent to a $0^{(n)}$-computable disjunction of finitary Σ_1 formulas about $(\mathcal{A}, \vec{R})$, and this equivalent disjunction can be found uniformly in ψ.*
(3) *If $X \geq_T 0^{(n)}$ computes a copy of $(\mathcal{A}, \vec{R})$, then there exists Y with $Y^{(n)} \equiv_T X$ which computes a copy of $\mathcal{A}$, and, furthermore, X computes an isomorphism between $\mathcal{A}$ and its copy.*

Sketch of the Proof. That (2) implies (1) follows from the fact that both $\vec{\varnothing}^{(n)}_{\mathcal{A}}$ and $\neg\vec{\varnothing}^{(n)}_{\mathcal{A}}$ are $\Sigma^{\mathsf{c}}_{n+1}$ definable. To prove that (1) implies (2) one needs to use, first, that every $\Sigma^{\mathsf{c}}_{n+1}$ formula about $\mathcal{A}$ is equivalent to a Σ^{c}_1 formula about $\mathcal{A}^{(n)} = (\mathcal{A}, \vec{\varnothing}^{(n)}_{\mathcal{A}})$, and, second, that $\vec{\varnothing}^{(n)}_{\mathcal{A}}$ is both Σ^{c}_1- and Π^{c}_1-definable in $(\mathcal{A}, \vec{R}, \overrightarrow{0^{(n)}})$, to get that every $\Sigma^{\mathsf{c}}_{n+1}$ formula is equivalent to a Σ^{c}_1 formula about $(\mathcal{A}, \vec{R}, \overrightarrow{0^{(n)}})$. Such Σ^{c}_1 formula is equivalent to a $0^{(n)}$-computable disjunction of finitary Σ_1 formulas about $(\mathcal{A}, \vec{R})$.

That (1) implies (3) follows from Theorem 5.11 iterated n times.

Let us now assume that (3). To show that (1) holds, we need to show that for every copy $\mathcal{B}$ of $\mathcal{A}$, $\vec{R}^{\mathcal{B}} \oplus 0^{(n)}$ computes $\vec{\varnothing}^{(n)}_{\mathcal{B}}$. Let X compute $(\mathcal{B}, \vec{R}^{\mathcal{B}}, \overrightarrow{0^{(n)}})$. By (3), there exists Y, with $Y^{(n)} \equiv_T X$, which computes a copy $\mathcal{C}$ of $\mathcal{B}$ and X computes the isomorphism. Then X computes $\vec{\varnothing}^{(n)}_{\mathcal{C}}$, and through the isomorphism, it computes $\vec{\varnothing}^{(n)}_{\mathcal{B}}$. □

The following particular case of the previous lemma was independently proved by Frolov.

Corollary 6.4 ([Fro10, Theorem 6])**.** *If $\mathcal{A}$ is a linear ordering and $(\mathcal{A}, Adj)$ has a $0'$-computable copy, then $\mathcal{A}$ has a low copy.*

It is not always the case that there is a nice complete set of rice relations. Conditions for when this is the case, under a certain interpretation of "nice," are given in [Mona]. There, the author studies sequences of $\Pi^{\mathtt{c}}_n$ *formulas* which define complete sets of Π_n relations for all the structures inside a class $\mathbb{K}$, and that, furthermore, are complete relative to any oracle. It is proved in [Mona] that such a set of formulas exists if and only if the number of n-back-and-forth types of tuples in structures in $\mathbb{K}$ is countable, and some mild effectiveness conditions hold on these types.

6.2. **The author's original definition of jump.** The definition given in this paper is better than the one in [Mon09] in the sense that it matches the other definitions of jump. However, the definition from [Mon09] has the advantage that it is more practical, and aesthetic, when looking at particular examples of jumps.

Definition 6.5. A *structural jump* of $\mathcal{A}$ is a structure of the form $(\mathcal{A}, \vec{R})$ where $\vec{R} \in \mathrm{RSeq}(\mathcal{A})$ is structurally rice complete.

Notice that $\mathcal{A}'$, as defined in 5.1, is a structural jump of $\mathcal{A}$, and that the only essential difference between $\mathcal{A}'$ and any other structural jump $\hat{\mathcal{A}}$ of $\mathcal{A}$ is that $\mathcal{A}'$ codes the information sequence $\overrightarrow{0'}$ while $\hat{\mathcal{A}}$ might not. This difference is not *structural*; it just involves information unrelated to the structure $\mathcal{A}$.

The following examples show how simple the structural jump of a structure can be. More examples are given in the next section.

