FRAÏSSÉ'S CONJECTURE IN Π^1_1-COMPREHENSION

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ABSTRACT. We prove Fraïssé's conjecture within the system of Π^1_1-comprehension. Furthermore, we prove that Fraïssé's conjecture follows from the Δ^0_2-bqo-ness of 3 over the system of Arithmetic Transfinite Recursion, and that the Δ^0_2-bqo-ness of 3 is a Π^1_2 statement strictly weaker than Π^1_1-comprehension.

1. INTRODUCTION

Among the longest-standing open questions in reverse mathematics is whether Fraïssé's conjecture is equivalent to Arithmetic Transfinite Recursion or not [Clo90] [Sim99, Remark X.3.31] [Mar05]. By Fraïssé's conjecture, we mean the following statement of second order arithmetic, conjectured by Fraïssé in [Fra48]:

> FRA: The class of countable linear orderings is well-quasi-ordered under embeddability.

A *well-quasi-ordering* is a quasi-ordering without infinite descending sequences or infinite antichains. The study of well-quasi-orderings has been of interest to reverse mathematicians for a long time because it seems to require stronger axioms than most other areas of mathematics. Many of the proofs use Π^1_2-CA_0, which is beyond the highest of the big five systems, Π^1_1-CA_0, where a large majority of mathematics can be developed. However, none of these theorems has been proved to be equivalent to Π^1_2-CA_0, and for many of them, the exact proof-theoretic strength is unknown. Other than Fraïssé's conjecture, the two most interesting well-quasi-ordering results are: Kruskal's theorem on finite trees [Kru60], which Friedman showed cannot be proved in ATR_0 (see [Sim85, RW93]); and the graph minor theorem of Robertson and Seymour, which Friedman, Robertson and Seymour [FRS87] showed cannot be proved in Π^1_1-CA_0. Neither of these two is equivalent to any of the big five systems. The reader can find a survey on the theory of well-quasi-orderings from the viewpoint of reverse mathematics in [Mar05]. Results of the author [Mon06, Mon07] show that FRA is a particularly interesting statement because it is *robust*: It is equivalent to other variations of itself, and equivalent to many other statements that involve embeddability of linear orderings; RCA_0+FRA is the least system where one can reasonably work with linear orderings and the embeddability relation, the same way ATR_0 is for ordinals.

FRA was a conjecture from 1948 until Richard Laver proved it in 1971 [Lav71], using Nash-Williams' notion of better-quasi-ordering (bqo) [NW68] — it is now a theorem. The exact proof-theoretic strength of FRA is unknown. It is known that Laver's proof of FRA can be carried out in Π^1_2-CA_0, and that since FRA is a true Π^1_2 statement, it cannot imply Π^1_1-CA_0. Shore [Sho93] proved that the assumption that the class of well orderings is well-quasi-ordered under embeddablity implies ATR_0, getting as a corollary that FRA implies ATR_0. But we still do not know whether FRA can be proved using just ATR_0, as has been conjectured by Peter Clote [Clo90], Stephen Simpson [Sim99, Remark X.3.31] and Alberto Marcone [Mar05]. Whether FRA can be proved in Π^1_1-CA_0 has also been an open question for that long — we prove that it can:

Theorem 1.1. *Π^1_1-CA_0 proves Fraïssé's conjecture.*

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Furthermore, we prove FRA from the Δ^0_2-bqo-ness of 3 (Definition 1.2) over ATR_0. This is a combinatorial statement that we will show is strictly below Π^1_1-CA_0. It was know that a proof of FRA from a statement below Π^1_1-CA_0 had to be completely different from the proof we knew — this one is indeed. The reason we knew that is that the key lemma in the original proof of FRA is the *minimal bad array lemma*, which is actually used all throughout better-quasi-ordering theory, and Marcone [Mar96] had proved that the *minimal bad array lemma* implies Π^1_1-CA_0. Instead, our new proof uses the work of the author from [Mon06] connecting FRA and the well-quasi-orderness of a certain class of trees, then ideas of Selivanov [Sel07] connecting another class of trees to the Δ^0_2-Wadge degrees of k-partitions of $\mathbb{R}$, and then some ideas from Engelen, Miller, and Steel's proof [vEMS87] that the Borel functions from $\mathbb{R}$ to a countable bqo Q are bqo under the Wadge reducibility (they use Borel determinacy).

We will review the various equivalent definitions of better-quasi-ordering in Section 2. For now, let us say that it is a strengthening of the notion of well-quasi-ordering that enjoys better closure properties. It was introduced by Nash-Williams [NW68] as a tool to prove well-quasi-ordering results. Let us describe what we mean by Δ^0_2-bqo, which is the relevant version for our statement. We use $[\mathbb{N}]^{\mathbb{N}}$ to denote the set of infinite increasing sequences of natural numbers. When we refer to a Δ^0_2-function $F\colon [\mathbb{N}]^{\mathbb{N}} \to \mathbb{N}$, we refer to a class function whose graph is defined by a $\boldsymbol{\Sigma}^0_2$ formula, allowing for real parameters.

