THERE IS NO ORDERING ON THE CLASSES IN THE GENERALIZED HIGH/LOW HIERARCHIES.

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Abstract. We prove that the existential theory of the Turing degrees, in the language with Turing reduction, 0, and unary relations for the classes in the generalized high/low hierarchy, is decidable.

1. Introduction

The high/low hierarchy was introduced by Soare in [Soa74] and independently by Cooper in a preprint of [Coo74] with the idea of classifying the Turing degrees below $0'$ depending on how close they are to being recursive and how close they are to being complete. This classification has been very helpful in the study of the structure of the Δ^0_2 Turing degrees. A generalization of this classification to all the Turing degrees is the generalized high/low hierarchy introduced by Jockusch and Posner in [JP78]. Many properties have been proved about members of certain classes in this hierarchy. To cite a few: every 1-generic set is GL_1 (see [Ler83, IV.2]); every minimal degree is GL_2 [JP78]; every non-GL_2 cups to every degree above it [JP78]; every GH_1 degree bounds a minimal degree [Joc77] but not every GH_2 [Ler86]; every GH_1 degree has the complementation property [GMS].

In [Ler85], Lerman proved that the $\exists$-theory of the Turing degrees in the language $\mathcal{L}_H$, which has a relation for the Turing reduction, constants for 0 and $0'$, and one unary relation for each class in the high/low hierarchy, is decidable. In that paper he leaves as an open question the decidability of the $\exists$-theory of the Turing Degrees in the language with predicates for the classes in the generalized high/low hierarchy. We prove here that $\exists$-theory of the Turing Degrees in the language $\mathcal{L}_{\mathrm{GH}}$, which has relations for the classes in the generalized high/low hierarchies instead of the high/low hierarchy and does not have a constant for $0'$ is decidable. The language $\mathcal{L}_{\mathrm{GH}}$ does not contain a relation symbol for GH_0 ($\mathbf{x} \in \mathrm{GH}_0 \iff \mathbf{x} \geq 0'$), and whether the $\exists$-theory of the Turing Degrees in the language $\mathcal{L}_{GH_0}$ with a symbol for GH_0 is decidable or not is unknown. A proof of this decidability result would probably use different techniques than ours.

The result we are proving, as Lerman's, is also interesting because it helps to understand how the degrees from the various classes of the generalized high/low hierarchy are located in the poset of the Turing Degrees. To prove it we show that every finite poset labeled with elements of $\mathcal{G}^*$, satisfying certain trivial conditions, can be embedded in the Turing Degrees. $\mathcal{G}^*$ is the partition of $\mathcal{D}$ induced by the generalized high/low hierarchy (see Definition 1.1). The proof is divided into two parts. In section 2 we analyze the problem and reduce it to a technical proposition

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which is an extension of Harrington's ZBC Lemma. One of the main tools in simplifying the problem is Lerman's Bounding Lemma [Ler85, 2.8]. We prove our technical result in section 3.

We have to note that the decidability of the $\exists$-theory of the Turing degrees in the language $\mathcal{L}_{GH}$, without a symbol for $\mathrm{GI}(\cdot)$, would follow from the decidability of the $\exists$-theory of $\langle \mathbf{D}, \leq, \vee, ', 0\rangle$. But this problem is still open. Another observation is that the $\exists$-theory of the Turing degrees in the language which has a relation for the Turing reduction and one binary relation for each class in the Relativized generalized high/low hierarchy (but does not have a constant for 0), is decidable. If we remove the symbol $\mathrm{GI}(\cdot,\cdot)$ from the language, this follows from the decidability of the $\exists$-theory of $\langle \mathbf{D}, \leq, \vee, '\rangle$, which was proved in [Mon03]. Otherwise we have to use that every countable Jump upper semilattice can be embedded in the Turing degrees, which was also proved in [Mon03].

Basic Notions. Define

$$\mathcal{C} = \{L_1, L_2, ...\} \cup \{I\} \cup \{H_1, H_2, ...\},$$

where L_n is the class of low_n degrees, I the class of intermediate degrees, and H_n the class of high_n degrees. A degree $\mathbf{x} \leq 0'$ is *low_n* if $\mathbf{x}^{(n)} = 0^{(n)}$, is *$high_n$* if $\mathbf{x}^{(n)} = 0^{(n+1)}$, and is *intermediate* if $\forall n\big(0^{(n)} <_T \mathbf{x}^{(n)} <_T 0^{(n+1)}\big)$. Note that for all n, $L_n \subseteq L_{n+1}$, $H_n \subseteq H_{n+1}$, and L_n, H_n and I are disjoint. These classes induce a partition, $\mathcal{C}^*$, of the degrees $\leq 0'$.

$$\mathcal{C}^* = \{L_1^*, L_2^*, ...\} \cup \{I^*\} \cup \{H_1^*, H_2^*, ...\},$$

where $L_1^* = L_1$, $H_1^* = H_1$, $I^* = I$ and for $n > 1$, $L_n^* = L_n \smallsetminus L_{n-1}$, and $H_n^* = H_n \smallsetminus H_{n-1}$. We define an ordering, $\prec$, on $\mathcal{C}^*$ as follows:

$$L_1^* \prec L_2^* \prec \cdots \prec I^* \prec \cdots \prec H_2^* \prec H_1^*.$$

Observe that if $\mathbf{x} \leq \mathbf{y}$, $\mathbf{x} \in X \in \mathcal{C}^*$ and $\mathbf{y} \in Y \in \mathcal{C}^*$, then $X \preceq Y$. Let $\mathcal{L}_H$ be the first order language with a binary relation $\leq$, two constant symbols 0 and $0'$, and an unary relation for each class in $\mathcal{C}$. Lerman proved that every existential formula of $\mathcal{L}_H$ which is consistent with the observation above, and consistent with the axioms of partial orderings with bottom and top elements, 0 and $0'$, is true about the degrees below $0'$, and also about the r.e. degrees.

As a generalization of these notions to all the Turing degrees we get the *generalized high/low hierarchy.*

Definition 1.1. For $n \geq 1$ we say that a degree $\mathbf{x}$ is *generalized low_n*, or GL_n, if $\mathbf{x}^{(n)} = (\mathbf{x} \vee 0')^{(n-1)}$. We say that a degree $\mathbf{x}$ is a *generalized $high_n$ degree*, or GH_n, if $\mathbf{x}^{(n)} = (\mathbf{x} \vee 0')^{(n)}$, and it is *generalized intermediate*, or GI, if $\forall n\big((\mathbf{x} \vee 0')^{(n-1)} <_T \mathbf{x}^{(n)} <_T (\mathbf{x} \vee 0')^{(n)}\big)$. Let

$$\mathcal{G} = \{\mathrm{GL}_1, \mathrm{GL}_2, ...\} \cup \{\mathrm{GI}\} \cup \{\mathrm{GH}_1, \mathrm{GH}_2, ...\}.$$

and

$$\mathcal{G}^* = \{\mathrm{GL}_1^*, \mathrm{GL}_2^*, ...\} \cup \{\mathrm{GI}^*\} \cup \{\mathrm{GH}_1^*, \mathrm{GH}_2^*, ...\},$$

where $\mathrm{GL}_1^* = \mathrm{GL}_1$, $\mathrm{GH}_1^* = \mathrm{GH}_1$, $\mathrm{GI}^* = \mathrm{GI}$ and for $n > 1$, $\mathrm{GL}_n^* = \mathrm{GL}_n \smallsetminus \mathrm{GL}_{n-1}$, and $\mathrm{GH}_n^* = \mathrm{GH}_n \smallsetminus \mathrm{GH}_{n-1}$. Let $\mathcal{L}_{\mathrm{GH}}$ be the first order language with a binary relation $\leq$, a constant symbol 0, and an unary relation for each class in $\mathcal{G}$.