Example 6.6. If $\mathcal{A}$ is a linear ordering, then $(\mathcal{A}, \mathit{Adj})$ is a structural jump of $\mathcal{A}$. If $\mathcal{A}$ is a $\mathbb{Q}$-vector space, $(\mathcal{A}, \vec{LD})$ is a structural jump of $\mathcal{A}$. If $\mathcal{A}$ is a Boolean algebra, $(\mathcal{A}, \mathit{atom})$ is a structural jump of $\mathcal{A}$, and $(\mathcal{A}, \mathit{atom}, \mathit{inf}, \mathit{atomless})$ is a structural double-jump of $\mathcal{A}$.

7. Finite complete sets of rice relations

In this last section we look at the following question: For which structures, and $n \in \mathbb{N}$, is there a finite structurally $\Sigma^{\mathtt{c}}_n$ complete set of relations? We do not have a general answer for this question; but we have some interesting examples.

7.1. **Linear orderings.** As we said a few times already, for linear orderings we do have a finite complete set of rice relations, namely the singleton $\{\neg \mathit{Adj}\}$, and a proof of this can be found in [Mon09]. Linear orderings also enjoy a nice structural double jump. The following is a complete set of $\Pi^{\mathtt{c}}_2$ relations for linear orderings with endpoints:

(1) $\mathit{Adj}(x, y)$;
(2) $\mathit{limleft}(x)$; where $\mathit{limleft}(x)$ holds if x is a limit from the left, that is, if $\exists y < x \ \&\ \forall y < x \exists z\ (y < z < x)$.
(3) $\mathit{limright}(x)$; where $\mathit{limright}(x)$ holds if x is a limit from the right, that is, if $\exists y > x \ \&\ \forall y > x \exists z\ (x < z < y)$.
(4) $D_n(x, y)$ for $n \geq 1$; where $D_n(x, y)$ holds if there is no string of $n+1$ adjacent elements somewhere between x and y, that it, if $x < y \ \&\ \neg \exists z_0, ..., z_n\ (x \leq z_0 < \cdots < z_n \leq y \ \&\ \bigwedge_{i<n} \mathit{Adj}(z_i, z_{i+1}))$.

That these form a complete set of $\Pi^{\mathtt{c}}_2$ relations follows from work of Frolov [Fro10, Theorem 7] who proved that part (3) of Lemma 6.3 holds for these relations. In the case when the linear ordering has no end points the situation is not much different; we just need to consider extra relations $D_{n,+\infty}(x)$ and $D_{n,-\infty}(x)$ which are like $D_n(\cdot, \cdot)$, but look at end and initial segments, $(x, +\infty)$ and $(-\infty, x)$, respectively.

This is a nice set of relations, but it is not finite.

Theorem 7.1 (Knight, R. Miller, Montalbán, Soskov, A. Soskova, M. Soskova, VanDen-Driessche, and Vatev; unpublished)**.** *There is a finite complete set of Π^c_2 relations for linear orderings with end points. These relations are:*

- $Adj(x, y)$,
- $limleft(x)$,
- $limright(x)$,
- $P(x, y, z, w) \equiv \bigwedge_{n \in \mathbb{N}} (Succ^n(z) = w \to D_{n+1}(x, y))$, *where* $Succ^n(z) = w$ *is a shorthand for* $\exists z_0, ..., z_n \ (z = z_0 < z_1 < \cdots < z_n = w \ \& \ \bigwedge_{i<n} Adj(z_i, z_{i+1}))$.

The theorem also holds for linear orderings without end points, although one needs to consider a couple added relations.

Proof. Let $\mathcal{A}$ be a presentation of a linear orderings with end points. We need to show that the relations D_n are uniformly computable in the relations above. Fix n. If $n = 1$, then $D_1(x, y) \equiv P(x, y, x, x)$. Suppose now that $n > 1$, and consider $x, y \in A$. Suppose that, recursively, we know whether or not $D_n(x, y)$ holds, and we want to check if $D_{n+1}(x, y)$ holds. If $D_n(x, y)$ holds, then we know that $D_{n+1}(x, y)$ holds too. Suppose now that $\neg D_n(x, y)$ holds. Then, we can search for z, w such that $Succ^n(z) = w$ and we have that $P(x, y, z, w) \iff D_{n+1}(x, y)$. □

7.2. **Boolean algebras.** Harris and Montalbán [HM] proved that, for all $n \in \mathbb{N}$, there is a finite structurally Π^c_n-complete set of relations. They describe a recursive procedure to build these sets of relations, but the construction is too involved to give here. Up to Boolean combination, and for the cases $n = 1, 2, 3, 4$, relations that are structurally Π^c_n complete for Boolean algebras were considered by Downey and Jockusch [DJ94] for $n = 1$, Thurber [Thu95] for $n = 2$, and Knight and Stob [KS00] for $n = 3, 4$. They did not mention their completeness, but they proved that (3) of Lemma 6.3 holds. Moreover, they showed that Boolean algebras have the low$_4$ property by showing that they admit strong 4th jump inversion (see Corollary 5.12).