Definition 1.2. A quasi-ordering $(Q; \leqslant_Q)$ is a Δ^0_2-*better-quasi-ordering* (Δ^0_2-*bqo*) if, for every Δ^0_2 function $F\colon [\mathbb{N}]^{\mathbb{N}} \to Q$, there exists an $X \in [\mathbb{N}]^{\mathbb{N}}$ such that $F(X) \leqslant_Q F(X^-)$, where X^- is X with its first element removed (i.e., $X^-(n) = X(n+1)$).

In Section 2, we will see that Π^1_1-CA_0 can prove the equivalence between this definition and the standard definition of bqo. Given a natural number n, we use "n" to also denote the partial ordering that consists of n incomparable elements. Here is our main result.

Theorem 1.3. *(ATR_0) Fraïssé's conjecture follows from 3 being a Δ^0_2-bqo.*

Marcone [Mar05, Section 3] discussed the bqo-ness of 3. He showed that ATR_0 proves 3 is a bqo, and it is open whether it is provable in a weaker system, even in RCA_0 — Marcone showed that 2 is a bqo in RCA_0. It follows from Lemma 2.3 that Π^1_1-CA_0 can prove 3 is a Δ^0_2-bqo. Thus, Theorem 1.1 follows from Theorem 1.3 and Lemma 2.3. Since saying that 3 is a Δ^0_2-bqo is a Π^1_2 statement, it must be strictly weaker than Π^1_1-CA_0. We do not know if ATR_0 can prove that 3 is a Δ^0_2-bqo. In the last section, we prove in ATR_0 that 2 is a Δ^0_2-bqo.

2. Better-quasi-orderings

Before introducing the notion of bqo, we need to settle on some notation. For a set $X \subseteq \mathbb{N}$, we use $[X]^{\mathbb{N}}$ to denote the set of infinite subsets of X. We sometimes think of an element $Y \in [X]^{\mathbb{N}}$ as an increasing sequence, namely the sequence enumerating its elements. Thus, we may write $Y(n)$ for the n-th element of Y. We use $[X]^{<\mathbb{N}}$ for the set of finite subsets of X, which we often also think of as finite increasing sequences. For $\sigma \in [X]^{<\mathbb{N}}$ and $Y \in [X]^{\leqslant\mathbb{N}}$, we write $\sigma \subseteq Y$ to mean that σ is a subset of Y, and $\sigma \sqsubseteq Y$ to mean that σ is an initial segment of Y as sequences. Suppose the maximum element of σ is below the least of Y. We then write $\sigma^\frown Y$, when we are thinking of sequences, for what we would write $\sigma \cup Y$, when we are thinking of sets. Before giving the definitions of bqo, we need to introduce blocks, barriers, and arrays.

Definition 2.1. A *block* is a subset $B \subseteq [\mathbb{N}]^{<\mathbb{N}}$ such that

(1) for every $X \in [\mathbb{N}]^{\mathbb{N}}$, there exists $\sigma \in B$ with $\sigma \sqsubset X$, and
(2) every two elements of B are $\sqsubseteq$-incomparable.

We say that B is a *barrier* if we also any two elements of B are $\subseteq$-incomparable.

If B is a block and Q a set, a map $b\colon B \to Q$ is called an *array*.

Note that there is a one-to-one correspondence between continuous functions $F\colon [\mathbb{N}]^{\mathbb{N}} \to Q$ (where Q is given the discrete topology) and arrays $f\colon B \to Q$: Given a continuous function $F\colon [\mathbb{N}]^{\mathbb{N}} \to Q$, let B be the set of $\sqsubseteq$-minimal strings $\sigma \in [\mathbb{N}]^{<\mathbb{N}}$ such that, for all $X \sqsupseteq \sigma$, the value of $F(X)$ is the same, and let $f(\sigma)$ be that value. It is not hard to see that B is a block and that, for every $\sigma \in B$ and $X \in [\mathbb{N}]^{\mathbb{N}}$ with $\sigma \sqsubseteq X$, $f(\sigma) = F(X)$.

Given $X \in [\mathbb{N}]^{\leqslant\mathbb{N}}$, we let X^- be X with its first element removed (i.e., $X^-(n) = X(n+1)$). For $\sigma, \tau \in [\mathbb{N}]^{<\mathbb{N}}$, we write $\sigma \lhd \tau$ if $\sigma \sqsubseteq \tau^-$.

Definition 2.2. Let $(Q; \leqslant_Q)$ be a quasi-ordering. We say that an array $b\colon B \to Q$ is *bad* if, for every $\sigma, \tau \in B$ with $\sigma \lhd \tau$, $b(\sigma) \not\leqslant_Q b(\tau)$. We say that a function $F\colon [\mathbb{N}]^{\mathbb{N}} \to Q$ is *bad* if, for every $X \in [\mathbb{N}]^{\mathbb{N}}$, $F(X) \not\leqslant_Q F(X^-)$. We are now ready to define *better-quasi-orderings (bqo)*.