We make two observations. The first one is that, as in the high/low hierarchy, for all n, $\mathrm{GL}_n \subseteq \mathrm{GL}_{n+1}$, $\mathrm{GH}_n \subseteq \mathrm{GH}_{n+1}$, and GL_n, GH_n and GI are disjoint. The second one is that 0 is GL_1. We will prove that every existential formula of $\mathcal{L}_{\mathrm{GH}}$ which is consistent with the observations above, and consistent with the axioms of partial orderings with bottom element 0, is true about the Turing degrees.

By relativizing these notions we get the *Relativized generalized high/low hierarchy.* We say that $\mathbf{a}$ is GH_n *relative to* $\mathbf{b}$, and we write $\mathbf{a} \in \mathrm{GH}_n(\mathbf{b})$ (or $\mathrm{GH}_n(\mathbf{a}, \mathbf{b})$) if $\mathbf{a} \geq_T \mathbf{b}$ and $\mathbf{a}^{(n)} = (\mathbf{a} \vee \mathbf{b}')^{(n)}$. Analogously we can define $\mathrm{GL}_n(\mathbf{b})$ and $\mathrm{GI}(\mathbf{b})$.

2. GH-posets.

In this section we show how to use our technical result, which extends Harrington's ZBC lemma, to prove our main result. (Harrington's ZBC lemma will be stated, and its extension will be proved, in the next section.) First we define GH-posets as the generalized version of Lerman's H-posets (see [Ler85]).

Definition 2.1. A *GH-poset* is a structure $\mathcal{P} = \langle P, \leq, 0, \mathrm{GL}_1, \mathrm{GL}_2, ..., \mathrm{GI}, ..., \mathrm{GH}_1 \rangle$ where $\langle P, \leq \rangle$ is a partial ordering, $0 \in P$ and GL_1, GL_2,...,GI,...,GH_1 are unary relations such that

- for all n, GL_n, GI and GH_n are mutually disjoint,
- for all n, $GL_n \subseteq GL_{n+1}$ and $GH_n \subseteq GH_{n+1}$,
- 0 is the least element of $\mathcal{P}$, and
- $\mathrm{GL}_1(0)$ holds.

For $C \in \mathcal{G}^*$ and $x \in P$ we define $C(x)$ in the obvious way. A GH-poset $\mathcal{P}$ is *standard* if for all $x \in P$, there is a $C \in \mathcal{G}^*$ such that $C(x)$. A standard GH-poset can be represented as a quadruple $\langle P, \leq, 0, f \rangle$ where $f\colon P \to \mathcal{G}^*$ takes $x \in P$ to the unique $C \in \mathcal{G}^*$ such that $C(x)$. Note that every GH-poset can be extended to a standard GH-poset on the same universe.

Of course, the main example that we are interested in is the GH-poset of the Turing Degrees.

Theorem 2.2. *The existential theory of*

$$\mathcal{D} = \langle \mathbf{D}, \leq_T, 0, \mathrm{GL}_1, \mathrm{GL}_2, ..., \mathrm{GI}, ..., \mathrm{GH}_2, \mathrm{GH}_1 \rangle$$

is decidable.

Proof. From the following proposition we get that an existential formula about $\mathcal{D}$ is true if and only if it does not contradicts the definition of GH-poset. It is not hard to show that one can check that effectively. □

Proposition 2.3. *Every finite GH-poset can be embedded into $\mathcal{D}$. (Of course via a GH-poset embedding.)*

Proof. Let $\mathcal{P} = \langle P, \leq, ... \rangle$ be a finite GH-poset. Without lost of generality we can assume that $\mathcal{P}$ is standard. The following lemma will allow us to consider only standard GH-posets where all the elements are either GL_1 or GH_1.

Bounding Lemma (Lerman, [Ler85, 2.8])**.** *Let $\mathbf{a} \notin \mathrm{GL}_2$ be given, and fix $X \in \mathcal{G}^*$ such that $\mathbf{a} \in X$. Let $Y \prec X$ be given. Then there is a degree $\mathbf{b} \leq_T \mathbf{a}$ such that $\mathbf{b} \in Y$.*

Here $\prec$ refers to the following ordering on $\mathcal{G}^*$:

$$\mathrm{GL}_1^* \prec \mathrm{GL}_2^* \prec \mathrm{GL}_3^* \prec \cdots \prec GI^* \prec \cdots \prec \mathrm{GH}_3^* \prec \mathrm{GH}_2^* \prec \mathrm{GH}_1^*.$$

Corollary 2.4. *If $\mathbf{x} \leq_T \mathbf{y} \in \mathbf{D}$, $\mathbf{x} \in \mathrm{GL}_1$, $\mathbf{y} \in \mathrm{GH}_1$, and $X \in \mathcal{G}^*$, then there exists a degree $\mathbf{z}$ such that $\mathbf{x} \leq_T \mathbf{z} \leq_T \mathbf{y}$ and $\mathbf{z} \in X$.*

Proof. First, we observe that for $Y \in \mathcal{G}^*$, and $\mathbf{a} \geq_T \mathbf{x}$, since $\mathbf{x} \in \mathrm{GL}_1$, we have that $\mathbf{a} \in Y \iff \mathbf{a} \in Y(\mathbf{x})$. This is because $\mathbf{a} \vee \mathbf{x}' = \mathbf{a} \vee (\mathbf{x} \vee 0') = \mathbf{a} \vee 0'$. Then just apply the previous lemma relativized to $\mathbf{x}$. □

Let $\mathcal{Q} = \langle Q, \leq \rangle$, where $Q = (P \smallsetminus \{0\}) \times \{0,1\}$, and

$$\langle x, i \rangle \leq \langle y, j \rangle \iff x \leq y \vee (x = y \;\&\; i \leq j)$$

From the corollary above, we get that if we had an embedding $\psi\colon \mathcal{Q} \to \langle \mathbf{D}, \leq \rangle$, such that for all $x \in P$, $\psi(\langle x, 0\rangle) >_T 0$, $\psi(\langle x, 0\rangle)$ is GL_1 and $\psi(\langle x, 1\rangle)$ is GH_1, we could get an embedding $\varphi\colon \mathcal{P} \to \mathcal{D}$. Just let $\varphi(x)$ be some degree in between $\psi(\langle x, 0\rangle)$ and $\psi(\langle x, 1\rangle)$ which is in the class $f(x)$, and let $\varphi(0) = 0 \in \mathbf{D}$. Now we have to show how to construct such a ψ.

Let $\{E_i : i \in P\}$ be a uniformly low, independent set of r.e. sets. For $F \subseteq P$, let $E_F = \bigoplus_{i \in F} E_i$ We will construct a sequence of sets $\{X_i\}_{i \in \omega}$ such that

(X.1) For all i, X_{i+1} is r.e. in and above X_i.
(X.2) For all i and $F \subseteq P$, $X_{2i} \oplus E_F$ is GL_1 and $X_{2i+1} \oplus E_F$ is GH_1.
(X.3) For $i, j \in \omega$ and $F_1, F_2 \subseteq P$ we have that

$$X_i \oplus E_{F_1} \leq_T X_j \oplus E_{F_2} \iff i \leq j \;\&\; F_1 \subseteq F_2.$$

Then, we define $\psi\colon Q \to \mathbf{D}$ by

$$\psi(\langle x, i\rangle) = X_{2\mathrm{rk}(x)+i} \oplus E_{\{y \in P : y \leq x\}},$$

where rk is some increasing function from $\mathcal{P}$ to ω. It is not hard to check, using (X.2), and (X.3), that ψ is an embedding $\mathcal{Q} \to \langle \mathbf{D} \leq \rangle$ and that for all $x \in P \smallsetminus \{0\}$, $\psi(\langle x, 0\rangle)$ is GL_1 and $\psi(\langle x, 1\rangle)$ is GH_1.

