EFFECTIVELY EXISTENTIALLY-ATOMIC STRUCTURES

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1. Introduction

The notions we study in this paper, those of *existentially-atomic structure* and *effectively existentially-atomic structure*, are not really new. The objective of this paper is to single them out, survey their properties from a computability-theoretic viewpoint, and prove a few new results about them. These structures are the simplest ones around, and for that reason alone, it is worth analyzing them. As we will see, they are the simplest ones in terms of how complicated it is to find isomorphisms between different copies and in terms of the complexity of their descriptions. Despite their simplicity, they are very general in the following sense: every structure is existentially atomic if one takes enough jumps, the number of jumps being (essentially) the Scott rank of the structure. That balance between simplicity and generality is what makes them important.

Existentially atomic structures are nothing more than atomic structures, as in model theory, except that the generating formulas for the principal types are required to be existential. They were analyzed by Simmons in [Sim76, Section 2] who calls them $\exists_1$-*atomic*, or *strongly existentially closed*. Simmons referred to [Pou72] as an earlier occurrence of these structures in the literature. Here is the formal definition:

Definition 1.1. Let $\mathcal{A}$ be a structure. We define the *automorphism orbit* of a tuple $\bar{a} \in A^{<\omega}$ to be the set

$$\mathrm{orb}_{\mathcal{A}}(\bar{a}) = \{\bar{b} \in A^{|\bar{a}|} : \text{there is an automorphism of } \mathcal{A} \text{ mapping } \bar{a} \text{ to } \bar{b}\}.$$

We say that $\mathcal{A}$ is *existentially atomic* or $\exists$-*atomic* if, for every tuple $\bar{a} \in A^{<\omega}$, there is an $\exists$-formula $\varphi_{\bar{a}}(\bar{x})$ which defines the automorphism orbit of $\bar{a}$; that is, such that

$$\mathrm{orb}_{\mathcal{A}}(\bar{a}) = \{\bar{b} \in A^{|\bar{a}|} : \mathcal{A} \models \varphi_{\bar{a}}(\bar{b})\}.$$

For instance, a linear ordering is $\exists$-atomic if and only if it is either dense or finite. A field is $\exists$-atomic if and only if it is algebraic over its prime subfield. A good source of examples of $\exists$-atomic structures are the $\exists$-algebraic structures which we introduce in Section 3. Other than algebraic fields, other examples of $\exists$-algebraic structures are connected graphs of finite valence with a named root and finite-dimensional torsion-free abelian groups with a named basis (see Example 3.2).

Existentially atomic structures can be characterized in various different ways as stated in the following theorem. We will review the notions involved in the theorem later in this introduction.

Theorem 1.2. *Let $\mathcal{A}$ be a countable structure. The following are equivalent:*

- (A1) *$\mathcal{A}$ is $\exists$-atomic.*
- (A2) *$\mathcal{A}$ has an infinitary Π_2 Scott sentence.*
- (A3) *$\mathcal{A}$ is uniformly continuously categorical.*
- (A4) *Every first-order type realized in $\mathcal{A}$ is $\exists$-supported in $\mathcal{A}$.*
- (A5) *Every $\forall$-type realized in $\mathcal{A}$ is $\exists$-supported in $\mathcal{A}$.*

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[0] Saved: April 15, 2016.

Compiled: April 15, 2016

The author was partially supported by the Packard Fellowship and NSF grant DMS-0901169.

(A6) $\mathcal{A}$ *is* 1-prime.

We prove this theorem in parts throughout the paper. The ideas for the proof are a combination of ideas from the literature which we will refer to as we use them.

This theorem is the particular case $\alpha = 1$ of [Mon, Theorem 1.1], which was for all $\alpha \in \omega_1$ and had a slightly different terminology. However, we can also view [Mon, Theorem 1.1] as a particular case of the theorem above: [Mon, Theorem 1.1] is essentially equivalent to the theorem above applied to the (relativized) $(<\alpha)$th-jump of $\mathcal{A}$, where the *$(<\alpha)$th-jump of $\mathcal{A}$* is defined to be the structure obtained by adding to $\mathcal{A}$ one relation for each computable infinitary Σ_β formula for $\beta < \alpha$. (See [Mon12, Mon13] for more on the jump of structures.) In [Mon], we defined the *Scott rank* of a structure to be the least α such that all its orbits are infinitary Σ_α-definable, and we argued that this is the best-behaved definition of Scott rank among the many in the literature. We thus have that the Scott rank of $\mathcal{A}$ is the least α such that, relative to some oracle X, the $(<\alpha)$th-jump of $\mathcal{A}$ is $\exists$-atomic. It follows that all the results we show about $\exists$-atomic structures apply to any structure so long as we take enough jumps.

Types and Scott families. Let us now review the notions used in Theorem 1.2. Part (A4) is the definition of $\exists$-atomic structure a model theorist would give. Part (A5) states that it is enough to look at $\forall$-types instead of full first-order types. Recall that a *$\forall$-type on the variables* $x_1, ..., x_n$ is a set $p(\bar{x})$ of $\forall$-formulas with free variables among $x_1, ..., x_n$ that is *satisfieable*: We say that a $\forall$-type is *realized in* a structure $\mathcal{A}$ if it is satisfied by some tuple in $\mathcal{A}$. Given $\bar{a} \in A^{<\omega}$, *the $\forall$-type of $\bar{a}$ in $\mathcal{A}$* is the set of $\forall$-formulas true of $\bar{a}$:

$$\forall\text{-}tp_{\mathcal{A}}(\bar{a}) = \{\varphi(\bar{x}) : \varphi \text{ is a } \forall\text{-formula and } \mathcal{A} \models \varphi(\bar{a})\}.$$

Note that by *type*, we do not mean *complete type*, as $\forall$-types are necessarily partial. For that same reason, instead of principal types, we have to deal with supported types:

Definition 1.3. A type $p(\bar{x})$ is *$\exists$-supported within a class of structures* $\mathbb{K}$ if there exists an $\exists$-formula $\varphi(\bar{x})$ which is realized in some structure in $\mathbb{K}$ and which implies all of $p(\bar{x})$ within $\mathbb{K}$; that is, $\mathcal{A} \models \forall \bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))$ for every $\psi(\bar{x}) \in p(\bar{x})$ and $\mathcal{A} \in \mathbb{K}$. We say that $p(\bar{x})$ is *$\exists$-supported in a structure* $\mathcal{A}$ if it is $\exists$-supported in $\mathbb{K} = \{\mathcal{A}\}$.

It is not hard to see that (A1) implies (A4) and that (A4) implies (A5). The proof that (A5) implies (A1) is given in Section 4.

Part (A2) states that $\exists$-atomic structures are among the simplest ones in terms of the complexity of their Scott sentences:

Definition 1.4. A sentence ψ is a *Scott sentence* for a structure $\mathcal{A}$ if $\mathcal{A}$ is the only countable structure satisfying ψ.

Scott [Sco65] proved that every countable structure has a Scott sentence in the infinitary language $L_{\omega_1,\omega}$. His proof used what we now call Scott families.

Definition 1.5. A *Scott family for a structure* $\mathcal{A}$ is a set S of formulas such that each $\bar{a} \in A^{<\omega}$ satisfies some formula $\varphi(\bar{x}) \in S$, and if $\bar{a}$ and $\bar{b}$ satisfy the same formula $\varphi(\bar{x}) \in S$, they are automorphic.

The set of all the defining formulas $\{\varphi_{\bar{a}} : \bar{a} \in A^{<\omega}\}$ from Definition 1.1 makes a Scott family. Thus, a structure is $\exists$-atomic if and only if it has a Soctt family of $\exists$-formulas. The proof that (A1) implies (A2) is essentially Scott's original construction of a Scott sentence. The proof that (A2) implies (A5) uses a variation of the type-omitting theorem which we present in Section 5.