- Q is a *barrier-bqo* if there is no bad array $b\colon B \to Q$ for any barrier B.
- Q is a *block-bqo* if there is no bad array $b\colon B \to Q$ for any block B.
- Q is a *continuous-bqo* if there is no bad continuous function $F\colon [\mathbb{N}]^{\mathbb{N}} \to Q$.
- Q is a Δ^0_2*-bqo* if there is no bad Δ^0_2-function $F\colon [\mathbb{N}]^{\mathbb{N}} \to Q$.
- Q is a *Borel-bqo* if there is no bad Borel function $F\colon [\mathbb{N}]^{\mathbb{N}} \to Q$.

The standard definition of bqo is the first one, so we use bqo to mean barrier-bqo. Using ideas from Marcone [Mar94], Cholak, Marcone, and Solomon [CMS04] [CMS04, Theorem 5.12] showed that WKL_0 can prove block-bqos and barrier-bqos are the same thing. The equivalence between block-bqos and continuous-bqos is immediate from the translation between bad continuous functions and bad arrays. The notion of Borel-bqos was introduced by Simpson [Sim85]. His proof that they are the same as barrier-bqos uses a lemma of Mathias's [Mat77] that says that, for every Borel function $F\colon [\mathbb{N}]^{\mathbb{N}} \to Q$, there is a set of the form $[X]^{\mathbb{N}}$ on which it is continuous. We do not know how to prove Mathias's lemma in Π^1_1-CA_0. But, to prove the equivalence between barrier-bqos and Δ^0_2-bqos, we only need Mathias's lemma for Δ^0_2 functions, which we show in Lemma 3.2 follows from Π^1_1-CA_0.

Lemma 2.3. *Let Q be a quasi-ordering. The following are equivalent over Π^1_1-CA_0:*

1. *Q is barrier-bqo.*
2. *Q is Δ^0_2-bqo.*

Proof. It is not hard to see that (2) implies (1), as a bad barrier array easily give a bad continuous array, which is in particular Δ^0_2. For the other direction, given a bad Δ^0_2-function $F\colon [\mathbb{N}]^{\mathbb{N}} \to Q$, use Lemma 3.2 to get X so that F is continuous on $[X]^{\mathbb{N}}$, and then using a bijection from $\mathbb{N}$ to X, we get a bad continuous function to Q. □

3. Δ^0_2 functions

Δ^0_2 functions can be coded by second order objects. The following lemma gives such codes. The following version of Kuratowski's Theorem states that every Δ^0_2 function is α-Σ^0_1 for some ordinal α. Medsalem and Tanaka [MT07] analyzed Kuratowski's Theorem from a reverse mathematics viewpoint, and our proof is not much different from theirs. We include the lemma as it gives a better intuition of what Δ^0_2 functions are.

Lemma 3.1. *(ATR_0) For every Δ^0_2 function $F\colon [\mathbb{N}]^{\mathbb{N}} \to \mathbb{N}$, there exist an ordinal α and two functions $g\colon [\mathbb{N}]^{<\mathbb{N}} \to \alpha$ and $f\colon [\mathbb{N}]^{<\mathbb{N}} \to \mathbb{N}$ such that:*

1. *if $\sigma \sqsubseteq \tau$, then $g(\sigma) \geqslant g(\tau)$;*
2. *if $\sigma \sqsubset \tau$ and $f(\sigma) \neq f(\tau)$, then $g(\sigma) > g(\tau)$; and*

(3) *for every* $X \in [\mathbb{N}]^{\mathbb{N}}$, $F(X) = \lim_n f(X \upharpoonright n)$.

Proof. Since F has a Σ^0_2 graph (maybe using real parameters), there is a bounded formula $\varphi(q, n, \sigma)$ such that

$$F(X) = q \Leftrightarrow \exists n \forall m\ \varphi(q, n, X \upharpoonright m).$$

For $\sigma \in [\mathbb{N}]^{<\mathbb{N}}$ of length k, define $\tilde{f}(\sigma)$ to be the least pair $(q, n) \in \mathbb{N}^2$ (in some reasonable ordering of pairs of order type $\mathbb{N}$) with $n, q < k$ such that $\forall m \leqslant k\ \varphi(q, n, \sigma \upharpoonright m)$, and define $\tilde{f}(\sigma)$ to be $(0, 0)$ if no such pair exists. Define $f(\sigma)$ to be the first coordinate of $\tilde{f}(\sigma)$. To see that F is the limit of f, suppose that $F(X) = q$. There exists a least n such that $\forall m\ \varphi(q, n, X \upharpoonright m)$. For every pair (q', n') that comes before (q, n), there is an m' such that $\neg\varphi(q', n', X \upharpoonright m')$. Let $\tilde{m}$ be the maximum of all these m's. We then have that for σ with $X \upharpoonright \tilde{m} \sqsubseteq \sigma \sqsubseteq X$, $\tilde{f}(\sigma) = (q, n)$ and $f(\sigma) = q$. Hence, $F(X) = \lim_n f(X \upharpoonright n)$.