To construct the sequence $\{X_i\}_i$, the main tool is the following proposition that we will prove in the next section.

Proposition 3.1

Proposition. *Let $\{D_i : i \in G\}$ be a finite, uniformly low, independent set of r.e. sets. For $F \subseteq G$, let $D_F = \bigoplus_{j \in F} D_j$. Then, there exist an r.e. set A and an A-r.e. set B such that*

$$A' \equiv_T 0'' \equiv_T B \oplus 0' \equiv_T B'_G \text{ and}$$
$$\forall F \subset G \; \forall i \in G \smallsetminus F \; \big(D_i \not\leq_T B_F\big),$$

where $B_F = A \oplus B \oplus D_F$.

Let $P = G$, $\{D_i : i \in G\} = \{E_i : i \in P\}$ and A and B be as above. We let $X_0 = \emptyset$, $X_1 = A$ and $X_2 = A \oplus B$. Observe that for all $F \subseteq P$, $X_0 \oplus E_F$ is GL_1 (actually, it is low). We have that $X_1 \oplus E_F$ is GH_1 because it is r.e. and $(A \oplus D_F)' \geq_T 0''$. We have that $X_2 \oplus E_F$ is GL_1 because

$$X_2 \oplus E_F \oplus 0' \geq_T B \oplus 0' \equiv_T (A \oplus B \oplus D_F)' = (X_2 \oplus E_F)'.$$

We construct the rest of the sequence by induction. Suppose we have defined the sequence up to X_{2i} satisfying the conditions (X.1)-(X.3). For each $i \in P$, let $D_i = E_i \oplus X_{2i}$. Since X_{2i} satisfies (X.2), we have that $\{D_i : i \in P\}$ is a finite, uniformly low, independent set of r.e. sets relative to X_{2i}. By the relativized version of the Proposition 3.1 we have sets A and B, both $\geq X_{2i}$, A r.e. in X_{2i} and B r.e. in A, such that

$$A' \equiv_T X''_{2i} \equiv_T B \oplus X'_{2i} \equiv_T (A \oplus B \oplus D_P)' \text{ and} \tag{2.1}$$

$$\forall F \subset G\ \forall i \in G \setminus F\ \big(D_i \not\leq_T A \oplus B \oplus D_F\big). \tag{2.2}$$

Let $X_{2i+1} = A$ and $X_{2i+2} = A \oplus B$. As above, we get that $X_{2i+1} \oplus E_F \in \mathrm{GH}_1(X_{2i})$ and $X_{2i+2} \oplus E_F \in \mathrm{GL}_1(X_{2i})$. Since X_{2i} is GL_1, we have that $X_{2i+1} \oplus E_F \in \mathrm{GH}_1$ and $X_{2i+2} \oplus E_F \in \mathrm{GL}_1$.

Now, let us prove that (X.3) holds. It is clear that if $k \leq j$ and $F_1 \subseteq F_2$ then $X_k \oplus D_{F_1} \leq_T X_j \oplus D_{F_2}$. Now suppose that either $k \not\leq j$ or $F_1 \not\subseteq F_2$. In the latter case, from (2.2) we get that $X_k \oplus D_{F_1} \not\leq_T X_j \oplus D_{F_2}$. In the former case we divide into two possible cases. First assume that $j = 2i$. We cannot have that $X_{2i} \oplus D_{F_2} \geq_T X_{2i+1}$ because

$$(X_{2i} \oplus D_{F_2})'' \equiv_T X''_{2i} \equiv_T X'_{2i+1}.$$

Hence $X_k \oplus D_{F_1} \not\leq_T X_j \oplus D_{F_2}$. Second, assume that $j = 2i+1$. It cannot happen that $X_{2i+1} \oplus D_{F_2} \geq X_{2i+2}$ because $X_{2i+1} \oplus D_{F_2}$ is r.e. in X_{2i} but, since $X_{2i+2} \oplus X'_{2i} \equiv_T X''_{2i}$, $X_{2i+2} \not\leq_T X'_{2i}$.

We have proved that every finite GH-poset $\mathcal{P}$ can be embedded into $\mathcal{D}$. □

3. The main Lemma.

In this section we prove the extension of Harrington's ZBC Lemma that we need to prove Proposition 2.3.

Harrington ZBC Lemma. *Given a set W, r.e. in and above Z', there exist sets B and C, such that, B is r.e. in Z, C is r.e. in B, and*

$$(Z \oplus B)' \equiv_T (Z \oplus B \oplus C)' \equiv_T Z' \oplus B \oplus C \equiv_T Z' \oplus W.$$

Proofs of Harrington's ZBC Lemma can be found in [Sim85, Lemma 2.1] and in [HS91, Theorem 2.5]. It consists of a finite injury construction on top of an infinite injury construction. Instead, to prove our extension, we needed two infinite injury tree constructions, one in top of the other.

Proposition 3.1. *Let $\{D_i : i \in G\}$ be a finite, uniformly low, independent set of r.e. sets. For $F \subseteq G$, let $D_F = \bigoplus_{j \in F} D_j$. Then, there exist an r.e. set A and an A-r.e. set B such that*

$$A' \equiv_T 0'' \equiv_T B \oplus 0', \tag{3.1}$$

$$B'_G \equiv_T 0'' \text{ and} \tag{3.2}$$

$$\forall F \subset G\ \forall i \in G \setminus F\ \big(D_i \not\leq_T B_F\big), \tag{3.3}$$

where $B_F = A \oplus B \oplus D_F$.

We will do two constructions. First we show how to construct an r.e. operator, that, when applied to A, will give us B. Then, we show how to construct A. During the construction of A we use the r.e. operator constructed to guess how B is going to look at the end. Both constructions are going to be $0''$-priority arguments over a tree of strategies.

Although the proof we give does not formally assume knowledge of $0''$-priority arguments over a tree of strategies, familiarity with this kind of arguments would be extremely useful in understanding the proof. The reader might look at [Soa87, Chapter XIV] for an introduction to tree constructions.

We have to satisfy various requirements. To get $A' \equiv_T 0''$, we will construct A such that $\forall n \in \omega\big(0''(n) = 1 - \lim_s A(\langle n, s\rangle)\big)$. Let E be an r.e. set such that if $n \in 0''$ then $E^{[n]} = m$ for some $m \in \omega$ and if $n \notin 0''$ then $E^{[n]} = \omega$. (We write $E^{[n]}$ for $\{x : \langle n, x\rangle \in E\}$ and by $E^{[n]} = m$ we mean $E^{[n]} = \{0, ..., m-1\}$.) Let $\{E_s\}_s$ be a recursive enumeration of E such that for all s and n, $E_s^{[n]}$ is an initial segment of ω. We will have that $A' \equiv_T 0''$ if, for every $n \in \omega$, the following requirement is satisfied:

$$P_n^A : A^{[n]} =^* E^{[n]}.$$

To get $0'' \equiv_T B \oplus 0'$, we will try to code the modulus of convergence of $A(\langle n, s\rangle)$ into B. We let $\tilde{A}$ be the A-r.e. set such that for all n, $\tilde{A}^{[n]} = k$ where k is the least such that $\forall x \geq k\big(A(\langle n, x\rangle) = A(\langle n, k\rangle)\big)$. The requirement P_n^B will try to enumerate the elements of $\tilde{A}^{[n]}$ into B, as long as it is permitted by higher priority negative requirements. We will prove later that, with the help of $0'$, we will be able to decode $\tilde{A}$ from B, and hence we will get that $0'' \leq_T 0' \oplus B$. To get $D_i \not\leq_T B_F$, for $i \in G \smallsetminus F$, we have the negative requirements:

$$N_{\langle F,i,e\rangle} : \{e\}^{B_F} \neq D_i.$$

To satisfy these requirements we will use the Sacks preservation method (see [Soa87, VII.3]). Each requirement N_n is going to be split in two requirements N_n^A and N_n^B, the former working in the construction of A, and the latter in the construction of B. As in the Sacks jump theorem (see [Soa87, Remark VII.3.3]), these requirements help us keep the jump of B_F down, because they preserve computations of the form $\{e\}^{B_F}(0)\downarrow$. We will prove later that, because of this,

$$\forall F \subset G\big(B_F' \equiv_T 0''\big). \tag{3.4}$$

Well, we actually wanted $B_G' \equiv_T 0''$. There are two possible approaches to obtain this. The first one is to add requirements which preserve computations of the form $\{e\}^{B_G}(0)\downarrow$. The second one, is just to prove that $B_F' = 0''$ for all $F \subset G$. In the latter, we would be proving a weaker result, but it implies the statement of the theorem as follows: Let D_{-1} be an r.e. set such that $\{D_i : i \in G_1\}$ is an independent, uniformly low set, where $G_1 = G \cup \{-1\}$. To get D_{-1} construct a low r.e. set $D >_T D_G$ using Sacks Jump Inversion theorem (as in [Soa87, Remark VII.3.2]), and then construct $D_{-1} \leq_T D$ so that $\{D_i : i \in G_1\}$ is independent (using [Rob71, Corollary 6]). The weaker result we would be proving will give us an r.e. set A and an A-r.e. set B such that (3.1),(3.4) and (3.3) hold for G_1 instead of G. Since $G \subset G_1$, we have that $B_G' \equiv_T 0''$. We will take this second approach.

3.1. **True Stages.** Suppose that we are doing a construction using a tree of strategies and that γ is a node in the tree. For the strategy at γ, only the stages at which γ is accessible are relevant. Here we define the notion of being a true stage with respect to a given set of stages.

Given a recursive set S of stages and a recursive enumeration $\{D_s\}_s$ of an r.e. set D, we say that $s \in S$ is an *S-D-true stage* if $\exists x\big(s = \mu s' \in S(D_{s'} \upharpoonright x = D \upharpoonright x)\big)$. We are interested in true stages because of the following property. If $\sigma \in 2^{<\omega}$ is an initial segment of both D_s and $D_{p(s)}$, where $p(s) = \max t < s(t \in S)$, and s is S-D-true, then σ is an initial segment of D. Hence, if we have a computation with oracle $D_{p(s)}$ which remains unaltered if we change the oracle to D_s, it will remain unaltered if we change the oracle to D.

Note that the set of S-D-true stages is recursive in D. However, at a given stage $t \geq s$ we can guess recursively whether s is S-D-true as follows. We say that $s \in S$ *looks S-D-true at t*, and we write $s \preceq_S t$, if $\exists x\big(s = \mu s' \in S(s \leq t \;\&\; D_{s'} \upharpoonright x = D_t \upharpoonright x)\big)$. Note that $\langle S, \preceq_S \rangle$ is a partial order. Moreover, it is a tree in the sense that for all $s \in S$, $\langle \{s' : s' \preceq_S s\}, \preceq_S \rangle$ is a linear order. Also note that if s is S-D-true, then for all $t \geq s$, $s \preceq_S t$ and for all $t \leq s$ we have that $t \preceq_S s$ iff t is S-D-true.

3.2. **Tree Constructions.** Now, we show how to construct an r.e. set C using a tree of strategies the way we are going to construct A and B later. When we construct A and B, all we are going to do is to specify certain parameters of the construction of C.

Assume that we want to construct C satisfying certain positive and negative requirements. Suppose that there is a positive requirement P^C which wants to enumerate the elements of an r.e. set Y into C, for which we have a recursive enumeration $\{Y_s\}_s$. P^C is divided into infinitely many sub-requirements P^C_n, $n \in \omega$. Each P^C_n is in charge of enumerating the elements of $Y^{[n]}$ into $C^{[n]}$. We assume that the enumeration of Y satisfies that for all s and n, $Y^{[n]}_s$ is a finite initial segment of ω. Hence $Y^{[n]}$ is either ω or a finite initial segment of it.

We also have negative requirements N^C_n which want to preserve certain computations by imposing a restraint on the enumeration of C. At each stage s, N^C_n computes $l^C(n, s) \in \omega$. (In the constructions of A and B, $l^C(n, s)$ is an approximation to the length of agreement between $\{e\}^{B_F}$ and D_i.) When computing $l^C(n, s)$, N^C_n wants to approximate a computation which uses a certain r.e. set D_{F_n} as an oracle. So, N^C_n will be interested in D_{F_n}-true stages.

We arrange the strategies in a tree: $\mathbb{T} = (\{\mathtt{i}\} \cup \omega)^{<\omega}$. The nodes at level $2n$ work for N^C_n and the ones a level $2n+1$ work for P^C_n. The outcome of N^C_n is the restraint it imposes, and the outcome of P^C_n is $\mathtt{i}$ if $Y^{[n]}$ is infinite, and the first number not in $Y^{[n]}$ otherwise. We order each level as follows: $\mathtt{i} <_L 0 <_L 1 <_L \ldots$. This induces a lexicographic order $<_L$ on $\mathbb{T}$ as in [Soa87, Definition XIV.1.1].

At each stage s we define $\gamma_s \in \mathbb{T}$, and we say that γ is *accessible* at s if $\gamma \subseteq \gamma_s$. We define:

- $S^C_\gamma = \{s : \gamma \subseteq \gamma_s\} \cup \{0\}$; we call the stages in S_γ, *γ-stages*^{C}.
- $T^C_\gamma = \{t : t$ is an S^C_γ-D_{F_n}-true stage$\}$ where $2n = |\gamma|$; we call the stages in T^C_γ, *γ-true stages*^{C}.
- we say that $t \prec^C_\gamma s$ if s looks S^C_γ-D_{F_n}-true at t, where $2n = |\gamma|$.
- $p^C_\gamma(t) = \max \bar{t} < t(\bar{t} \in S^C_\gamma)$, the last γ-stage^C before t.
- Let $\mathtt{TP}^C$ be maximal in $\mathbb{T} \cup [\mathbb{T}]$ such that for all $k < |\mathtt{TP}^C|$,
$$\mathtt{TP}^C(k) = \liminf_{t \in S^C_{TP^C \upharpoonright k}} \gamma_t(k);$$
we call $\mathtt{TP}^C$, the *true path* of the construction of C.
- s is a *γ-expansionary stage*^{C} iff $s \in S^C_\gamma$ and $l^C(n, s) > l^C(n, t)$ for all γ-stage^C $t < s$.

The superscript C in S^C_γ, T^C_γ, stage^C, etc. denotes that these objects correspond to the C-construction. We include the superscript in the notation because later on we will be considering more than one construction at the same time. We might drop it if it is clear from the context which construction we are referring to.

Construction of C. Stage 0. Let $C_0 = \emptyset$ and $\gamma_0 = \emptyset$.

Stage $s+1$. Define $C_{s,0} = C_s$. For $k = 1, ..., s$, run sub-stage k.

Substage k. Suppose we have already defined $\gamma_s \restriction k = \gamma$ and $C_{s,k-1}$.