7.3. **Vector spaces.** As we mentioned before, $\vec{LD}$ is structurally rice complete for $\mathbb{Q}$-vector spaces. This is again a very nice sequence of relations that is not finite.

Theorem 7.2 (Knight, R. Miller, Montalbán, Soskov, A. Soskova, M. Soskova, VanDen-Driessche, and Vatev; unpublished)**.** *Let $\mathcal{V}$ be the infinite dimensional countable $\mathbb{Q}$-vector space. There is no finite complete set of rice relations in $\mathcal{V}$.*

Sketch of the Proof. The proof has two parts. First, we give a different, more hands on, proof that $\vec{LD}$ is structurally rice complete. A proof that lets us observe that every rice relation $R(\bar{x})$ is $\leq^{\mathcal{V}}_T$-reducible to $0'$ and finitely many instances of $\vec{LD}$. Let n be the arity of R and let $(v_1, ...v_n) \in \mathcal{V}^n$. Then, using $(LD_2, ..., LD_n)$ we can find out all the linear dependencies among the vectors $v_1, ..., v_n$. So we can find a linearly independent subset of $v_1, ..., v_n$ which generates the rest, and then we can search for the equations witnessing these dependencies. Let us now assume that the v_i's are all linearly independent, as otherwise we can reduce the problem to a smaller set. By standard arguments using disjunctive normal forms one can show that every finitary Σ_1 formula about $v_1, ..., v_n$ can be effectively decided by solving systems of linear equations and inequations, and hence $0'$ can then decide infinitary Σ^c_1 formulas about $v_1, ..., v_n$ in a uniform way.

Second, we show that no finite sequence $(LD_2, ..., LD_{n-1})$ can $\leq^{\mathcal{V}}_T$-compute the whole sequence $\vec{LD}$. We build a copy $\mathcal{A}$ of $\mathcal{V}$ where $(LD_2, ..., LD_{n-1})$ is computable, but $\vec{LD}$ is not. We will define a computable subspace W of $\mathbb{Q}^\omega$ and then let $\mathcal{A}$ be the quotient $\mathbb{Q}^\omega / W$.

Let $b_0, b_1, ...$ be the standard basis of $\mathbb{Q}^\omega$. To make sure LD_n is not computable we will let $LD_n(b_{kn}, ..., b_{k(n+1)-1})$ hold in $\mathcal{A}$ if and only if $k \in 0'$. Given $s \in \mathbb{N}$, let A_s be the set of all $v \in \mathbb{Q}^\omega$ which are a linear combinations of $b_0, ..., b_s$ using coefficients among the first s rational numbers. At stage s of the construction, if we see that k enters $0'$ we enumerate a non-trivial linear combination of $b_{kn}, ..., b_{k(n+1)-1}$ to W without adding to W any vector of A_s that was not in W already, and without adding any new dependence between any set of vectors of A_s of size less than n. Proving that this can be done requires a little linear algebra, and proving that W and $LD_2, ..., LD_{n-1}$ end up being computable is a rather standard argument (for a fully spelled out argument of a similar kind, see [DHK+07]).

Relativizing this proof to $0'$, we get a $0'$-computable copy of $\mathcal{V}$, where $(LD_2, ..., LD_{n-1}) \oplus \overrightarrow{0'}$ is $0'$-computable, but $\vec{LD}$ computes $0''$. Thus $\vec{LD}$ is not r.i. computable in $(LD_2, ..., LD_{n-1}) \oplus \overrightarrow{0'}$.

Finally, if $R_1, ..., R_m$ were structurally rice complete, then by the first part of the argument we would have that finitely may of the LD_i would also be rice complete, but the second part shows this is not the case. □

7.4. **Equivalence structures.** An *equivalence structure* is a structure with a binary relation E which is an equivalence relation. For equivalence structures, the following is a complete set of rice relations:

(1) $F_k(x)$ for $k \in \mathbb{N}$, where $F_k(x)$ holds if there are $\geq k$ elements equivalent to x, and
(2) the information sequence $\vec{G}$ (called the *character* of E), where
$G = \{\langle n, k\rangle \in \mathbb{N}^2 :$ there are $\geq n$ equivalence classes with $\geq k$ elements$\}$.