Let us now define α and g. Let $T \subseteq [\mathbb{N}]^{<\mathbb{N}}$ be the the set of σ's such that $f(\sigma) \neq f(^-\sigma)$, where $^-\sigma$ is σ without its last element, i.e., $^-\sigma = \sigma \upharpoonright |\sigma| - 1$. Also, include the empty string in T. Since for every $X \in [\mathbb{N}]^{<\mathbb{N}}$, $f(X \upharpoonright n)$ stabilizes as $n \to \infty$, T has no infinite paths; that is, $(T; \sqsupseteq)$ is well-founded. Let α be the well-founded rank of T. For $\sigma \in T$, let $g(\sigma)$ be the well-founded rank of $T_\sigma = \{\tau \in T : \tau \sqsupseteq \sigma\}$. For $\sigma \notin T$, define $g(\sigma)$ by recursion on the length of σ by letting $g(\sigma) = g(^-\sigma)$. □

We now prove Mathias's lemma [Mat77] for Δ^0_2-functions. The proof is a small variation of the proof of the Σ^0_2-Ramsey theorem, which was carried ou in Π^1_1-CA_0 by Simpson, using ideas of [Sol88] (see also Tanaka [Tan89]).

Lemma 3.2. *(Π^1_1-CA_0) For every Δ^0_2 function $F\colon [\mathbb{N}]^{\mathbb{N}} \to \mathbb{N}$, there exists $X \in [\mathbb{N}]^{\mathbb{N}}$ such that the restriction of F to $[X]^{\mathbb{N}}$ is continuous.*

Proof. We essentially rewrite the proof in [Sim99, Lemma VI.6.2] that the Σ^0_2-Ramsey theorem is provable in Π^1_1-CA_0. In that theorem, we essentially have a function $F\colon [\mathbb{N}]^{\mathbb{N}} \to 2$ (i.e., a set), and we want the restriction of F to be constant. The modification to the case when F is a function $F\colon [\mathbb{N}]^{\mathbb{N}} \to \mathbb{N}$ is quite minimal.

Let (f, g, α) be a code for the Δ^0_2 function F as in the previous lemma. Let M be a coded β-model containing the code (f, g, α). It is known that the existence of such a model can be proved within Π^1_1-CA_0 (Friedman [Fri75]; see [Sim99, Theorem VII.2.10]). Consider the following Π^0_1 formula:

$$\psi(n, q, Y) \quad \equiv \quad \forall m \geqslant n f(Y \upharpoonright m) = q.$$

Notice that

$$F(Y) = q \quad \Leftrightarrow \quad \exists n \psi(n, q, Y).$$

By recursion on $n \in \mathbb{N}$, using the code for M as a parameter, define sequences $\sigma_n \in [\mathbb{N}]^{<\mathbb{N}}$ and $X_n \in M \cap [\mathbb{N}]^{\mathbb{N}}$ as follows. For each n, we shall have that the last entry of σ_n is smaller than the first of X_n. These are essentially Mathias-forcing conditions; at the end of stages we will define X so that $\sigma_n \sqsubset X \subseteq \sigma_n \cup X_n$ for all n. Begin with $\sigma_0 = \emptyset$ and $X_0 = \omega$. Given σ_n and X_n, define $\sigma_{n+1} = \sigma_n \cup \min(X_n)$. Notice that $\sigma_n \cup X_n = \sigma_{n+1} \cup X_n^-$. We are now going to apply the open Ramsey Theorem. This theorem states that if $\varphi(Y)$ is a Σ^0_1 formula and $V \in [\mathbb{N}]^{\mathbb{N}}$, then there is a $W \in [V]^{\mathbb{N}}$ such that either $\varphi(Y)$ for all $Y \in [W]^{\mathbb{N}}$, or $\neg\varphi(Y)$ for all $Y \in [W]^{\mathbb{N}}$. It is known that the open Ramsey Theorem is provable in ATR_0 (due to Friedman, McAloon and Simpson [FMS82]; see [Sim99, Theorem V.9.7]). It is also known that β-models are models of ATR_0 (Friedman [Fri71]; see also [Sim99, Theorem VII.2.7]), and hence we can apply the open Ramsey Theorem within M. By finitely many applications of the open Ramsey Theorem, we obtain $X_{n+1} \in [X_n^-]^{\mathbb{N}}$ such that, for all subsets $\sigma \subseteq \sigma_n$ and all $q \leqslant n$,

$$(\forall Y \in [X_{n+1}]^{\mathbb{N}})\ \psi(n, q, \sigma^\frown Y) \quad \text{or} \quad (\forall Y \in [X_{n+1}]^{\mathbb{N}})\ \neg\psi(n, q, \sigma^\frown Y).$$