$k = 2n+1$:

- ▷ Let $R(n,s) = \max\{\gamma_{s'}(2i) : s' \leq s \ \& \ (\gamma_{s'} \restriction 2i <_L \gamma \vee \gamma_{s'} \restriction 2i \subseteq \gamma)\}$, the maximum of all the restraints imposed by higher priority negative requirements.
- ▷ If, since the last γ-stage, something has been enumerated into $Y^{[n]}$, set $\gamma_s(k) = \mathtt{i}$ and enumerate all the elements of $\{n\} \times Y_s^{[n]}$ not less than $R(n,s)$ into $C_{s,k}$.
- ▷ Otherwise set $\gamma_s(k)$ to be the smallest number not in $Y_s^{[n]}$.

$k = 2n$:

- ▷ First, N_n^C computes $l^C(n,s)$, and hence it determines whether s is γ-expansionary or not.
- ▷ Let $\gamma_s(k)$ be the last γ-expansionary stage which is $\preceq_\gamma s$.

At the end of stage s define $C_{s+1} = C_{s,s}$.

Let $C = \bigcup_s C_s$. ♢

Note that, to construct an r.e. set C this way, all we have to do is specify $\{Y_s\}_s$, $l^C(n,s)$ and $\{D_{F_n,s}\}_s$ for each $n \in \omega$.

We define $\hat{C}_{s,k}$ as the best approximation to C that we have at sub-stage k of stage s:

$$\hat{C}_{s,k} = C_{s,k} \cup \bigcup \{(\{j\} \times \omega \smallsetminus R(j,s)) : j < \frac{k}{2} \ \& \ \gamma_s(2j+1)\downarrow = \mathtt{i}\}.$$

(Here we use $R(j,s)$ as $\{x : x < R(j,s)\}$.)

In the following lemma we state and prove some basic properties of this construction.

Lemma 3.2. *Suppose that $\gamma = \gamma_{s_0} \restriction k \subseteq TP^C$.*

1. *For all $s \geq s_0$, $\gamma_s \not<_L \gamma$.*
2. *If $k = 2n+1$ then*
   - (a) *for all $s \geq s_0$, $R(n,s) \geq R(n,s_0)$, and if $s \in S_\gamma$ then $R(n,s) = R(n,s_0)$;*
   - (b) $TP^C(k) = \begin{cases} \mathtt{i} & \text{if } Y^{[n]} \text{ is infinite} \\ l & \text{if } Y^{[n]} = l < \omega; \end{cases}$
   - (c) *if $\gamma_{s_0} \restriction k+1 \subseteq TP^C$, then $\hat{C}^{[\leq n]}_{s_0,k} = C^{[\leq n]}$;*
   - (d) $C^{[\leq n]} =^* Y^{[\leq n]}$.
3. *If $k = 2n$ then*
   - (a) *If $\gamma_{s_0}(k) = s_1 \in T_\gamma$, then for all γ-stages $s \geq s_0$, $\gamma_s(k) \geq s_1$.*
   - (b) *$TP^C(k) = \lim_{s \in T_\gamma} \gamma_s(k)$ = last γ-expansionary true stage, if such a stage exists.*
   - (c) $R(n,s_0) = \gamma_{s_0}(k)$.
   - (d) *If $\gamma_{s_0}(k) = s_1 \in T_\gamma$, then $\hat{C}_{s,k} \restriction s_1 = C \restriction s_1$.*

Proof. The proof is by simultaneous induction on k. Suppose the lemma is true for all γ with $|\gamma| < k$. First suppose that $k = 2n+1$ for some n. Let $\gamma' = \gamma \restriction k-1$. By part (1) of the induction hypothesis we have that, for all $s \geq s_0$, $\gamma_s \not<_L \gamma'$. By (3b), $\gamma(k-1)$ is a γ'-true stage, and then by (3a), for all $s \geq s_0$, if $s \in S_{\gamma'}$, $\gamma_s \not<_L \gamma$. This proves (1). Part (2a) follows from the previous one and the definition of R. Part (2b) is immediate from the construction. For (2c), we know, from the induction hypothesis, that $\hat{C}^{[<n]}_{s_0,k} = C^{[<n]}$. If $\gamma_{s_0}(k) = \mathtt{i}$, since $\liminf_s R(n,s) = R(n,s_0)$, $C^{[n]} = C^n_{s_0,k} \cup (\omega \smallsetminus R(n,s_0)^{[n]}) = \hat{C}^{[n]}_{s_0,k}$, where $R(n,s_0)^{[n]} = \{x : \langle n,x\rangle < R(n,s_0)\}$. If $\gamma_{s_0}(k) = y_n = \mathtt{TP}^C(k) < \omega$, then nothing else is enumerated into $Y^{[n]}$ after s_0, and hence nothing is enumerated into $C^{[n]}$ after s_0. So $C^{[n]} = C^{[n]}_{s_0} = \hat{C}^{[n]}_{s,k}$. Part (2d) follows from the fact that for some s, $\gamma_s \restriction k+1 \subset \mathtt{TP}^C$, and that $\hat{C}^{[\leq n]}_{s,k} =^* Y^{[\leq n]}$.

Now suppose that $k = 2n$ for some n. To prove (1), assume that $k > 0$, (it is trivial otherwise) and let $\gamma' = \gamma \restriction k-1$. By induction hypothesis, we have that for all $s \geq s_0$, $\gamma_s \not<_L \gamma'$. If $\gamma_s(k-1) = \mathtt{i}$ we clearly never go left again. Otherwise, we do not enumerate anything in $Y^{[n]}$ any more, and hence, we never move left again either. For part (3a) observe that s_1 is γ-expansionary and that for all $s \geq s_1$, $s_1 \preceq_\gamma s$. Part (3b) follows from (3a). Now, let us prove (3c). Let $s_1 = \gamma_{s_0}(k)$. By (2a), $R(n,s_0) = R(n,s_1)$, and $R(n,s_1)$ can not be $> s_1$, but $R(n,s_1) \geq \gamma_{s_1}(k) = s_1$. So $R(n,s_0) = R(n,s_1) = s_1 = \gamma_{s_0}(k)$. For the last part we have that $\hat{C}^{[<n]}_{\gamma,s} = C^{[<n]}$ by (2c), and that $\hat{C}^{[\geq n]}_{\gamma,s} \restriction s_1 = C^{[\geq n]} \restriction s_1$ because for all $s \geq s_0$, $R(n,s) \geq s_1$. □

3.3. **Construction of B.** Now we construct an r.e. operator that, when applied to a set Z, returns a set $B[Z]$ r.e. in Z. Later, when we define A, we will let $B = B[A]$. We use the framework defined 3.2.

Construction of $B[Z]$. All we need to do is to specify the parameters needed in the tree construction of 3.2. We let $\tilde{Z}$ be the set that P^B wants to enumerate. $\tilde{Z}$ has the following recursive enumeration:

$$\tilde{Z}_t = \{\langle e,x\rangle < t : \exists y > x\big(\langle e,y\rangle < t \;\&\; Z(\langle e,x\rangle) \neq Z(\langle e,y\rangle)\big)\}.$$

For each negative requirement N^B_n we have to define D_{F_n} and $l^{B[Z]}(n,t)$. For $n = \langle F,i,e\rangle$ let $D_{F_n} = D_F$. We will use the letter t for the stages in the B-construction and $\beta_t \in \mathbb{T}^B$ for the approximation to $\mathtt{TP}^B$ at t. Now suppose that we are at stage t, sub-stage k of the construction, where $k = 2n$ and $n = \langle F,i,e\rangle$. Assume we have already defined $B_{t,k-1}$ and $\beta = \beta_t \restriction k$. Define

$$l^{B[Z]}(n,t) = \begin{cases} x + \frac{1}{2} & \text{if } \{e\}^{B_{F,t,k}[Z]}_t(x)\downarrow = \{e\}^{B_{F,p_\beta(t)}[Z]}_{p_\beta(t)}(x) \text{ with} \\ & \text{the same computation} \\ x & \text{otherwise.} \end{cases}$$

where x is maximal such that $D_{i,t} \restriction x = \{e\}^{B_{F,t,k}[Z]}_t \restriction x = \{e\}^{B_{F,p_\beta(t)}[Z]}_{p_\beta(t)} \restriction x$ with the same computation. Recall that $p_\beta(t)$ is the last β-stage before t and that $B_{F,t,k}[Z] = Z \oplus B_{t,k}[Z] \oplus D_{F,t}$. ◊

First, observe that the construction is recursive in Z, and hence $B[Z]$ is Z-re. Second, observe that $\tilde{Z}_t$ depends only on $Z \restriction t$, and hence so do the first t stages of the construction of $B[Z]$.