Having access to a Scott family for a structure $\mathcal{A}$ allows us to recognize the different tuples in $\mathcal{A}$ up to automorphism. This is exactly what is necessary to build isomorphisms between different copies of $\mathcal{A}$, as we will see below. If we want to build a computable isomorphism, we need the Scott family to be computably enumerable.

Definition 1.6. We say that a Scott family is *c.e.* if the set of indices for its formulas is c.e. A structure $\mathcal{A}$ is *effectively* $\exists$*-atomic* if it has a c.e. Scott family of $\exists$-formulas.

A reader familiar with computable structure theory has surely heard of structures with c.e. Scott families of $\exists$-formulas before and their connection with relative computable categoricity.

Primality. In model theory, a *prime* model is one that elementary embeds into every model of its theory. We look at the one-quantifier version of this notion.

Definition 1.7. A structure $\mathcal{A}$ is *1-prime* if, for every countable model $\mathcal{B}$ of the $\forall_2$-theory of $\mathcal{A}$, there is an embedding from $\mathcal{A}$ to $\mathcal{B}$ which preserves $\forall$-formulas. We call such embeddings preserving $\forall$-formulas *1-embeddings*.

The proof that $\exists$-atomic implies 1-prime (i.e., (A1) $\Rightarrow$ (A6)) is quite straightforward. The reversal needs the $\forall$-type omitting theorem. We give these proofs in Lemma 6.1.

We will also consider an effective version:

Definition 1.8. A computable structure $\mathcal{A}$ is *uniformly effectively 1-prime* if there is a computable operator Φ such that, for every computable model $\mathcal{B}$ of the $\forall_2$-theory of $\mathcal{A}$, $\Phi^{D(\mathcal{B})}$ is a 1-embedding from $\mathcal{A}$ to $\mathcal{B}$.

We will prove that the notion of uniformly effectively 1-prime is equivalent to that of effectively $\exists$-atomic. The notion of *uniformly effectively prime* for full-first order theories and elementary embeddings (instead of just one-quantifier formulas) was introduced by Cholak and McCoy in [CM]. There, they showed that it is equivalent to that of effectively atomic and that a theory can have at most one uniformly effectively prime model up to computable isomorphism. Their results follow from Theorem 1.11 below if one adds to the language relations for all first-order formulas, although the proofs are quite different.

Categoricity. Part (A3) is very different in form from the rest in the sense that it is computational in nature, rather than syntactical.

Definition 1.9. A structure $\mathcal{A}$ is *uniformly continuously categorical* if there is a continuous operator $\Phi\colon 2^\omega \to \omega^\omega$ that, when given as input the atomic diagram $D(\mathcal{B})$ of a copy $\mathcal{B}$ of $\mathcal{A}$, outputs an isomorphism $\Phi^{D(\mathcal{B})}$ form $\mathcal{B}$ to $\mathcal{A}$. [1]

The definition above is one of the many variations of the notion of *computable categoricity*, a notion that tries to measure the complexity of a structure in terms of how difficult it is to build isomorphisms between its different presentations. A structure is *computably categorical* if any two computable copies are computably isomorphic. Despite computable categoricity being the most natural definition for most computability theorists, the definition above is the one that has the cleanest syntactical characterization — it is equivalent to the structure being $\exists$-atomic. The connection between categoricity and atomicity was first noticed by Nurtazin [Nur74], who showed that a decidable structure is *computably categorical for decidable copies*[2]

[1]Melnikov and the author [MM] proved the equivalence between (A2) and (A3) in a much more general setting, that of Polish groups (S_∞ in this case) acting continuously on Polish spaces (the space of presentations of structures in this case). Furthermore, they showed that the equivalence is an easy corollary of a theorem of Effros from 1965 [Eff65].

[2]$\mathcal{A}$ is *computably categorical for decidable copies* if every decidable copy of $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$.

if and only if it is effectively atomic[3] over a finite set of parameters. Goncharov then improved this result and showed that a 2-decidable structure is computably categorical if and only if it is effectively $\exists$-atomic over a finite set of parameters. Ash, Knight, Manasse, Slaman [AKMS89, Theorem 4], and Chisholm [Chi90, Theorem V.10], removed the 2-decidability assumption and proved that a structure is *relatively computably categorical* if and only if it is effectively $\exists$-atomic over a finite set of parameters. Relativizing their result, we get the following theorem, which is a version of Theorem 1.2, now with parameters:

Theorem 1.10. *Let $\mathcal{A}$ be a countable structure. The following are equivalent:*

(B1) *$\mathcal{A}$ is $\exists$-atomic over a finite set of parameters.*
(B2) *$\mathcal{A}$ has an infinitary Σ_3 Scott sentence.*
(B3) *$\mathcal{A}$ is computably categorical on a cone. That is, there is a $C \in 2^\omega$ such that, for every $X \geq_T C$, every X-computable copy of $\mathcal{A}$ is X-computably-isomorphic to $\mathcal{A}$.*

The equivalence between (B1) and (B3) is just the boldface version of [AKMS89, Theorem 4] and [Chi90, Theorem V.10]. That (B1) implies (B2) easily follows from the corresponding parts in Theorem 1.2. The opposite direction is a slightly more subtle and is proved in Lemma 7.2.

The following theorem is the effective version of the equivalence between (A1), (A3) and (A6). The notion of uniform computably categorical structure was introduced by Ventsov [Ven92]. Other notions of uniform categoricity were studied by Kudinov [Kud96a, Kud96b, Kud97] and by Downey, Hirschfeldt and Khoussainov [DHK03].

Theorem 1.11. *Let $\mathcal{A}$ be a computable structure. The following are equivalent:*

(C1) *$\mathcal{A}$ is effectively $\exists$-atomic.*
(C2) *$\mathcal{A}$ is uniformly relatively computably categorical; that is, the operator Φ in Definition 1.9 is computable.*
(C3) *$\mathcal{A}$ is uniformly computably categorical; that is, the operator Φ in Definition 1.9 is computable and is only required to work when the input $\mathcal{B}$ is a computable structure, i.e., when given as oracle the atomic diagram $D(\mathcal{B})$ of a computable copy $\mathcal{B}$ of $\mathcal{A}$, Φ outputs an isomorphism $\Phi^{D(\mathcal{B})}$ form $\mathcal{B}$ to $\mathcal{A}$.*
(C4) *$\mathcal{A}$ is uniformly effectively 1-prime.*

The equivalence between (C1), (C2) and (C3) was proved by Ventsov in [Ven92]. The fact that effectively atomic structures are the same as uniformly effectively prime structures was proved by Cholak and McCoy in [CM]. We prove the equivalence between (C1) and (C4) in Lemma 6.2 using a very different proof.

Turing degree and enumeration degree. The most common tool to measure the computational complexity of a structure is the degree spectrum. Prior to the introduction of the degree spectrum, Jockusch considered a much more natural notion, which unfortunately does not always apply:

Definition 1.12 (Jockusch)**.** A structure $\mathcal{A}$ has *Turing degree* $X \in 2^\omega$ if X computes a copy of $\mathcal{A}$, and every copy of $\mathcal{A}$ computes X.

It turns out that if we consider the same definition, but on the enumeration degrees (as Knight implicitly did in [Kni98]), we obtain a better-behaved notion.

Definition 1.13. A structure $\mathcal{A}$ has *enumeration degree* $X \subseteq \omega$ if every enumeration of X computes a copy of $\mathcal{A}$, and every copy of $\mathcal{A}$ computes an enumeration of X. Recall that an *enumeration of* X is an onto function $f\colon \omega \to X$.