Fix an equivalence structure $\mathcal{A}$, and a rice relation R of arity n. We will show how we can uniformly compute $R^{\mathcal{A}}$ using $\vec{F}^{\mathcal{A}}$, $\vec{G}^{\mathcal{A}}$ and $0'$. Let $(v_1, ...v_n) \in \mathcal{A}^n$. Using $(F_2, ..., F_n)$, find which of the v_i's are equivalent to which. Let us assume they are all nonequivalent, as otherwise we can reduce the problem to a maximal subset of nonequivalent v_i's. Given $\bar{k} \in \mathbb{N}^n$, let $F_{\bar{k}}(\bar{x}) = \bigwedge_{i=1}^n F_{k_i}(x_i)$. A standard argument shows that every finitary Σ_1 formula about $v_1, ..., v_n$ is equivalent to a disjunction of formulas of the form $F_{\bar{k}}(\bar{v})$ in conjunction with some formulas of the form "$(m, k) \in G$." So, using G, we can effectively transform every Σ_1^{c} formula into a disjunction of formulas $F_{\bar{k}}(\bar{v})$ for $\bar{k}$ in some computable set $C \subseteq \mathbb{N}^n$. Define the following ordering on $\mathbb{N}^n$: $\bar{k} \preccurlyeq \bar{l}$ if for all $i = 1, ..., n$, $k_i \leq l_i$. Note that if $\bar{k} \preccurlyeq \bar{l}$, then $F_{\bar{k}}(\bar{v}) \vee F_{\bar{l}}(\bar{v}) \equiv F_{\bar{k}}(\bar{v})$. It is not hard to show (by induction on n) that $(\mathbb{N}^n, \preccurlyeq)$ is a well-quasi ordering, that is, that for every sequence $\{\bar{k}_i : i \in \mathbb{N}\}$, there is $i < j$ with $\bar{k}_i \preccurlyeq \bar{k}_j$. Well-quasi orderings have the property that for every set of $C \subseteq \mathbb{N}^n$ there is a finite subset $C_0 \subseteq C$ such that every element of C is $\preccurlyeq$-above some element of C_0. It follows that $\bigvee_{\bar{k} \in C} F_{\bar{k}}(\bar{v}) \equiv \bigvee_{\bar{k} \in C_0} F_{\bar{k}}(\bar{v})$. Let us note that $0'$ is necessary to find C_0 from C. So, we have shown that every Σ_1^{c} formula can be re-written, with the help of $0'$ and G, as a finite disjunction of the form $\bigvee_{\bar{k} \in C_0} F_{\bar{k}}(\bar{v})$.

Theorem 7.3 (Knight, R. Miller, Montalbán, Soskov, A. Soskova, M. Soskova, VanDenDriessche, and Vatev; unpublished)**.** *Let $\mathcal{A}$ be an equivalence structure which has one equivalence class of each finite size. There is no finite complete set of rice relations in $\mathcal{A}$.*

Sketch of the Proof. First, observe that every rice relation $R(\bar{x})$ is $\leq_T^{\mathcal{A}}$-reducible to $0'$ and finitely many instances of $\vec{F}$, noting that G is computable in this case.

Second, we show that no finite sequence $(F_2, ..., F_{n-1})$ can $\leq_T^{\mathcal{A}}$-compute the whole sequence $\vec{F}$. We build a copy $\mathcal{B}$ of $\mathcal{A}$ where $(F_2, ..., F_{n-1})$ are computable, but $\vec{F}$ is not. Suppose n is even; if not take $n+1$ as n. Start by building a structure with equivalence classes of al even sizes and all sizes less than n; we will build the odd size classes beyond n by stages. Let b_k be a fixed element in a class of size $2k + n$. If k enters $0'$, we will make b_k's equivalence class bigger, so asking whether $F_{2k+n+1}(b_k)$ holds will tell us if $k \in 0'$. At each stage build one new odd size equivalence class. If k enters $0'$ add elements to b_k's equivalence class to make it have some large odd size not considered yet, and also make a new equivalence class of size $2k+n$. Notice that the value of F_i for $i < n$ is never changed during the construction, and hence these relations are all computable.

Relativize this proof to $0'$ to obtain a $0'$-computable copy of $\mathcal{A}$ where $(F_2, ..., F_{n-1}) \oplus \overrightarrow{0'}$ is $0'$-computable but $\vec{F}$ computes $0''$.

Finally, if there was a finite complete set of rice relations, we would get that a finite set of the F_k's would also make a complete set, but we just proved this can not be the case. □

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