As part of the same recursion, define $p\colon \mathbb{N}^2 \times [\mathbb{N}]^{<\mathbb{N}} \to \{0,1\}$ such that, for all n, q and $\sigma \subseteq \sigma_n$, $p(n,q,\sigma) = 1$ if the case above is the former, i.e., if $(\forall Y \in [X_{n+1}]^{\mathbb{N}})\ \psi(n,q,\sigma^\frown Y)$, and $p(n,q,\sigma) = 0$ otherwise. Notice that if $p(n,q,\sigma) = 1$, then for every $Y \in [X_{n+1}]^{\mathbb{N}}$, $F(\sigma^\frown Y) = q$.

Finally, define $X = \bigcup_n \sigma_n = \bigcap_n \sigma_n \cap X_n$. We claim that F is continuous when restricted to $[X]^{\mathbb{N}}$. For this, we need to show that for each $q \in Q$, $F^{-1}(q)$ is open: For $Z \in [X]^{\mathbb{N}}$,

$$Z \in F^{-1}(q) \Leftrightarrow \exists n \psi(n,q,Y) \Leftrightarrow (\exists n \geqslant q)\ p(n,q,\sigma) = 1,$$

where $\sigma = Z \cap [0, \min(X_n)]$ and $Y = Z \cap (\min(X_n), +\infty)$. This is clearly an open set. □

4. Labeled trees

In this section, we prove Theorem 1.3, that FRA is implied by the Δ^0_2-better-quasi-orderness of 3 over ATR_0.

Definition 4.1. Given a set Q, a *Q-tree* is a well-founded tree $T \subseteq \omega^{<\omega}$ that comes with an associated *labeling function* $q_T\colon T \to Q$. We let $\mathcal{T}r(Q)$ denote the set of Q-trees.

Let $\leqslant_Q$ be a partial ordering on Q. Given $T, S \in \mathcal{T}r(Q)$, a function $f\colon T \to S$ is a *weak homomorphism* if, for every $\sigma \sqsubseteq \tau \in T$, $f(\sigma) \sqsubseteq f(\tau)$ and $q_T(\sigma) \leqslant_Q q_S(f(\sigma))$. When such an f exists, we write $T \preccurlyeq_w S$. We say that f is a *homomorphism* if also, for every $\sigma \sqsubset \tau \in T$, $f(\sigma) \sqsubset f(\tau)$. When such an f exists, we write $T \preccurlyeq S$.

In [Mon06], the trees in $\mathcal{T}r(2)$ are called *signed trees*, thinking of 2 as $\{-,+\}$. The author proved in [Mon06] that there is a one-to-one correspondence between $(\mathcal{T}r; \preccurlyeq)$ and the class of scattered indecomposable linear orderings ordered by embeddability, in order to get the theorem below. Another application of the correspondence was Marcone and Montalbán's [MM09] analysis of FRA for linear orderings of finite Hausdorff rank.

Theorem 4.2. *([Mon06, Theorem 4.2]) The following are equivalent over RCA_0:*

1. *Fraïssé's conjecture.*
2. *$(\mathcal{T}r(2); \preccurlyeq)$ is well-quasi-ordered.*

Our proof below works for the weak-embeddability notion $\preccurlyeq_w$, while the theorem above is for $\preccurlyeq$. The next lemma connects the two notions. Given a partial ordering Q, let $Q \sqcup \{*\}$ be the partial ordering that consists of Q together with a new element $*$ that is incomparable to all the elements in Q. For instance, $2 \sqcup \{*\} = 3$.

Lemma 4.3. *(RCA_0) As quasi-orderings, $(\mathcal{T}r(Q); \preccurlyeq)$ embeds in $(\mathcal{T}r(Q \sqcup \{*\}); \preccurlyeq_w)$. Thus, the well-quasi-orderness of $(\mathcal{T}r(Q \sqcup \{*\}); \preccurlyeq_w)$ implies that of $(\mathcal{T}r(Q); \preccurlyeq)$.*

Proof. Given a tree $T \in \mathcal{T}r(Q)$, let $\Phi(T) \in \mathcal{T}r(Q \sqcup \{*\})$ be defined as follows. Given $\sigma \in \omega^{<\omega}$, let

$$\sigma^\circ = (\sigma(0), 0, \sigma(1), 0, ..., 0, \sigma(|\sigma|-1)) \quad \text{and} \quad \sigma^* = (\sigma(0), 0, \sigma(1), 0, ..., 0, \sigma(|\sigma|-1), 0).$$

Define

$$\Phi(T) = \{\sigma^\circ : \sigma \in T\} \cup \{\sigma^* : \sigma \in T\}.$$

It is not hard to see the $\Phi(T)$ is a subtree of $\omega^{<\omega}$. As for the labeling function, define

$$q_{\Phi(T)}(\sigma^\circ) = q_T(\sigma) \quad \text{and} \quad q_{\Phi(T)}(\sigma^*) = *.$$

Let us now show that Φ is an embedding from $(\mathcal{T}r(Q); \preccurlyeq)$ to $(\mathcal{T}r(Q \sqcup \{*\}); \preccurlyeq_w)$. If $S \preccurlyeq T$ via a homomorphism f, then we easily get that $\Phi(S) \preccurlyeq_w \Phi(T)$ via the homomorphism $\sigma^\circ \mapsto f(\sigma)^\circ$ and $\sigma^* \mapsto f(\sigma)^*$.