[3]A structure is *effectively atomic* if it has a c.e. Scott family of elementary first-order formulas.

Equivalently, $\mathcal{A}$ has enumeration degree X if and only if, for every $Y \in 2^\omega$, Y computes a copy of $\mathcal{A}$ if and only if X is c.e. in Y. Notice that, for $X, Z \subseteq \omega$, if $\mathcal{A}$ has enumeration degree X, then $\mathcal{A}$ has enumeration degree Z if and only if X and Z are enumeration equivalent.

As an example, we let the reader verify that the group $\mathcal{G}_X = \bigoplus_{i\in X} \mathbb{Z}_{p_i}$, where p_i is the ith prime number, has enumeration degree X.

The enumeration degree of a structure is indeed a good way to measure its computational complexity. Unfortunately, in general, a structure need not have enumeration degree. Furthermore, there are whole classes of structures, like linear orderings for instance, where no structure has enumeration degrees unless it is already computable (this was shown by Richter [Ric81]). On the other hand, there are whole classes of structures which all have enumeration degree. For instance, Frolov, Kalimullin and R. Miller [FKM09] proved that all fields of finite transcendence degree over $\mathbb{Q}$ have enumeration degree. Calvert, Harizanov, Shlapentokh [CHS07] showed that every torsion-free abelian groups of finite rank always has enumeration degree. Steiner [Ste13] showed that graphs of finite valance with finitely many connected components always have enumeration degree. The following theorem (which is new) shows how all these results fit in a much more general framework. All the examples above can be easily seen to be $\exists$-algebraic over a finite tuple of parameters, and are $\Pi^{\mathtt{c}}_2$-axiomatizable once that tuple of parameters is fixed.

Theorem 1.14. *Let $\mathbb{K}$ be a $\Pi^{\mathtt{c}}_2$ class, all whose structures are $\exists$-atomic. Then every structure in $\mathbb{K}$ has enumeration degree, and that enumeration degree is given by its $\exists$-theory.*

2. Background and notation

An $\exists$-*formula* is one of the form $\exists x_1 \exists x_2 ... \exists x_n\ \varphi(x_1, ..., x_n)$, where φ is finitary and quantifier free. A $\forall_2$-*formula* is one of the form $\forall y_1 \forall y_2 ... \forall y_m\ \psi(y_1, ..., y_m)$, where ψ is an $\exists$-formula. For background on infinitary formulas and computably infinitary formulas, see [AK00, Chapter 6 and 7]. We will use $\Sigma^{\mathtt{in}}_\alpha$ to denote the set of infinitary Σ_α-formulas, $\Sigma^{\mathtt{c}}_\alpha$ for the computable infinitary formulas, and $\Sigma^{\mathtt{c},X}_\alpha$ for the X-computable infinitary formulas.

Given a presentation of a structure $\mathcal{B}$, we define its *atomic diagram* $D(\mathcal{B}) \in 2^\omega$ as follows: First, consider an effective enumeration of $\{\varphi^{at}_i : i \in \omega\}$ of the atomic formulas on the variables $x_0, x_1, ...$, and assume φ^{at}_i only uses variables x_j for $j < i$. Then, define $D(\mathcal{B})(i) = 1$ if and only if $\mathcal{B} \models \varphi^{at}_i[x_j \mapsto j]$, and let $D(\mathcal{B})(i) = 0$ otherwise. Recall that the domain of $\mathcal{B}$ is a subset of the natural numbers, so we are assigning to x_j the natural number j. If φ^{at}_i uses a variable x_j and j is not in the domain of $\mathcal{B}$, we let $D(\mathcal{B})(i) = 0$.

Given a tuple $\bar{b} \in B^{<\omega}$, we define $D_{\mathcal{B}}(\bar{b})$ to be the length-$|\bar{b}|$ approximation to the atomic type of $\bar{b}$: That is, $D_{\mathcal{B}}(\bar{b})$ is the string $\sigma \in 2^{|\bar{b}|}$ defined by $\sigma(i) = 1$ if and only if $\mathcal{B} \models \varphi^{at}_i(x_j \mapsto b_j)$. For each $\sigma \in 2^{<\omega}$, there is a formula $\varphi^{at}_\sigma(\bar{x})$, where $|\bar{x}| = |\sigma|$, which states that the atomic diagram of $\bar{x}$ is σ. That is:

$$\varphi^{at}_\sigma(\bar{x}) \quad \equiv \quad \left(\bigwedge_{i<|\bar{x}|, \sigma(i)=1} \varphi^{at}_i(\bar{x}) \right) \wedge \left(\bigwedge_{i<|\bar{x}|, \sigma(i)=0} \neg\varphi^{at}_i(\bar{x}) \right)$$

3. Existentially algebraic structures

A important source of examples of $\exists$-atomic structure are the $\exists$-algebraic structures.

Definition 3.1. An element a of a structure $\mathcal{A}$ is $\exists$-*algebraic* if there is an $\exists$-formula $\varphi(x)$ true of a such that $\{b \in A : \mathcal{A} \models \varphi(b)\}$ is finite. A structure $\mathcal{A}$ is $\exists$-*algebraic* if all its elements are.

Here are some examples.

Example 3.2. A *field that is algebraic over its prime sub-field* is ∃-algebraic because every element is among the finitely many roots of some polynomial with coefficients on the prime sub-field, and the elements in the prime sub-field can be defined by quantifier-free formulas.

A *connected graph of finite valance with a selected root vertex* is ∃-algebraic because every element is among the finitely many that are at a given distance from the root.

An *abelian, torsion-free group with a selected basis* is ∃-algebraic because every element can be defined by a $\mathbb{Q}$-linear combination of the basis elements.

We prove that ∃-algebraic structures are ∃-atomic in two lemmas. The core of the argument is an application of König's lemma that appears in the first one.

Lemma 3.3. *Two countable structures that are ∃-algebraic and have the same ∃-theories are isomorphic.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be ∃-algebraic structures with the same ∃-theories. List the elements of A as $\{a_0, a_1, ...\}$. For each n, let $\varphi_n(x_0, ..., x_{n-1})$ be an ∃-formula which is true of the tuple $\langle a_0, ..., a_{n-1}\rangle$, has finitely many solutions, and implies $\varphi_{n-1}(x_0, ..., x_{n-2})$. (By *solution* for a formula, we mean a tuple that makes it true.) Consider the tree

$$T = \{\bar{b} \in B^{<\omega} : D_{\mathcal{B}}(\bar{b}) = D_{\mathcal{A}}(a_0, ..., a_{|\bar{b}|-1}) \quad \& \quad \mathcal{B} \models \varphi_{|\bar{b}|}(\bar{b})\}.$$

T is clearly a tree in the sense that it is closed under taking initial segments of tuples. It is finitely branching because, for each n, φ_n is true for only finitely many tuples. To show that T is infinite, notice that, for each n, the tuple $(a_0, ..., a_{n-1})$ itself witnesses that

$$\mathcal{A} \models \exists x_0, ..., x_{n-1}(\varphi^{at}_{\sigma}(\bar{x}) \;\&\; \varphi_n(\bar{x})), \qquad \text{where } \sigma = D_{\mathcal{A}}(a_0, ..., a_{n-1}) \in 2^n.$$

(Here, $\varphi^{at}_{\sigma}(\bar{x})$ is the formula that states that "$D(\bar{x}) = \sigma$," as defined in the background section.) Since $\mathcal{A}$ and $\mathcal{B}$ have the same ∃-theories, $\mathcal{B}$ models this sentence too, and the witness is an n-tuple that belongs to T. König's lemma states that every infinite, finitely branching tree must have an infinite path. Thus, T must have an infinite path $P \in B^{\omega}$. From this path, we obtain a map $a_n \mapsto P(n)\colon A \to B$, which we claim is an isomorphism. The map is an embedding because, by the definition of T, it preserves finite atomic diagrams. But then it must be an isomorphism: If $b \in B$ is a solution of an ∃-formula φ with finitely many solutions, then φ must have the same number of solutions in $\mathcal{A}$ (because $\exists\text{-}Th(\mathcal{A}) = \exists\text{-}Th(\mathcal{B})$), and since ∃-formulas are preserved under embeddings, one of those solutions has to be mapped to b. □