For the followings definition and lemma fix Z and drop the suffix $[Z]$ from the notation.

Definition 3.3. For $n = \langle F, i, e\rangle$, let

$$l_n^B = \begin{cases} x + \frac{1}{2} & \text{if } \{e\}^{B_F}(x)\downarrow \\ x & \text{otherwise.} \end{cases}$$

where x is maximal such that $D_i \restriction x = \{e\}^{B_F} \restriction x$ (x might be ω).

Lemma 3.4. *Let $\beta = TP^B \restriction k$, where $k = 2n$ and $n = \langle F, i, e\rangle$, and assume that $\tilde{Z}^{[<n]}$ is finite.*

(1) *If $\beta \subseteq \beta_{t_0}$, $\beta_{t_0}(k) = t_1 \in T_\beta$ and $l = \lceil l^B(\beta, t_0)\rceil$, then*

$$\{e\}_{t_0}^{B_{F,t_0,k}} \restriction l = \{e\}^{B_F} \restriction l.$$

(For $p \in \mathbb{Q}$, $\lceil p\rceil = \mu q \in \mathbb{Z}(p \leq q)$.)

(2) *If $l \in \frac{1}{2}\cdot\omega = \{\frac{n}{2} : n \in \omega\}$ and $l \leq l_n^B$, then there exists a β-expansionary true stage t_0 such that $l^B(\beta, t_0) \geq l$.*

(3) *If $\{e\}^{B_F} \neq D_i$ then*

$$TP^B(k) = \lim_{t \in T_\beta} \beta_t(k) < \omega.$$

Proof. For part (1) we have to show that $B_{F,t_0,k}$ is preserved up to the use, u, of the computations $\{e\}_{t_0}^{B_{F,t_0,k}} \restriction \lceil l^B(\beta, t_0)\rceil$. By Lemma 3.2.3d and the assumption on $\tilde{Z}^{[<n]}$, we have that $B_{t_0,k} \restriction t_1 = \hat{B}_{t,k} \restriction t_1 = B \restriction t_1$. Note that $t_1 > u$. Since the computations $\{e\}_{t_0}^{B_{F,t_0,k}} \restriction \lceil l^B(\beta, t_0)\rceil$ were there at stage $p_\beta(t_1)$ too, we have that $D_{F,p_\beta(t_1)} \restriction u = D_{F,t_1} \restriction u$. Since $t_1 \in T_\beta$, nothing below u is enumerated into D_F after t_1. So we have that $B_{F,t_0,k} \restriction u = B_F \restriction u$. This proves (1). Part (2) is clear.

For part (3), suppose toward a contradiction, that there are infinitely many β-expansionary true stages. This implies that there is a stage $t_0 \in T_\beta$ with $l^B(\beta, t_0) > l_n^B$. Let $l = \lfloor l_n^B\rfloor$. So we have that $\{e\}_{t_0}^{B_{F,t_0,k}}(l)\downarrow$, and that this computation is going to be preserved for ever by part (1). Since $D_i(l) \neq \{e\}^{B_F}(l)$, l had to be enumerated into D_i after stage t_0, and this disagreement is preserved for ever. So, for no t after that stage, we have that $l^B(\beta, t) > l^B(\beta, t_0)$. Hence, there are no more β-expansionary stages. □

3.4. **Construction of A.** We construct an r.e. set A satisfying the following requirements:

$$P_n^A: \quad A^{[n]} =^* E^{[n]},$$
$$N_{\langle F,i,e\rangle}^A: \quad \{e\}^{B_F} \neq D_i.$$

We will use a tree construction like the one in 3.2. At each stage s, N_n^A computes an approximation to

$$l_n = \begin{cases} x + \frac{1}{2} & \text{if } \{e\}^{B_F[A]}(x)\downarrow \\ x & \text{otherwise.} \end{cases}$$

where $x = \max x' \leq \omega\big(D_i \restriction x' = \{e\}^{B_F[A]} \restriction x'\big)$, and imposes a restraint on A to preserve these computations. To approximate $B_F[A]$, we run the construction of B for a few stages using $\hat{A}_{s,2n}$ as an oracle. (Recall that $\hat{A}_{s,2n}$ is the best approximation to A that N_n^A has at stage s.) So we have to decide for how many stages to run $B_t[\hat{A}_{s,2n}]$. For this purpose, along with the construction we define $\delta_s \in \mathbb{T}^B$ as an approximation to $\mathtt{TP}^B$.

Construction of A. All we need to do is to specify the parameters needed in the tree construction of 3.2. The set that P^A wants to enumerate is E, for which we have a recursive enumeration $\{E_s\}_s$. For each negative requirement N^A_n we have to define D_{F_n} and $l^A(n,s)$. For $n = \langle F, i, e\rangle$ let $D_{F_n} = D_F$. We will use the letter s for the stages of the A-construction and $\alpha_s \in \mathbb{T}^A$ for the approximation to $\mathtt{TP}^A$ at s. Now, for each s and $n < \frac{s}{2}$, we define $\delta_s \in \mathbb{T}^B$ and $l^A(n,s)$. Suppose that we are in stage s, sub-stage $k = 2n$ of the construction, and we have already defined $A_{s,k-1}$, $\alpha = \alpha_s \upharpoonright k$ and $\delta = \delta_s \upharpoonright k$.

- ▷ Let $t_{n,s} < p^A_\alpha(s)$ be maximal such that $t_{n,s} \prec_\delta^{B[\hat{A}_{s,k}]} s$. If there is no such $t_{n,s}$, let it be 0. (This is for how many stages we are going to run the computation of $B[\hat{A}_{s,k}]$.)
- ▷ $\delta_s(k) = \beta^{B[\hat{A}_{s,k}]}_{t_{n,s}}(k)$ (the last δ-expansionary stage$^{B[\hat{A}_{s,k}]}$ that is $< p^A_\alpha(s)$ and $\prec_\delta^{B[\hat{A}_{s,k}]} s$).
- ▷ $l^A(n,s) = l^{B[\hat{A}_{s,k}]}(n, t_{n,s})$ $(= l^{B[\hat{A}_{s,k}]}(n, \delta_s(k)))$.
- ▷ $\delta_s(k+1) = \mu x(x \notin \tilde{\hat{A}}^{[n]}_{s,k})$ (the place after which the nth column of $\hat{A}_{s,k}$ stabilizes).

$\diamondsuit$

Let $B = B[A]$. From now on, when we write the superscript B, we are referring to the construction of $B[A]$.

In the following lemma we show that δ_s is a good approximation to $\mathtt{TP}^B$.