Conversely, suppose that $g\colon \Phi(S) \to \Phi(T)$ is a weak homomorphism. The first observation is that g must actually be a homomorphism: for any two consecutive elements of S, one has a label in Q and the other one is labeled $*$. These two elements cannot be mapped through g to the same node in T, as there is no label in $Q \sqcup \{*\}$ that is above $*$ and an element of Q at the same time. The second observation is that elements of the form σ° must be mapped through g to elements of the form $\tau^\circ \in \Phi(T)$. We can then define $f(\sigma)$ to be the unique string τ for which $g(\sigma^\circ) = \tau^\circ$. It is not hard to see that f is a homomorphism.

Finally, as for the second part of the lemma, suppose $(\mathcal{T}r(Q); \preccurlyeq)$ is not a well-quasi-ordering. Let $\{T_i : i \in \omega\}$ be a bad sequence in $(\mathcal{T}r(Q); \preccurlyeq)$; i.e., a sequence such that $\forall i < j$, $T_i \not\preccurlyeq T_j$. We then have that $\{\Phi(T) : i \in \omega\}$ is a bad sequence in $(\mathcal{T}r(Q \sqcup \{*\}); \preccurlyeq_w)$, and hence that $(\mathcal{T}r(Q \sqcup \{*\}); \preccurlyeq_w)$ is not a well-quasi-ordering either. □

The next step is to capture the embeddability relation $\preccurlyeq_w$ in terms of games – we use two games. Both games are equivalent, but the first game is the one we will use in our main proof, and the second one is clopen and hence determined within ATR_0.

Definition 4.4. Given $T, S \in \mathcal{T}r(Q)$, consider the following game $G(T, S)$: Player I plays a sequence $\tau_0 \sqsubseteq \tau_1 \sqsubseteq \tau_2 \sqsubseteq ... \in T$ and player II a sequence $\sigma_0 \sqsubseteq \sigma_1 \sqsubseteq \sigma_2 \sqsubseteq ... \in S$, playing alternatively. Since T and S are well-founded, these sequences must stabilize, say at t_∞ and s_∞. Player II wins the game if $q_T(t_\infty) \leqslant_Q q_S(s_\infty)$.

Lemma 4.5. *(RCA_0) Player II has a strategy for $G(T, S)$ if and only if $T \preccurlyeq_w S$.*

Proof. Suppose first that we have a weak homomorphism $f : T \to S$. We define the strategy for II as follows. If player I plays $\tau_i \in T$, let II reply with $f(\tau_i)$. Since f is a week homomorphism, if I plays $\tau_0 \sqsubseteq \tau_1 \sqsubseteq \tau_2 \sqsubseteq ... \in T$, then $f(\tau_0) \sqsubseteq f(\tau_1) \sqsubseteq f(\tau_2) \sqsubseteq ... \in S$. Also, for every i, $q_T(\tau_i) \leqslant_Q q_S(f(\tau_i))$. Thus, at the limit, we get $q(\tau_\infty) \leqslant_Q q(f(\tau_\infty))$.

Suppose now that we have a winning strategy for II. We define $f(\tau)$ as follows. Let $\tilde{\tau}_i = \tau \upharpoonright i$ for $i \leqslant |\tau|$. Let player I play $\tilde{\tau}_0$ repeatedly until the strategy for player II answers with σ_{j_0} satisfying $q_T(\tilde{\tau}_0) \leqslant_Q q_S(\sigma_{j_0})$ — let $\tilde{\sigma}_0 = \sigma_{j_0}$. Notice that II might not answer right away with such a $\tilde{\sigma}_0$, but since it is a winning strategy, he must eventually. Once that happens, let player I play $\tilde{\tau}_1$ repeatedly until player II answers with $\sigma_{j_1} = \tilde{\sigma}_1$ satisfying $q_T(\tilde{\tau}_1) \leqslant_Q q_S(\tilde{\sigma}_1)$. Continue like this until we define $\tilde{\sigma}_{|\tau|}$ satisfying $q_T(\tau) \leqslant_Q q_S(\tilde{\sigma}_{|\tau|})$. Let $f(\tau) = \tilde{\sigma}_{|\tau|}$. It is clear that f preserves inclusion because if $\rho \sqsubseteq \tau \upharpoonright i$, then $f(\rho) = \tilde{\sigma}_{|\rho|} \sqsubseteq \tilde{\sigma}_{|\tau|}$. It is clear that it preserves labels, as it was defined to do so. □

Consider the following variation of the game.