Lemma 3.4. *Every ∃-algebraic structure is ∃-atomic.*

Proof. Let $\mathcal{A}$ be ∃-algebraic and take $\bar{a} \in A^{<\omega}$. Let $\varphi(\bar{x})$ be an ∃-formula true of $\bar{a}$ with the least possible number of solutions, say k solutions. We claim that every solution to φ is automorphic to $\bar{a}$, and hence that φ defines the orbit of $\bar{a}$. Suppose, toward a contradiction, that $\bar{b}$ satisfies φ but is not automorphic to $\bar{a}$. Then there must exist an ∃-formula $\psi(\bar{x})$ that is true of either $\bar{a}$ or $\bar{b}$, but not of both: This follows from the previous lemma, as $(\mathcal{A}, \bar{a})$ and $(\mathcal{A}, \bar{b})$ are not isomorphic and are both ∃-algebraic. If $\psi(\bar{x})$ is true of $\bar{a}$, then $\varphi(\bar{x}) \wedge \psi(\bar{x})$ would be true of $\bar{a}$ and have fewer solutions than φ, contradicting our choice of φ. Suppose now that $\psi(\bar{x})$ is not true of $\bar{a}$. Let i be the number of solutions of $\psi(\bar{x}) \wedge \varphi(\bar{x})$. Then the formula about $\bar{y}$ saying

"$\varphi(\bar{y})$ and there are i solutions to $\varphi \wedge \psi$ all different from $\bar{y}$"

is an ∃-formula true of $\bar{a}$ with $k - i$ solutions, again contradicting our choice of φ. □

The statements of the lemmas in this section are new, but the ideas behind them are not. Proofs like that of Lemma 3.3 using König's lemma have appeared in many other places before, for instance [HLZ99]. The ideas for the proof of Lemma 3.4 are similar to those one would use in a proof that algebraic structures are atomic (without the ∃-), except that here one has to be slightly more careful.

4. Existentially atomicity in terms of types

In this short section, we prove that if every ∀-type is ∃-supported in a structure $\mathcal{A}$, the structure is ∃-atomic (that is, that (A5) ⇒ (A1)). The proof is an adaptation of classical arguments with back-and-forth relations.

Definition 4.1. Given structures $\mathcal{A}$ and $\mathcal{B}$, we say that a set $I \subseteq \mathcal{A}^{<\omega} \times \mathcal{B}^{<\omega}$ has the *back-and-forth property* if, for every $\langle \bar{a}, \bar{b} \rangle \in I$,

- $D_{\mathcal{A}}(\bar{a}) = D_{\mathcal{B}}(\bar{b})$ (i.e., $|\bar{a}| = |\bar{b}|$ and $\bar{a}$ and $\bar{b}$ satisfy the same atomic formulas among the first $|\bar{a}|$ many);
- for every $c \in A$, there exists $d \in B$ such that $\langle \bar{a}c, \bar{b}d \rangle \in I$; and
- for every $d \in B$, there exists $c \in A$ such that $\langle \bar{a}c, \bar{b}d \rangle \in I$.

A standard back-and-forth argument shows that if I has the back-and-forth property and $\langle \bar{a}, \bar{b} \rangle \in I$, then there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ mapping $\bar{a}$ to $\bar{b}$. Furthermore, if I is c.e., then there is a computable such isomorphism.

Proof of (A5) ⇒ (A1) in Theorem 1.2. For each $\bar{a} \in A^{<\omega}$, let $\varphi_{\bar{a}}(\bar{x})$ be an ∃-formula supporting the ∀-type of $\bar{a}$. We need to show that $S = \{\varphi_{\bar{a}} : \bar{a} \in A^{<\omega}\}$ is a Scott family for $\mathcal{A}$. Consider the set

$$I_{\mathcal{A}} = \{\langle \bar{a}, \bar{b} \rangle \in A^{<\omega} \times A^{<\omega} : \mathcal{A} \models \varphi_{\bar{a}}(\bar{b})\}.$$

We will show that $\mathcal{I}_{\mathcal{A}}$ has the *back-and-forth* property.

Before we prove these three properties, we need to prove a couple of smaller facts. First, notice that, for every $\bar{a} \in A^{<\omega}$, $\mathcal{A} \models \varphi_{\bar{a}}(\bar{a})$: This is because otherwise $\neg\varphi_{\bar{a}}$ would be part of the ∀-type of $\bar{a}$, and hence implied by $\varphi_{\bar{a}}$, which cannot be the case, as $\varphi_{\bar{a}}$ is realizable in $\mathcal{A}$. Second, let us show that $I_{\mathcal{A}}$ is symmetric; that is, that if $\mathcal{A} \models \varphi_{\bar{a}}(\bar{b})$, then $\mathcal{A} \models \varphi_{\bar{b}}(\bar{a})$. Suppose $\mathcal{A} \models \varphi_{\bar{a}}(\bar{b})$. Then, we cannot have $\mathcal{A} \models \varphi_{\bar{a}}(\bar{x}) \to \neg\varphi_{\bar{b}}(\bar{x})$, as the negation is witnessed by $\bar{b}$. It thus follows that $\neg\varphi_{\bar{b}}(\bar{x})$ is not part of the ∀-type of $\bar{a}$, and hence that $\mathcal{A} \models \varphi_{\bar{b}}(\bar{a})$.

We can now prove that $I_{\mathcal{A}}$ has the back-and-forth property. Suppose $\langle \bar{a}, \bar{b} \rangle \in I_{\mathcal{A}}$. Notice that $\bar{a}$ and $\bar{b}$ must then satisfy the same ∀-formulas. In particular, they must satisfy the same atomic formulas, and hence have the same atomic diagrams. To show the second condition, take $c \in A$. If there was no $d \in A$ with $\langle \bar{a}c, \bar{b}d \rangle \in I_{\mathcal{A}}$, we would have that $\mathcal{A} \models \neg\exists y \varphi_{\bar{a}c}(\bar{b}, y)$. This formula would be part of the ∀-type of $\bar{b}$, and hence implied by $\varphi_{\bar{b}}$. But then, since $\mathcal{A} \models \varphi_{\bar{b}}(\bar{a})$, we would have $\mathcal{A} \models \neg\exists y \varphi_{\bar{a}c}(\bar{a}, y)$, which is not true as witnessed by c. The third condition of the back-and-forth property follows from the symmetry of $I_{\mathcal{A}}$.

Now that we know that $I_{\mathcal{A}}$ has the back-and-forth property, through a standard back-and-forth argument we get that if $\mathcal{A} \models \varphi_{\bar{a}}(\bar{b})$, then there exists an automorphism of $\mathcal{A}$ taking $\bar{a}$ to $\bar{b}$. In particular, we get that if $\varphi_{\bar{a}}(\bar{b})$ and $\varphi_{\bar{a}}(\bar{c})$ both hold, then $\bar{b}$ and $\bar{c}$ are automorphic. This proves that S is a Scott family for $\mathcal{A}$. □

5. Building structures and omitting types

Before we continue studying the properties of ∃-atomic structures, we need to make a stop to prove some general lemmas that will be useful in future sections. First, we prove a lemma that will allow us to find computable structures in a given class of structures. Second, using similar techniques, we prove the type-omitting lemma for ∀-types and its effective version.