Lemma 3.5. *Let $\alpha = \mathtt{TP}^A \upharpoonright k$ and $\beta = \mathtt{TP}^B \upharpoonright k$, where $k = 2n$ and $n = \langle F, i, e\rangle$.*

1. *If $\alpha \subseteq \alpha_{s_0}$, then $\beta \subseteq \delta_{s_0}$.*
2. *If s_0 is an α-expansionary true stage^A, then $t_0 = \delta_{s_0}(k)$ is a β-expansionary true stage^B. Moreover, for every α-stage^A $s \geq s_0$, $t_0 \leq \delta_s(k)$.*
3. *If s_0 is an α-expansionary true stage^A, and s_1 is and α-stage^A, $s_0 < s_1$, then $\delta_{s_0}(k) < \delta_{s_1}(k) \iff \alpha_{s_0}(k) < \alpha_{s_1}(k)$.*
4. *If s_0 is a true stage^A, then $t_0 = \delta_{s_0}(k)$ is a β-expansionary true stage^B.*
5. *If $\mathtt{TP}^A \upharpoonright k+1 \subseteq \alpha_s$, then $\mathtt{TP}^B \upharpoonright k+1 \subseteq \delta_s$.*

Proof. We prove the lemma by simultaneous induction on n. Let $\alpha' = \alpha \upharpoonright k-1$ and $\beta' = \beta \upharpoonright k-1$. From part (5) of the induction hypothesis, we have that if $\alpha' \subseteq \alpha_{s_0}$, then $\beta' \subseteq \delta_{s_0}$ (because $\alpha' = \mathtt{TP}^A \upharpoonright 2(n-1)+1$). If also $\alpha \subseteq \alpha_{s_0}$, then, by Lemma 3.2.2c, $\hat{A}^{[<n]}_{s_0,k} = A^{[<n]}$, and hence $\tilde{\hat{A}}^{[n-1]}_{s_0,k} = \tilde{A}^{[n-1]}$. Therefore, the computation of $\delta_{s_0}(k-1)$ is correct. This proves part (1).

To prove (2), we start by showing that t_0 is a β-expansionary true stage$^{B[\hat{A}_{s_0,k}]}$. Since $t_0 = \beta^{B[\hat{A}_{s_0,k}]}_{t_{n,s}}(k)$, it is clear that it is β-expansionary$^{B[\hat{A}_{s_0,k}]}$. It is a β-true stage$^{B[\hat{A}_{s_0,k}]}$ because $t_0 < p^A_\alpha(s_0)$, $t_0 \prec_\beta^{B[\hat{A}_{s_0,k}]} s_0$ and $s_0 \in T^A_\alpha$. The second observation is that since $\hat{A}_{s_0,k} \upharpoonright s_0 = A \upharpoonright s_0$ (this is by Lemma 3.2.3d), the first s_0 ($\geq t_0$) stages of the computations of $B[\hat{A}_{s_0,k}]$, and of $B[A]$ are the same. This implies that t_0 is a β-expansionary true stage^B. Moreover, if $s \in S^A_\alpha$ and $s > s_0$, then

$\hat{A}_{s_0,k}\upharpoonright s_0 = \hat{A}_{s,k}\upharpoonright s_0 = A\upharpoonright s_0$. As above, this implies that t_0 is a β-expansionary true stage$^{B[\hat{A}_{s,k}]}$. Then, since $t_0 \prec_\beta^{B[\hat{A}_{s,k}]} s$ and $t_0 < p_\alpha^A(s)$, $t_0 \le \delta_s(k)$.

Let us prove part (3). Let $t_0 = \delta_{s_0}(k)$ and $t_1 = \delta_{s_1}(k)$. From the proof of part (2) we get that t_0 is a β-expansionary true stage$^{B[\hat{A}_{\alpha,s_1}]}$ and $t_0 \le t_1$. So, $t_1 > t_0$ iff $l^{B[\hat{A}_{\alpha,s_1}]}(n,t_1) > l^{B[\hat{A}_{\alpha,s_1}]}(n,t_0)$. Note that $l^A(n,s_1) = l^{B[\hat{A}_{\alpha,s_1}]}(n,t_1)$ and that, since the first s_0 stages in the computations of $B[\hat{A}_{s_0,k}]$, and of $B[\hat{A}_{s_1,k}]$ are the same, $l^{B[\hat{A}_{\alpha,s_1}]}(n,t_0) = l^{B[\hat{A}_{\alpha,s_0}]}(n,t_0) = l^A(n,s_0)$. So, $t_1 > t_0$ iff $l^A(n,s_1) > l^A(n,s_0)$. We have that $l^A(n,s_1) > l^A(n,s_0)$ iff there is and α-expansionary stage^A $\bar{s}$, $s_0 \prec_\alpha^A \bar{s} \preceq_\alpha^A s_1$, which happens iff $\alpha_{s_1}(k) > \alpha_{s_0}(k)$.

For part (4), let $s_1 = \alpha_{s_0}(k)$. s_1 is an α-expansionary stage^A, and it is $\preceq_\alpha^A s_0$. Since s_0 is α-true, so is s_1. Then, by part (2), $\delta_{s_1}(k)$ is a β-expansionary true stage^B. Since $\alpha_{s_0} = s_1 = \alpha_{s_1}(k)$, by part (3), $t_0 = \delta_{s_0}(k) = \delta_{s_1}(k)$. So t_0 is a β-expansionary true stage^B.

For part (5), let $s_0 = \mathtt{TP}^A(k)$ and $t_0 = \delta_{s_0}(k)$. We claim that $t_0 = \mathtt{TP}^B(k)$. Since s_0 is and α-expansionary true stage^A, from (2), we have that t_0 is a β-expansionary true stage^B. We have to show that it is the last one. Suppose, toward a contradiction, that $t_1 > t_0$ is a β-expansionary true stage^B. Let $s_1 \in T_\alpha^A$ be such that $t_1 < p_\alpha^A(s_1)$ and $A_{s_1}\upharpoonright t_1 = A\upharpoonright t_1$. Then we have that t_1 is a β-expansionary true stage$^{B[\hat{A}_{s_1,k}]}$ and $t_1 \le t_{n,s_1}$. So $\delta_{s_1}(k) \ge t_1 > t_0 = \delta_{s_0}(k)$, and hence $\alpha_{s_1}(k) > \alpha_{s_0}(k)$, which contradicts the fact that s_0 is the last α-expansionary true stage. Now, we have to show that for any $(\alpha^\wedge s_0)$-stage^A s, $\delta_s(k) = t_0$. Since $\alpha_s(k) = s_0 = \alpha_{s_0}(k)$, $\delta_s(k) = \delta_{s_0}(k) = t_0$. $\square$

Lemma 3.6. *Let $\alpha = \mathtt{TP}^A\upharpoonright k$ and $\beta = \mathtt{TP}^B\upharpoonright k$, where $k = 2n$ and $n = \langle F,i,e\rangle$, and assume that $\tilde{A}^{[<n]}$ is finite.*

(1) *If $s_0 \in T_\alpha$, $t_0 = \delta_{s_0}(k)$ and $l = \lceil l^A(n,s_0)\rceil$, then*
$$\{e\}_{t_0}^{B_{F,t_0,k}[\hat{A}_{s_0,k}]}\upharpoonright l = \{e\}^{B_F}\upharpoonright l.$$
(2) *If $l \in \frac{1}{2}\cdot\omega$ and $l \le l_n$, then there exists an α-expansionary true stage^A s_0 such that $l^A(n,s_0) \ge l$.*

Proof. Let $s_1 = \alpha_{s_0}(k)$. By Lemma 3.2.3d, $\hat{A}_{s_0,k}\upharpoonright s_1 = A\upharpoonright s_1$. Then, since $t_0 \le s_1$ (this is because $\alpha_{s_0}(k) = \alpha_{s_1}(k)$, and hence, by Lemma 3.5.3, $t_0 = \delta_{s_0}(k) = \delta_{s_1}(k) \le s_1$),
$$\{e\}_{t_0}^{B_{F,t_0,k}[\hat{A}_{s_0,k}]}\upharpoonright\lceil l^A(n,s_0)\rceil = \{e\}_{t_0}^{B_{F,t_0,k}}\upharpoonright\lceil l^A(n,s_0)\rceil.$$
By Lemma 3.4.1, and since $t_0 \in T_\beta$ (this is by 3.5.4),
$$\{e\}_{t_0}^{B_{F,t_0,k}}\upharpoonright\lceil l^B(n,t_0)\rceil = \{e\}^{B_F}\upharpoonright\lceil l^B(n,t_0)\rceil.$$
Part (1) now follows, since $l^A(n,s_0) = l^B(n,t_0)$.