Definition 4.6. Let $G'(T, S)$ be the following game. Players I′ and II′ play $\sqsubset$-ascending sequences of strings in the trees T and S. The sequences need to be strictly ascending and they must satisfy:

(1) I′ is required to satisfy $q_T(\tau_{i+1}) \not\leqslant_Q q_S(\sigma_i)$.
(2) II′ is required to satisfy $q_T(\tau_i) \leqslant_Q q_S(\sigma_i)$.

The first player who cannot make a legal move looses.

Notice that since T and S are well-founded, one of the players must eventually run out of moves, and then the game is over. Thus, this is a finitely terminating game, or in other words, a clopen game. Recall Steel's Theorem [Ste78] that clopen determinacy is equivalent to ATR_0 over RCA_0. Furthermore, given an infinite sequence of clopen games, we can get a sequence of winning strategies for all of them in ATR_0.

Lemma 4.7. *(RCA_0) These two games are equivalent in the following sense: There is a computable operator mapping winning strategies for I′ in $G'(T, S)$ to winning strategies for I in $G(T, S)$, and the same for the second player.*

Proof. Suppose first that we have a winning strategy for I′ in $G'(S,T)$ — we want to define one in $G(S,T)$. Let the first move for the strategy in $G(S,T)$ be the same move for the strategy in $G'(S,T)$, say $\tilde\tau_0$. Let I continue playing this same string $\tilde\tau_0$ repeatedly in $G(S,T)$, and do it until II plays $\tilde\sigma_0$ with $q_T(\tilde\tau_0) \leqslant_Q q_S(\tilde\sigma_0)$. If II never does and always plays strings σ satisfying $q_T(\tilde\tau_0) \nleqslant_Q q_S(\sigma)$, then I wins the game. So suppose II eventually plays such a $\tilde\sigma_0$. Feed $\tilde\sigma_0$ to the next move of II′ in $G'(S,T)$ and let $\tilde\tau_1$ be the answer given by I′'s strategy in $G'(S,T)$. Back in $G(S,T)$, let I play this same string $\tilde\tau_1$ repeatedly until II plays $\tilde\sigma_1$ with $q_T(\tilde\tau_1) \leqslant_Q q_S(\tilde\sigma_1)$. As before, if II never plays such a $\tilde\sigma_a$, he loses. Otherwise, continue as above, feeding $\tilde\sigma_1$ as the next II′ move in $G'(S,T)$, etc. Notice that since $q_T(\tilde\tau_1) \nleqslant_Q q_S(\tilde\sigma_0)$, $\tilde\sigma_1$ must strictly extend $\tilde\sigma_0$. At some point, II's moves in $G(S,T)$ will stabilize and will not find a $\tilde\sigma_i$ with $q_T(\tilde\tau_i) \leqslant_Q q_S(\tilde\sigma_i)$. As we argued above, that implies I wins.

Suppose now that we have a winning strategy for II′ in $G'(S,T)$ — we want to define one in $G(S,T)$. The construction is exactly the same as above, replacing $\leqslant_Q$ by $\nleqslant_Q$ and $\nleqslant_Q$ by $\leqslant_Q$. □

Theorem 4.8. *(ATR_0) If $\mathcal{Q}$ is a Δ^0_2-bqo, then $(\mathcal{T}r(Q);\leqslant_w)$ is a bqo.*

Proof. Suppose $(\mathcal{T}r(Q);\leqslant_w)$ is not a bqo. Then, there exist a barrier $B \subseteq \mathbb{N}^{<\mathbb{N}}$ and a bad array $b\colon B \to \mathcal{T}r(Q)$. Thus, for every $\sigma,\tau \in B$ with $\sigma \lhd \tau$, we have $b(\sigma) \nleqslant_w b(\tau)$. We will define a bad Δ^0_2 function $F\colon [\mathbb{N}]^{\mathbb{N}} \to Q$, contradicting that Q is a Δ^0_2-bqo. We will start by defining F as a third-order object, as it is intuitively clearer, and show that it is Δ^0_2 later.

Consider the games $G(b(\sigma), b(\tau))$ for each $\sigma \lhd \tau \in B$. Since $b(\sigma) \nleqslant_w b(\tau)$, player II does not have a winning strategy, and hence II′ does not have a winning strategy for $G'(b(\sigma), b(\tau))$ either. By clopen determinacy, I′ has a winning strategy in $G'(b(\sigma), b(\tau))$. Furthermore, within ATR_0, we can get a sequence of winning strategies for I′ in all the games $G'(b(\sigma), b(\tau))$ for all $\sigma,\tau \in B$ with $\sigma \lhd \tau$. By the previous lemma, we get a uniform sequence of winning strategies of I in the games $G(b(\sigma), b(\tau))$.