Assume, without loss of generality, we are working with a relational vocabulary τ. Given a class of structure $\mathbb{K}$, we let $\mathbb{K}^{fin}$ be — essentially — the set of all the finite substructures of the structures in $\mathbb{K}$:

$$\mathbb{K}^{fin} = \{D_{\mathcal{A}}(\bar{a}) : \mathcal{A} \in \mathbb{K}, \bar{a} \in \mathcal{A}^{<\omega}\} \subseteq 2^{<\omega}.$$

Lemma 5.1. *Let $\mathbb{K}$ be a $\Pi^{\mathtt{c}}_2$ class for which $\mathbb{K}^{fin}$ is c.e. Then there is at least one computable structure in $\mathbb{K}$.*

Proof. We build a structure $\mathcal{A}$ in $\mathbb{K}$ by building a finite approximation to it. That is, we build a nested sequence of finite structures $\mathcal{A}_s$ for $s \in \omega$. Formally, that is not precisely correct: We will build the diagram $D(\mathcal{A})$ as the limit of a nested sequence $\sigma_s \in 2^{<\omega}$ for $s \in \omega$, where each σ_s is in $\mathbb{K}^{fin}$. We then think of $\mathcal{A}_s$ as the *partial* finite structure with domain $|\sigma_s|$, where only the atomic formulas φ^{at}_i for $i < |\sigma_s|$ are decided, and the rest are not decided yet. Working with the $\mathcal{A}_s$'s is closer to our intuition of what is going on, but a formal proof would only use the σ_s's.

Of course, we require that $\mathcal{A}_s \subseteq \mathcal{A}_{s+1}$ (that is, that $\sigma_s \subseteq \sigma_{s+1}$ as binary strings). At the end of stages, we define the structure $\mathcal{A} = \bigcup_{s\in\omega} \mathcal{A}_s$, and hence $D(\mathcal{A}) = \bigcup_s \sigma_s$.

Let $\bigwedge\!\!\!\bigwedge_{i\in I} \forall \bar{y}_i \psi_i(\bar{y}_i)$ be the $\Pi^{\mathtt{c}}_2$ sentence that axiomatizes $\mathbb{K}$, where each ψ_i is $\Sigma^{\mathtt{c}}_1$. To get $\mathcal{A} \in \mathbb{K}$, we need to guarantee that, for each i and each $\bar{a} \in A^{|\bar{y}_i|}$, we have $\mathcal{A} \models \psi_i(\bar{a})$. For this, when we build $\mathcal{A}_{s+1}$, we will make sure that,

$$\text{for every } i < s \text{ and every } \bar{a} \in A_s^{|\bar{y}_i|},\ \mathcal{A}_{s+1} \models \psi_i(\bar{a}). \tag{1}$$

Notice that, since ψ_i is $\Sigma^{\mathtt{c}}_1$, $\mathcal{A}_{s+1} \models \psi_i(\bar{a})$ implies $\mathcal{A} \models \psi_i(\bar{a})$. Thus, we would end up with $\mathcal{A} \models \bigwedge\!\!\!\bigwedge_{i\in I} \forall \bar{y}_i \psi_i(\bar{y}_i)$.

Now that we know what we need, let us build the sequence of $\mathcal{A}_s$'s. Suppose we have already built $\mathcal{A}_0, ..., \mathcal{A}_s$ and we want to define $\mathcal{A}_{s+1} \supseteq \mathcal{A}_s$. All we need to do is search for a partial finite structure in $\mathbb{K}^{fin}$ satisfying (1). Notice that, given a finite diagram σ for a finite partial structure, we can check if it satisfies (1). Since $\mathbb{K}^{fin}$ is c.e., all we have to do is search for such a $\sigma_{s+1} \in \mathbb{K}^{fin}$ — well, except that we need to show that at least one such structure exists. Since $\mathcal{A}_s \in \mathbb{K}^{fin}$, there is some $\mathcal{B} \in \mathbb{K}$ which has a partial finite substructure $\mathcal{B}_s$ isomorphic to $\mathcal{A}_s$. (That is, modulo a permutation of the presentation, we can assume that σ_s is an initial segment of the atomic diagram of $\mathcal{B}$.) Since $\mathcal{B} \models \bigwedge\!\!\!\bigwedge_{i\in I} \forall \bar{y}_i \psi_i(\bar{y}_i)$, for every $i < s$ and every $\bar{b} \in \mathcal{B}_s^{|\bar{y}_i|}$, there exists a tuple in $\mathcal{B}$ witnessing that $\mathcal{B} \models \psi_i(\bar{b})$. Let $\mathcal{B}_{s+1}$ be a finite substructure of $\mathcal{B}$ containing $\mathcal{B}_s$ and all those witnessing tuples. Let σ_{s+1} be the initial segment of the atomic diagram of $\mathcal{B}$, witnessing that $\mathcal{B}_{s+1}$ satisfies (1) with respect to $\mathcal{B}_s$. □

Corollary 5.2. *Let $\mathbb{K}$ be a $\Pi^{\mathtt{c}}_2$ class of structures, and S be the $\exists$-theory of some structure in $\mathbb{K}$. If S is c.e. in a set X, then there is an X-computable presentation of a structure in $\mathbb{K}$ with $\exists$-theory S.*

Proof. Add to the $\Pi^{\mathtt{c}}_2$ axiom for $\mathbb{K}$ the $\Pi^{\mathtt{c},X}_2$ sentence saying that the structure must have $\exists$-theory S:

$$\left(\bigwedge\!\!\!\bigwedge_{\text{“}\exists\bar{y}\psi(\bar{y})\text{”}\in S} \exists \bar{y}\psi(\bar{y}) \right) \wedge \left(\forall \bar{x} \bigvee_{\substack{\sigma\in 2^{|\bar{x}|} \\ \text{“}\exists\bar{y}\varphi_\sigma(\bar{y})\text{”}\in S}} \varphi_\sigma(\bar{x}) \right),$$

where $\varphi^{at}_\sigma(\bar{x})$ is the formula “$D(\bar{x}) = \sigma$” (as in the background section). Let $\mathbb{K}_S$ be the new $\Pi^{\mathtt{c},X}_2$ class of structures. All the models in $\mathbb{K}_S$ have $\exists$-theory S, and hence $\mathbb{K}^{fin}_S$ is enumeration reducible to S, and hence is c.e. in X too. Applying Lemma 5.1 relative to X, we get an X-computable structure in $\mathbb{K}_S$ as wanted. □

Not only can we build a computable structure in such a class $\mathbb{K}$, we can build one omitting certain types.

Lemma 5.3. *Let $\mathbb{K}$ be a $\Pi^{\mathtt{in}}_2$ class of structures. Let $\{p_i(\bar{x}_i) : i \in \omega\}$ be a sequence of $\forall$-types which are not $\exists$-supported in $\mathbb{K}$. Then there is a structure $\mathcal{A} \in \mathbb{K}$ which omits all the types $p_i(\bar{x}_i)$ for $i \in \omega$.*

Furthermore, if $\mathbb{K}$ is $\Pi^{\mathtt{c}}_2$, $\mathbb{K}^{fin}$ is c.e. and the list $\{p_i(\bar{x}_i) : i \in \omega\}$ is c.e., we can make $\mathcal{A}$ computable.