For part (2) we use Lemma 3.4.2. So we get a β-expansionary true stage^B t_0 such that $l^B(n,t_0) \ge l$. As in the proof of 3.5.5, there is a stage $s_0 \in T_\alpha^A$ such that the first t_0 stages of the computations of $B[\hat{A}_{s_0,k}]$, and of $B[A]$ are the same and $\delta_{s_0}(k) \ge t_0$. Therefore $l^A(n,s_0) = l^B(n,\delta_{s_0}(k)) \ge l^B(n,t_0) \ge l$. $\square$

3.5. **Verifications.** Now we show that A and $B = B[A]$ are as we wanted.

Lemma 3.7. *If $|\mathtt{TP}^A| \ge 2n$, where $n = \langle F,i,e\rangle$, and $\tilde{A}^{[<n]}$ is finite, then $\{e\}^{B_F} \ne D_i$.*

Proof. Suppose, toward a contradiction, that $\{e\}^{B_F} = D_i$. We will show that then $D_i \leq_T D_F$ contradicting the hypothesis. Let $\alpha = \mathtt{TP}^A \restriction k$, where $k = 2n$. Given $p \in \omega$, we want to find $D_i(p)$ recursively in D_F. Find $s_0 \in T^A_\alpha$, such that $l^A(\alpha, s_0) > p$. Such an s_0 exists because of Lemma 3.6.2, and we can find it recursively in $T^A_\alpha \leq D_F$. Then, by Lemma 3.6.1,

$$D_i(p) = \{e\}^{B_F}(p) = \{e\}^{B_{F,t_0,k}[\hat{A}_{s_0,k}]}_{t_0},$$

where $t_0 = \delta_{s_0}(k)$. □

Lemma 3.8. *For all n, if $n = \langle F, i, e\rangle$, then*

1. $\{e\}^{B_F} \neq D_i$;
2. $|\mathtt{TP}^B| \geq 2n + 1$.
3. $|\mathtt{TP}^A| \geq 2n + 1$;
4. $A^{[n]} =^* E^{[n]}$ *and* $|\mathtt{TP}^A| \geq 2n + 2$
5. $B^{[n]}$ *is finite,* $|\mathtt{TP}^B| \geq 2n + 2$.

Proof. We prove the lemma by induction on n. Suppose the lemma is true for all $m < n$. By part (4) of the inductive hypothesis we have that $|\mathtt{TP}^A| \geq 2n$ and $\tilde{A}^{[<n]}$ is finite. So, Lemma 3.7 implies (1). Then, Lemma 3.4.3 implies (2). To prove (3) we observe that if $s_0 < s_1$ are α-expansionary true stages^A, then $\delta_{s_0}(2n) < \delta_{s_1}(2n)$ (this is by 3.5.3). Then, by Lemma 3.5.2 both $\delta_{s_0}(2n)$ and $\delta_{s_1}(2n)$ are β-expansionary true stages^B. But then, since there are only finitely many β-expansionary true stages^B, there are only finitely many α-expansionary true stages^A. Hence $\mathtt{TP}^A(2n)$ exists. Part (4) is now implied by Lemma 3.2, parts 2d and 2b. This also implies that $\tilde{A}^{[n]}$ is finite and the last part follows, again by Lemma 3.2, parts 2d and 2b. □

From the previous lemma we get that $|\mathtt{TP}^A| = |\mathtt{TP}^B| = \omega$, and that all the requirements N_n and P^A_n are satisfied. What is left to show is that for all $F \subset G$, $B'_F \equiv_T 0''$ and that $0'' \equiv_T B \oplus 0'$.

Lemma 3.9. *For all $F \subset G$, $\mathtt{TP}^A \equiv_T 0'' \equiv_T (B_F)'$.*

Proof. Clearly $\mathtt{TP}^A \leq_T 0''$. We can compute $0''$, from $(B_F)'$, because $A \leq_T B_F$ and $0'' \leq_T A'$. Now we show how to compute $(B_F)'$ from $\mathtt{TP}^A$. Fix F and $i \in G \setminus F$. We claim that

$$\{e\}^{B_F}(0)\downarrow \iff \{e\}^{B_{F,t_0,k}[\hat{A}_{s_0,k}]}_{t_0}(0)\downarrow,$$

where $s_0 = \mathtt{TP}^A(k)$, $t_0 = \mathtt{TP}^B(k)$, $k = 2n$ and $n = \langle F, i, e\rangle$. The implication from right to left is because of Lemma 3.6.2. For the other direction, is because of Lemma 3.6.1. □

Lemma 3.10. $\mathtt{TP}^A \equiv_T 0'' \equiv_T B \oplus 0'$.

Proof. We know that $\mathtt{TP}^A \equiv_T 0''$ and that $B \oplus 0' \leq_T 0''$ because B is r.e. in an r.e. set. Now we prove that $\mathtt{TP}^A \leq_T B \oplus 0'$. By induction on n we compute

- $\mathtt{TP}^A(2n)$,
- $\mathtt{TP}^B(2n)$,
- $\mathtt{TP}^B(2n + 1)$, and
- $\mathtt{TP}^A(2n + 1)$,

recursively in $B \oplus 0'$. By Lemma 3.2.3b, we have that

$$\mathtt{TP}^A(2n) = \lim_{s \in T^A_\alpha} \alpha_s(2n),$$

where $\alpha = \mathtt{TP}^A \restriction 2n$. Since $T^A_\alpha \leq D_F$ (where F is such that $n = \langle F, i, e \rangle$), we can compute this recursively in $D'_F \equiv_T 0'$. Then we compute $\mathtt{TP}^B(2n) = \delta_s(2n)$, where $s = \mathtt{TP}^A(2n)$ (this is because of Lemma 3.5.5). Now, let

$$z_n = \mu z\big(\langle n, z\rangle \geq \mathtt{TP}^B(2n) \;\&\; z \notin B^{[n]}\big).$$

From the construction, since $\mathtt{TP}^B(2n) = \liminf_t R^B(n,t)$, it has to be the case that $z_n \notin \tilde{A}^{[n]}$ (otherwise it would be eventually enumerated into $B^{[n]}$). Having this information, we can compute $\mathtt{TP}^B(2n+1) = \mu z(z \notin \tilde{A}^{[n]})$ recursively in $A \leq_T 0'$. Then we can compute $\lim_x E(\langle n, x\rangle) = \lim_x A(\langle n, x\rangle) = A(\langle n, z_n\rangle)$. If $\lim_x E(\langle n, x\rangle)$ is 1, then $\mathtt{TP}^A(2n+1) = \mathtt{i}$, and if it is 0, then $\mathtt{TP}^A(2n+1) = \mu x(x \notin E^{[n]})$ which can be computed from $E \leq_T 0'$. □

This finishes the proof of Proposition 3.1.

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