Let us now move into the definition of $F(X)$ for $X \in [\mathbb{N}]^{\mathbb{N}}$. Define X^{-i} to be X without its first i elements. For each i, let σ_i be the initial segment of X^{-i} that is in B — note that $\sigma_i \lhd \sigma_{i+1}$. We will play all the games $G(b(\sigma_i), b(\sigma_{i+1}))$ simultaneously as follows: In $G(b(\sigma_i), b(\sigma_{i+1}))$, player I follows his strategy and player II copies I's moves from the following game $G(b(\sigma_{i+1}), b(\sigma_{i+2}))$. This determines runs in all the games: First, the players I in all the games make their first move according to their own strategies; then all the players II copy the moves I just made in the adjacent games; then all the players I answer by following their respective strategies; then all the players II copy; etc. Since the trees $b(\sigma_i) \in \mathcal{T}r(Q)$ are well-founded, the moves by I and II in $G(b(\sigma_i), b(\sigma_{i+1}))$ must eventually stabilize — say to $t_{i,\infty} \in b(\sigma_i)$ and $s_{i,\infty} \in b(\sigma_{i+1})$ respectively. Since I wins all these games, we have that the label of $t_{i,\infty}$ in the Q-tree b_{σ_i} is not $\leqslant_Q$-below the label of $s_{i,\infty}$ in $b_{\sigma_{i+1}}$ (i.e., $q_{b(\sigma_i)}(t_{i,\infty}) \nleqslant_Q q_{b(\sigma_{i+1})}(s_{i,\infty})$.) Since all II does is to copy the move from the following game, we get that $s_{i,\infty} = t_{i+1,\infty}$.

Define $F(X)$ be the label of $t_{0,\infty}$ in $b(\sigma_0)$; that is, $F(X) = q_{b(\sigma_0)}(t_{0,\infty}) \in Q$.

Before showing how F is Δ^0_2, let us show why it is bad. The point to notice here is that the run of the game $G(b(\sigma_i), b(\sigma_{i+1}))$ depends on the runs of the games $G(b(\sigma_j), b(\sigma_{j+1}))$ for $j > i$, but not for $j < i$. Thus, we can still define the run of $G(b(\sigma_1), b(\sigma_2))$, even if we never consider the game $G(b(\sigma_0), b(\sigma_1))$. This is exactly what we do when we define $F(X^-)$. Thus $F(X^-) = q_{b(\sigma_1)}(t_{1,\infty})$. By our observation above, since player I wins $G(b(\sigma_0), b(\sigma_1))$, we get that $F(X) \nleqslant_Q F(X^-)$.

We need to argue that F is Δ^0_2. The function that, given $k, i \in \mathbb{N}$, $J \in \{\mathrm{I}, \mathrm{II}\}$ and $X \in [\mathbb{N}]^{\mathbb{N}}$, outputs the kth move of player J in the game $G(b(\sigma_i), b(\sigma_{i+1}))$ for X, as we described it above, is computable in the sequence of strategies. Thus, the function that calculates $t_{0,\infty}$ is Δ^0_2, and so is the function that gives its label, namely $F(X)$. $\square$

5. Better-quasi-orderness of 2

Marcone [Mar05, Lemma 3.2] proved that 2 is a bqo in RCA_0. We extend his ideas to Δ^0_2-bqos.

Lemma 5.1. *(ATR_0) 2 is a Δ^0_2-bqo.*

Proof. Let $F\colon [\mathbb{N}]^{\mathbb{N}} \to 2$ be a Δ^0_2 function, and let f, g, and α be as in Lemma 3.1. Suppose F is a bad array, and hence that for every $Z \in [\mathbb{N}]^{<\mathbb{N}}$, $F(Z) = 1 - F(Z^-)$. Here is the key point Marcone exploited in his proof: If we let Z^{-k} be Z without its first k elements, then $F(Z) = F(Z^{-k})$ if k is even, and $F(Z) = 1 - F(Z^{-k})$ if k is odd.

Let $\sigma \in [\mathbb{N}]^{<\mathbb{N}}$ be such that $g(\sigma) = 0$. (If there is no such σ, let $g(\sigma)$ be the least possible.) Let $n > \max(\sigma)$ and let

- $Z = [n+1, +\infty)$,
- $X = \sigma^\frown n^\frown Z$, and
- $Y = \sigma^\frown Z$.

Since $g(\sigma)$ is minimal, f never changes after σ, and hence $F(X) = F(Y)$. Let $k = |\sigma|$. We then have that

$$X = Z^{-k+1} \quad \text{and} \quad Y = Z^{-k}.$$

Therefore, if k is odd, $F(X) = F(Z)$ and $F(Y) = 1 - F(Z)$, and the other way around if k is odd. This contradicts that $F(X) = F(Y)$. $\square$

Let us note that the only part in the proof above that uses more than RCA_0 is the use of Lemma 3.1 to find the codes f, g, and α for F.

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