Proof. We construct $\mathcal{A}$ by stages as in the proof of Lemma 5.1, the difference being that now we need to omit the types p_i. So, on the even stages s, we do exactly the same thing we did in Lemma 5.1, and we use the odd stages to omit the types. At stage $s+1 = 2\langle i, j\rangle+1$, we ensure that the jth tuple $\bar{a}$ does not satisfy p_i as follows. Let $\bar{b} = \mathcal{A}_s \setminus \bar{a}$, and let $\sigma = D_{\mathcal{A}_s}(\bar{a}, \bar{b})$. So we have that $\bar{a}$ satisfies $\exists \bar{y}\ \varphi^{at}_\sigma(\bar{a}, \bar{y})$. Since p_i is not $\exists$-supported in $\mathbb{K}$, there exists a $\forall$-formula $\psi(\bar{x}) \in p_i$ which is not implied by $\exists \bar{y}\ \varphi^{at}_\sigma(\bar{a}, \bar{y})$ within $\mathbb{K}$. That means that, for some finite $\mathcal{B} \in \mathbb{K}^{fin}$ and some $\bar{d} \in \mathcal{B}^{<\omega}$, we have $\mathcal{B} \models \exists \bar{y}\ \varphi^{at}_\sigma(\bar{b}, \bar{y}) \wedge \neg\psi(\bar{a})$. Since $\mathcal{B} \models \exists \bar{y}\ \varphi^{at}_\sigma(\bar{b}, \bar{y})$, we can assume $\mathcal{B}$ extends $\mathcal{A}_s$. Since such $\mathcal{B}$ and ψ exist, we can wait until we find them and then define $\mathcal{A}_{s+1}$ to be such $\mathcal{B}$. □

6. 1-prime structures

In this section, we prove the equivalences that have to do with the notion of 1-prime structures. The first lemma proves the equivalence between (A1) and (A6), and the second lemma the equivalence between (C1) and (C4).

Lemma 6.1. *A structure is $\exists$-atomic if and only if it is 1-prime.*

Proof. Suppose first that $\mathcal{A}$ is $\exists$-atomic. Let $\mathcal{B}$ be a model of the $\forall_2$-theory of $\mathcal{A}$. Let $\{a_1, a_2, ...\}$ be an enumeration of A and, for each $s \in \omega$, let φ_s be an $\exists$-formula defining the orbit of $(a_1, ..., a_s)$. We define a 1-embedding f from $\mathcal{A}$ to $\mathcal{B}$ by stages. We define $f(a_s)$ at stage s, always making sure that $\mathcal{B} \models \varphi_s(f(a_1), ..., f(a_s))$. To see we can do this, notice that the formula

$$\forall x_1, ..., x_s\ \ (\varphi_s(x_1, ..., x_s) \to \exists x_{s+1}\ \varphi_{s+1}(x_1, ..., x_s, x_{s+1}))$$

is true of $\mathcal{A}$, and hence part of the $\forall_2$-theory of $\mathcal{B}$ too. To see that f is a 1-embedding, notice that for every $\forall$-formula $\psi(x_1, ...x_s)$ true of $(a_1, ..., a_s)$ in $\mathcal{A}$, we have that

$$\forall x_1, ..., x_s\ (\varphi_s(x_1, ..., x_s) \to \psi(x_1, ..., x_s))$$

is part of the $\forall_2$ theory of $\mathcal{A}$, and hence of $\mathcal{B}$ too.

It is the reverse direction that uses the type-omitting theorem. Suppose $\mathcal{A}$ is not $\exists$-atomic. We have already proved that (A1) implies (A5), so we have that the $\forall$-type of some tuple $\bar{a} \in A^{<\omega}$ is not $\exists$-supported within $\mathcal{A}$. Let ψ be the conjunction of the $\forall_2$-theory of $\mathcal{A}$. Since ψ is $\Pi^{\mathtt{in}}_2$, by Lemma 5.3, there is a model $\mathcal{B}$ of ψ which omits the $\forall$-type of $\bar{a}$. But then we cannot have a 1-embedding of $\mathcal{A}$ into $\mathcal{B}$, as 1-embeddings preserve $\forall$-types, and hence there is nowhere to map $\bar{a}$ in $\mathcal{B}$. Thus, $\mathcal{A}$ is not 1-prime. □

Lemma 6.2. *A computable structure $\mathcal{A}$ is effectively $\exists$-atomic if and only if it is uniformly effectively 1-prime.*

Proof. For the left-to-right direction, notice that, given a computable model $\mathcal{B}$ of the $\forall_2$-theory of $\mathcal{A}$, we can use the c.e. Scott family of $\mathcal{A}$ to build a 1-embedding f for $\mathcal{A}$ to $\mathcal{B}$ exactly as in the proof of the lemma above. Notice also that f can be computed uniformly in $D(\mathcal{B})$.

For the right-to-left direction, let Φ be a computable operator witnessing that $\mathcal{A}$ is 1-prime. Consider $\Phi^{D(\mathcal{A})}$, which is a 1-embedding form $\mathcal{A}$ into itself. Again, let $\{a_0, a_1, ...\}$ be an enumeration of $\mathcal{A}$, and let $\bar{a}$ be an initial segment of that enumeration; we will use Φ to find an $\exists$-formula defining the orbit of $\bar{a}$, effectively uniformly in $\bar{a}$. (We are assuming the domain of $\mathcal{A}$ is ω, so actually a_i is the natural number i, but we think of a_i as a member of

$\mathcal{A}$ rather than as a natural number.) Let $\tilde{a}$ be such that $\Phi^{D(\mathcal{A})}(\tilde{a}) = (0,1,...,|\bar{a}|-1)$. Thus, $\Phi^{D(\mathcal{A})}$ maps $\tilde{a}$ to $\bar{a}$ in $\mathcal{A}$. Let s be such that $\Phi^{D(\mathcal{A})\restriction s}$ is defined on $\tilde{a}$. (I.e., let s be the *use* of the computation. As a convention, when we run a computable functional on a finite oracle $\sigma \in 2^{<\omega}$, we only run it for $|\sigma|$ steps.) Let $\bar{c} = (a_{|\bar{a}|}, ..., a_{s-1})$, so $\bar{a}\bar{c} = (a_0, ..., a_{s-1})$. Recall from the background section that $\varphi^{at}_{D(\mathcal{A})\restriction s}$ is the conjunction of the first s atomic (and negation of atomic) facts about $a_0, ..., a_{s-1}$. Thus, $\mathcal{A} \models \varphi^{at}_{D(\mathcal{A})\restriction s}(\bar{a}, \bar{c})$. Finally, define

$$\varphi_{\bar{a}}(\bar{x}) \quad \equiv \quad \exists \bar{y}\ \varphi^{at}_{D(\mathcal{A})\restriction s}(\bar{x}, \bar{y}).$$

We claim that $\varphi_{\bar{a}}$ supports the $\forall$-type of $\bar{a}$, and hence that it defines the orbit of $\bar{a}$. Let $\bar{b}$ be another tuple in $\mathcal{A}$ satisfying $\varphi_{\bar{a}}$; we need to show that $\bar{a}$ and $\bar{b}$ satisfy the same $\forall$-types. Let $\bar{d}$ be the witnesses for $\varphi_{\bar{a}}(\bar{b})$, i.e., such that $\mathcal{A} \models \varphi^{at}_{D(\mathcal{A})\restriction s}(\bar{b}, \bar{d})$. Consider a new presentation of $\mathcal{A}$, call it $\tilde{\mathcal{A}}$, where we permute $\bar{a}\bar{c}$ for $\bar{b}\bar{d}$ and leave the rest the same. Since the first s elements of the presentation $\tilde{\mathcal{A}}$ are $\bar{b}\bar{d}$, we have that

$$D(\tilde{\mathcal{A}})\restriction s = \mathcal{D}_{\tilde{\mathcal{A}}}(\bar{b}\bar{d}) = \mathcal{D}_{\mathcal{A}}(\bar{a}\bar{c}) = D(\mathcal{A})\restriction s.$$

It follows that $\Phi^{D(\tilde{\mathcal{A}})}(\tilde{a}) = \Phi^{D(\mathcal{A})}(\tilde{a}) = (0,1,...,|\bar{a}|-1)$. But now, in the new presentation $\tilde{\mathcal{A}}$, $(0,1,...,|\bar{a}|-1)$ corresponds to $\bar{b}$. Since both $\Phi^{D(\mathcal{A})}$ and $\Phi^{D(\tilde{\mathcal{A}})}$ preserve $\forall$-types, and $\Phi^{D(\mathcal{A})}(\tilde{a}) = \bar{a}$ and $\Phi^{D(\tilde{\mathcal{A}})}(\tilde{a}) = \bar{b}$, we have that

$$\forall\text{-}tp_{\mathcal{A}}(\bar{a}) = \forall\text{-}tp_{\mathcal{A}}(\tilde{a}) = \forall\text{-}tp(\bar{b}).$$

This proves that $\varphi_{\bar{a}}$ supports the $\forall$-type of $\bar{a}$ and hence defines its orbit. Since the definition of $\varphi_{\bar{a}}$ was uniform, we can build a whole c.e. Scott family for $\mathcal{A}$. □

7. Scott sentences of existentially atomic structures.

Scott [Sco65] showed that every countable structure has a Scott sentence in $\mathcal{L}_{\omega_1,\omega}$. We prove it below for $\exists$-atomic structures. The same proof would show that if a structure has a Scott family of $\Sigma^{\mathtt{in}}_{\alpha}$-formulas, it has a $\Pi^{\mathtt{in}}_{\alpha+1}$-Scott sentence. The key remaining step in Scott's proof is to show that every orbit in a countable structure is $\mathcal{L}_{\omega_1,\omega}$-definable by showing that if two elements satisfy the same $\mathcal{L}_{\omega_1,\omega}$-formulas, they are automorphic.

Lemma 7.1. *Every $\exists$-atomic structure has a $\Pi^{\mathtt{in}}_2$ Scott sentence. Furthermore, every effectively $\exists$-atomic computable structure has a $\Pi^{\mathtt{c}}_2$ Scott sentence.*

Proof. Let S be a Scott family of $\exists$-formulas for $\mathcal{A}$. For each $\bar{a} \in A^{<\omega}$, let $\varphi_{\bar{a}}(\bar{x})$ be the $\exists$-formula in S defining the orbit of $\mathcal{A}$. (For the empty tuple, let $\varphi_{\emptyset}()$ be a sentence that is always true.) For any other structure $\mathcal{B}$, consider the set

$$I_{\mathcal{B}} = \{(\bar{a}, \bar{b}) \in \mathcal{A}^{<\omega} \times \mathcal{B}^{<\omega} : \mathcal{B} \models \varphi_{\bar{a}}(\bar{b})\}.$$

If $I_{\mathcal{B}}$ had the back-and-forth property (see Definition 4.1), we would know that $\mathcal{B}$ is isomorphic to $\mathcal{A}$. Since $I_{\mathcal{A}}$ has the back-and-forth property (see proof of Theorem 1.2), we get that $I_{\mathcal{B}}$ has the back-and-forth property if and only if $\mathcal{B}$ is isomorphic to $\mathcal{A}$. Recall from Definition 4.1 that $I_{\mathcal{B}}$ has the back-and-forth property if and only if:

$$\bigwedge\!\!\!\bigwedge_{\bar{a}\in A^{<\omega}} \forall \bar{x} \in B^{|\bar{a}|} \Bigg(\langle \bar{a}, \bar{x}\rangle \in I_{\mathcal{B}} \Rightarrow$$
$$\Bigg(\Big(D_{\mathcal{A}}(\bar{a}) = D_{\mathcal{B}}(\bar{x}) \Big) \wedge \Big(\forall y \in B \bigvee\!\!\!\bigvee_{c\in A} (\langle \bar{a}c, \bar{x}y\rangle \in I_{\mathcal{B}}) \Big) \wedge \Big(\bigwedge\!\!\!\bigwedge_{c\in A} \exists y \in B (\langle \bar{a}c, \bar{x}y\rangle \in I_{\mathcal{B}}) \Big) \Bigg) \Bigg).$$

The Scott sentence for $\mathcal{A}$ is a sentence that is true of a structure $\mathcal{B}$ if and only if $I_{\mathcal{B}}$ has the back-and-forth property:

$$\bigwedge\!\!\!\bigwedge_{\bar{a}\in A^{<\omega}} \forall x_1, ..., x_{|\bar{a}|}\Bigg(\varphi_{\bar{a}}(\bar{x}) \Rightarrow \Bigg(\Big(\varphi^{at}_{D_{\mathcal{A}}(\bar{a})}(\bar{x})\Big) \wedge \Big(\forall y \bigvee\!\!\!\bigvee_{c\in A} \varphi_{\bar{a}c}(\bar{x}y)\Big) \wedge \Big(\bigwedge\!\!\!\bigwedge_{c\in A} \exists y \varphi_{\bar{a}c}(\bar{x}y)\Big)\Bigg)\Bigg).$$

As for the effectivity claim, if $\mathcal{A}$ is a computable presentation and S is c.e., then the map $\bar{a} \mapsto \varphi_{\bar{a}}$ is computable, and the conjunctions and disjunctions in the Scott sentence above are all computable. ◻

To prove the other direction, we need to go through the type-omitting theorem for $\forall$-types.

Proof of (A2) ⇒ (A1) in Theorem 1.2. Suppose ψ is a $\Pi^{\mathtt{in}}_2$ Scott sentence for $\mathcal{A}$, but that $\mathcal{A}$ is not atomic. We have already shown that (A1) implies (A5). Thus, there is a $\forall$-type realized in $\mathcal{A}$ which is not $\exists$-supported. But then, by Lemma 5.3, there exists a model of ψ which omits that type. This structure could not be isomorphic to $\mathcal{A}$, as they do not realize the same types. This contradicts that ψ is a Scott sentence for $\mathcal{A}$. ◻

Lemma 7.2. *Let $\mathcal{A}$ be a structure. The following are equivalent:*

(1) *$\mathcal{A}$ is $\exists$-atomic over a finite tuple of parameters.*
(2) *$\mathcal{A}$ has a $\Sigma^{\mathtt{in}}_3$-Scott sentence.*

Proof. If $\mathcal{A}$ is $\exists$-atomic over a finite tuple of parameters $\bar{a}$, then $(\mathcal{A}, \bar{a})$ has a $\Pi^{\mathtt{in}}_2$ Scott sentence $\varphi(\bar{c})$. Then $\exists\bar{y}\varphi(\bar{y})$ is a Scott sentence for $\mathcal{A}$.

Suppose now that $\mathcal{A}$ has a Scott sentence $\bigvee\!\!\!\bigvee_{i\in\omega} \exists\bar{y}_i\psi_i(\bar{y}_i)$. $\mathcal{A}$ must satisfy one of the disjuncts, and that disjunct must then be a Scott sentence for $\mathcal{A}$ too. So, suppose the Scott sentence for $\mathcal{A}$ is $\exists\bar{y}\ \psi(\bar{y})$, where ψ is $\Pi^{\mathtt{in}}_2$. Let $\bar{c}$ be a new tuple of constants of the same size as $\bar{y}$. If $\varphi(\bar{c})$ were a Scott sentence for $(\mathcal{A}, \bar{a})$, we would know $\mathcal{A}$ is $\exists$-atomic over $\bar{a}$ — but this might not be the case. Suppose $(\mathcal{B}, \bar{b}) \models \varphi(\bar{c})$. Then $\mathcal{B}$ must be isomorphic to $\mathcal{A}$, as it satisfies $\exists\bar{y}\ \psi(\bar{y})$. But $(\mathcal{B}, \bar{b})$ and $(\mathcal{A}, \bar{a})$ need not be isomorphic. However, it is enough for us to show that one of the models of $\varphi(\bar{c})$ is $\exists$-atomic over $\bar{c}$. Since there are only countably many models of $\varphi(\bar{c})$, there are countably many $\forall$-types among the models of $\varphi(\bar{c})$. Thus, we can omit the non-$\exists$-supported ones while satisfying $\varphi(\bar{c})$. The resulting structure would be $\exists$-atomic over $\bar{c}$ and isomorphic to $\mathcal{A}$. ◻

We remark that in [Mon] we mentioned this fact, but did not give a proof, as we overlooked the fact that $(\mathcal{B}, \bar{b})$ and $(\mathcal{A}, \bar{a})$ in the proof above need not be isomorphic. The extra step in the proof above seems to be necessary.

8. Turing degree and enumeration degree

The proof of Theorem 1.14 needs a couple of lemmas that are interesting in their own right.

Lemma 8.1. *Let $\mathbb{K}$ be a $\Pi^{\mathtt{c}}_2$ class all of whose structures have different $\exists$-theories. Then every structure in $\mathbb{K}$ has enumeration degree given by its $\exists$-theory.*

Proof. Take a structure $\mathcal{A} \in \mathbb{K}$, and let S be its $\exists$-theory. By Corollary 5.2, if X can compute an enumeration of S, then it can compute a presentation of a structure $\mathcal{B} \in \mathbb{K}$ with $\exists$-theory S. Since both $\mathcal{A}$ and $\mathcal{B}$ have the same $\exists$-theory, they must be isomorphic. So, X is computing a copy of $\mathcal{A}$. Of course, every copy of $\mathcal{A}$ can enumerate S, and hence $\mathcal{A}$ has enumeration degree S. ◻

The following lemma is a strengthening of Lemma 3.3.

Lemma 8.2. *If $\mathcal{A}$ and $\mathcal{B}$ are $\exists$-atomic and have the same $\exists$-theory, then they are isomorphic.*

Proof. We prove that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic using a back-and-forth construction. Let

$$I = \{\langle \bar{a}, \bar{b}\rangle : \forall\text{-}tp_{\mathcal{A}}(a_0, ..., a_s) = \forall\text{-}tp_{\mathcal{B}}(b_0, ..., b_s)\}.$$

By assumption, $\langle \emptyset, \emptyset\rangle \in I$. We need to show that I has the back-and-forth property (Definition 4.1), as that would imply that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. Clearly, $\forall\text{-}tp_{\mathcal{A}}(a_0, ..., a_s) = \forall\text{-}tp_{\mathcal{B}}(b_0, ..., b_s)$ implies $D_{\mathcal{A}}(a_0, ..., a_s) = D_{\mathcal{B}}(b_0, ..., b_s)$. For the second condition in Definition 4.1, suppose $\langle \bar{a}, \bar{b}\rangle \in I$ and let $c \in A$. Let ψ be the principal $\exists$-formula satisfied by $\bar{a}c$. Since $\forall\text{-}tp_{\mathcal{A}}(\bar{a}) = \forall\text{-}tp_{\mathcal{B}}(\bar{b})$, there is a d in $\mathcal{B}$ satisfying the same formula over $\bar{b}$. We need to show that $\forall\text{-}tp_{\mathcal{A}}(\bar{a}c) = \forall\text{-}tp_{\mathcal{B}}(\bar{b}d)$. Let us remark that since we do not know $\mathcal{A}$ and $\mathcal{B}$ are isomorphic yet, we do not know that ψ generates a $\forall$-type in $\mathcal{B}$.

First, to show $\forall\text{-}tp_{\mathcal{A}}(\bar{a}c) \subseteq \forall\text{-}tp_{\mathcal{B}}(\bar{b}d)$, take $\theta(\bar{x}y) \in \forall\text{-}tp_{\mathcal{A}}(\bar{a}c)$. Then

$$\text{“}\forall y(\psi(\bar{x}y) \rightarrow \theta(\bar{x}y))\text{”} \ \in\ \forall\text{-}tp_{\mathcal{A}}(\bar{a}c) \ =\ \forall\text{-}tp_{\mathcal{B}}(\bar{b}d),$$

and hence $\theta \in \forall\text{-}tp_{\mathcal{B}}(\bar{b}d)$. Let us now prove the other inclusion. Let $\tilde{\psi}(\bar{x}y)$ be the $\exists$-formula generating $\forall\text{-}tp_{\mathcal{B}}(\bar{b}d)$ in $\mathcal{B}$. Then, since $\neg\tilde{\psi} \notin \forall\text{-}tp_{\mathcal{B}}(\bar{b}d)$, by our previous argument, $\neg\tilde{\psi} \notin \forall\text{-}tp_{\mathcal{A}}(\bar{a}c)$ either, and hence $\mathcal{A} \models \tilde{\psi}(\bar{a}c)$. The rest of the proof that $\forall\text{-}tp_{\mathcal{B}}(\bar{b}d) \subseteq \forall\text{-}tp_{\mathcal{A}}(\bar{a}c)$ is now symmetrical to the one of the other inclusion: For $\tilde{\theta}(\bar{x}y) \in \forall\text{-}tp_{\mathcal{A}}(\bar{b}d)$, we have that “$\forall y(\tilde{\psi}(\bar{x}y) \rightarrow \tilde{\theta}(\bar{x}y))$” $\in \forall\text{-}tp_{\mathcal{A}}(\bar{a}c)$, and hence $\theta \in \forall\text{-}tp_{\mathcal{B}}(\bar{a}c)$. □

Proof of Theorem 1.14. The proof is immediate from Lemmas 8.1 and 8.2. □

The following gives a structural property that is sufficient for a structure to have enumeration degree. The property is far from necessary though.

Corollary 8.3. *Suppose that a structure $\mathcal{A}$ has a $\Sigma^{\mathtt{c}}_3$ Scott sentence. Then $\mathcal{A}$ has enumeration degree.*

Proof. Let $\bigvee\!\!\!\bigvee_{i\in\omega} \exists \bar{x}_i\ \psi_i(\bar{x}_i)$ be the $\Sigma^{\mathtt{c}}_3$ Scott sentence for $\mathcal{A}$, where each ψ_i is $\Pi^{\mathtt{c}}_2$. $\mathcal{A}$ satisfies one of the disjuncts, say $\exists \bar{x}_i\ (\psi_i(\bar{x}_i))$, and hence this disjunct is also a Scott sentence for $\mathcal{A}$. Let $\tilde{\tau}$ be the vocabulary τ of $\mathcal{A}$, together with $|\bar{x}_i|$ many new constant symbols $\bar{c}$, and let $\tilde{\mathcal{A}}$ be the $\tilde{\tau}$-structure $(\mathcal{A}, \bar{a})$, where $\bar{a}$ is such that $\mathcal{A} \models \psi_i(\bar{a})$. Now, even if this sentence might not be a Scott sentence for $\tilde{\mathcal{A}}$, we can still work with it. We claim that $\mathcal{A}$ has enumeration degree given by $\exists\text{-}tp_{\mathcal{A}}(\bar{a})$, which is the same as $\exists$-theory$(\tilde{\mathcal{A}})$. Clearly, every copy of $\mathcal{A}$ can enumerate $\exists\text{-}tp_{\mathcal{A}}(\bar{a})$. On the other hand, using $\exists$-theory$(\tilde{\mathcal{A}})$ and the $\Pi^{\mathtt{c}}_2$ sentence $\psi_i(\bar{c})$, we can build a model of $\psi_i(\bar{c})$ by Corollary 5.2. Even if this model does not turn out to be isomorphic to $\tilde{\mathcal{A}}$, when we look at it as a τ-structure, it is isomorphic to $\mathcal{A}$. □

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